Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 5, No. 2, (March 2013), pp. 377–386
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On fuzzy soft topological spaces

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Received 30 April 2012; Revised 10 July 2012; Accepted 3 August 2012

Abstract. Topological structure of fuzzy soft sets has been recently introduced by Tanay and Kandemir in [23]. In this paper, we introduce the notion of soft quasi-coincidence for fuzzy soft sets and we use this notion to characterize fundamental concepts of fuzzy soft topological spaces such as fuzzy soft closures, fuzzy soft bases and fuzzy soft continuity. Some basic properties of this concept are also presented.

2010 AMS Classification: 54A40, 06D72

Keywords: Fuzzy soft set, Fuzzy soft topology, Soft quasi-coincidence.

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1. Introduction

Engineering, physics, computer sciences, economics, social sciences, medical sciences and many other diverse fields deal daily with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; two of them are fuzzy set theory, developed by Zadeh [24], and soft set theory, introduced by Molodtsov [19], that are related to this work. At present, works on the soft set theory and its applications are progressing rapidly. Maji et al [14] defined operations of soft sets to make a detailed theoretical study on the soft sets. By using these definitions, soft set theory has been applied in several directions, such as topology [5, 17, 22, 23, 25], various algebraic structures [2, 3, 7, 11], operations research [4, 9, 10] especially decision-making [6, 8, 13, 15, 20]. In recent times, researchers have contributed a lot towards fuzzification of soft set theory. Maji et al. [16] introduced the concept of fuzzy soft set and some properties regarding fuzzy soft union, intersection, complement of a fuzzy soft set, De Morgan Law etc. These results were further revised and improved by Ahmad and Kharal [1]. Tanay and Kandemir [23] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [21] gave the definition of fuzzy soft topology over the initial universe set.

In this paper, we have studied some properties related to fuzzy soft topological spaces. Our work is an attempt to introduce the concept on soft quasi-coincidence for fuzzy soft sets. By using the concept of soft quasi-coincidence, we characterize fundamental notions of fuzzy soft topological space such as fuzzy soft closure, fuzzy soft base and fuzzy soft continuity.

2. Preliminaries

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$.

Definition 2.1 ([24]). A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \rightarrow [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of x in A for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh as follows:
(1) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in X$.
(3) $C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)$, for all $x \in X$.
(4) $D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$, for all $x \in X$.
(5) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, for all $x \in X$.

A fuzzy point in X, whose value is α $(0 < \alpha \leq 1)$ at the support $x \in X$, is denoted by x_α [24]. A fuzzy point $x_\alpha \in A$, where A is a fuzzy set in X iff $\alpha \leq \mu_A(x)$ [24]. The class all fuzzy points will be denoted by $S(X)$.

Definition 2.2 ([18]). For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If A is not quasi-coincident with B, then we write $A\overline{q}B$.

Definition 2.3 ([19]). Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \longrightarrow P(X)$ and $A \subseteq E$.

In the other words, the soft set is a parametrized family of subsets of the set X. Every set $F(e)$, for every $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

Definition 2.4 ([16]). Let $A \subseteq E$. A pair (f, A) is called a fuzzy soft set over X if $f : A \longrightarrow I^X$ is a function.

We will use $FS(X, E)$ instead of the family of all fuzzy soft sets over X.

Roy and Samanta [21] did some modifications in above definition analogously ideas made for soft sets.

Definition 2.5 ([21]). Let $A \subseteq E$. A fuzzy soft set f_A over universe X is mapping from the parameter set E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subset E$ and $f_A(e) = 0_X$ if $e \notin A$, where 0_X denotes empty fuzzy set on X.

Definition 2.6 ([21]). The fuzzy soft set $f_\varnothing \in FS(X, E)$ is called null fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_\varnothing(e) = 0_X$.

Definition 2.7 ([21]). Let $f_E \in FS(X,E)$. The fuzzy soft set f_E is called universal fuzzy soft set, denoted by $\widetilde{1}_E$, if for all $e \in E$, $f_E(e) = 1_X$ where $1_X(x) = 1$ for all $x \in X$.

Definition 2.8 ([21]). Let $f_A, g_B \in FS(X,E)$. f_A is called a fuzzy soft subset of g_B if $f_A(e) \leq g_B(e)$ for every $e \in E$ and we write $f_A \sqsubseteq g_B$.

Definition 2.9 ([21]). Let $f_A, g_B \in FS(X,E)$. f_A and g_B are said to be equal, denoted by $f_A = g_B$ if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

Definition 2.10 ([21]). Let $f_A, g_B \in FS(X,E)$. Then the union of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \vee g_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $h_C = f_A \sqcup g_B$.

Definition 2.11 ([21]). Let $f_A, g_B \in FS(X,E)$. Then the intersection of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \sqcap g_B$.

Definition 2.12 ([23]). Let $f_A \in FS(X,E)$. The complement of f_A, denoted by f_A^c, is a fuzzy soft set defined by $f_A^c(e) = 1 - f_A(e)$ for every $e \in E$.

Let us call f_A^c to be fuzzy soft complement function of f_A. Clearly $(f_A^c)^c = f_A$, $(\widetilde{1}_E)^c = \widetilde{0}_E$ and $(\widetilde{0}_E)^c = \widetilde{1}_E$.

Definition 2.13 ([12]). Let $FS(X,E)$ and $FS(Y,K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then f_{up} is called a fuzzy soft mapping from X to Y and denoted by $f_{up} : FS(X,E) \to FS(Y,K)$.

(1) Let $f_A \in FS(X,E)$, then the image of f_A under the fuzzy soft mapping f_{up} is the fuzzy soft set over Y defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \begin{cases} \bigvee_{x \in u^{-1}(y)} \left(\bigvee_{e \in p^{-1}(k) \cap A} f_A(e) \right)(x) & \text{if } u^{-1}(y) \neq \varnothing,\ p^{-1}(k) \cap A \neq \varnothing, \\ 0_Y & \text{otherwise,} \end{cases}$$

(2) Let $g_B \in FS(Y,K)$, then the preimage of g_B under the fuzzy soft mapping f_{up} is the fuzzy soft set over X defined by $f_{up}^{-1}(g_B)$, where

$$f_{up}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & \text{for } p(e) \in B, \\ 0_X & \text{otherwise,} \end{cases}$$

If u and p are injective then the fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the fuzzy soft mapping f_{up} is said to be surjective. The fuzzy soft mapping f_{up} is called constant, if u and p are constant.

Theorem 2.14 ([12]). *Let $f_A, \in FS(X,E)$, $\{f_{A_i}\}_{i \in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, $\{g_{B_i}\}_{i \in J} \subset FS(Y,K)$, where J is an index set.*

(1) *If $(f_{A_1}) \sqsubseteq (f_{A_2})$, then $f_{up}(f_{A_1}) \sqsubseteq f_{up}(f_{A_2})$.*
(2) *If $(g_{B_1}) \sqsubseteq (g_{B_2})$, then $f_{up}^{-1}(g_{B_1}) \sqsubseteq f_{up}^{-1}(g_{B_2})$.*
(3) *$f_{up}(\sqcup_{i \in J}(f_{A_i})) = \sqcup_{i \in J} f_{up}(f_{A_i})$.*
(4) *$f_{up}(\sqcap_{i \in J}(f_{A_i})) \sqsubseteq \sqcap_{i \in J} f_{up}(f_{A_i})$*
(5) *$f_{up}^{-1}(\sqcup_{i \in J}(g_{B_i})) = \sqcup_{i \in J} f_{up}^{-1}(g_{B_i})$.*

(6) $f_{up}^{-1}(\sqcap_{i\in J}(g_{B_i})) = \sqcap_{i\in J} f_{up}^{-1}(g_{B_i})$.
(7) $f_{up}^{-1}(\widetilde{1}_K) = \widetilde{1}_E$, $f_{up}^{-1}(\widetilde{0}_K) = \widetilde{0}_E$.
(8) $f_{up}(\widetilde{0}_E) = \widetilde{0}_K$, $f_{up}(\widetilde{1}_E) \sqsubseteq \widetilde{1}_K$.

Theorem 2.15. *Let $\{f_{A_i}\}_{i\in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, where J is an index set.*
(1) $f_{up}(\sqcap_{i\in J}(f_{A_i})) = \sqcap_{i\in J} f_{up}(f_{A_i})$ *if f_{up} is injective.*
(2) $f_{up}(\widetilde{1}_E) = \widetilde{1}_K$ *if f_{up} is surjective.*
(3) $f_{up}^{-1}(g_B^c) = (f_{up}^{-1}(g_B))^c$

Proof. They are proved easily. □

3. Soft quasi-coincidence

Definition 3.1. The fuzzy soft set $f_A \in FS(X,E)$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X i.e. there exists $x \in X$ such that $f_A(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denote this fuzzy soft point $f_A = e_x^\alpha = \{(e, x_\alpha)\}$.

Definition 3.2. Let e_x^α, $f_A \in FS(X,E)$. We say that $e_x^\alpha \widetilde{\in} f_A$ read as e_x^α belongs to the fuzzy soft set f_A if for the element $e \in A$, $\alpha \leq f_A(e)(x)$.

Proposition 3.3. *Every non null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points which belong to f_A.*

Proof. This follows from the fact that any fuzzy set is the union of fuzzy points which belong to it [18]. □

Definition 3.4. Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. We say that $x_\alpha \in f_A$ read as x_α belongs to the fuzzy soft set f_A whenever $x_\alpha \in f_A(e)$, i.e. $\alpha \leq f_A(e)(x)$ for all $e \in A$.

Definition 3.5. Let f_A, $g_B \in FS(X,E)$. f_A is said to be soft quasi-coincident with g_B, denoted by $f_A q g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If f_A is not soft quasi-coincident with g_B, then we write $f_A \overline{q} g_B$.

Definition 3.6. Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. x_α is said to be soft quasi-coincident with f_A, denoted by $x_\alpha q f_A$, if and only if there exists an $e \in E$ such that $\alpha + f_A(e)(x) > 1$.

Proposition 3.7. *Let f_A, $g_B \in FS(X,E)$, Then the followings are true.*
(1) $f_A \sqsubseteq g_B \Leftrightarrow f_A \overline{q} g_B^c$.
(2) $f_A q g_B \Rightarrow f_A \sqcap g_B \neq \widetilde{0}_E$
(3) $x_\alpha \overline{q} f_A \Leftrightarrow x_\alpha \in f_A^c$
(4) $f_A \overline{q} f_A^c$.
(5) $f_A \sqsubseteq g_B \Rightarrow x_\alpha q f_A$ *implies* $x_\alpha q g_B$.
(6) $f_A q g_B \Leftrightarrow$*there exists an* $e_x^\alpha \widetilde{\in} f_A$ *such that* $e_x^\alpha q g_B$.
(7) $e_x^\alpha \overline{q} f_A \Leftrightarrow e_x^\alpha \widetilde{\in} f_A^c$.
(8) $f_A \sqsubseteq g_B \Leftrightarrow$ *If* $e_x^\alpha q f_A$, *then* $e_x^\alpha q g_B$ *for all* $e_x^\alpha \in FS(X,E)$.

Proof.

(1) $f_A \sqsubseteq g_B \Leftrightarrow$ for all $e \in E$ and all $x \in X$, $f_A(e)(x) \leq g_B(e)(x)$
$\Leftrightarrow$ for all $e \in E$ and all $x \in X$, $f_A(e)(x) - g_B(e)(x) \leq 0$
$\Leftrightarrow$ for all $e \in E$ and all $x \in X$, $f_A(e)(x) + 1 - g_B(e)(x) \leq 1$
$\Leftrightarrow f_A \overline{q} g_B^c$

(2) Let $f_A q g_B$. Then there exist an $e \in E$ and a $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$. This implies that $f_A(e)(x) \neq 0$ and $g_B(e)(x) \neq 0$ for $e \in E$ and $x \in X$. Hence $f_A \sqcap g_B \neq \widetilde{0}_E$.

(3) $x_\alpha \overline{q} f_A \Leftrightarrow$ for all $e \in E$, $\alpha + f_A(e)(x) \leq 1$
$\Leftrightarrow$ for all $e \in E$, $\alpha \leq 1 - f_A(e)(x)$
$\Leftrightarrow$ for all $e \in E$, $\alpha \leq f_A^c(e)(x)$
$\Leftrightarrow x_\alpha \in f_A^c$

(4) Suppose that $f_A q f_A^c$. Then there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + f_A^c(e)(x) > 1$. Then $f_A(e)(x) + 1 - f_A(e)(x) > 1$ and so $f_A(e)(x) > f_A(e)(x)$, this is contradiction.

(5) Let $x_\alpha q f_A$. Then there exists $e \in E$ such that $\alpha + f_A(e)(x) > 1$. Since $f_A \sqsubseteq g_B$, $\alpha + g_B(e)(x) > 1$ for some $e \in B$. Hence we have $x_\alpha q g_B$.

(6) If $f_A q g_B$, then there exist an $e \in E$ and a $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$. Put $f_A(e)(x) = \alpha$. Then we have $e_x^\alpha \in f_A$ and $e_x^\alpha q g_B$.

Conversely, suppose that $e_x^\alpha q g_B$ for some $e_x^\alpha \widetilde{\in} f_A$. Then $\alpha \leq f_A(e)(x)$ and $\alpha + g_B(e)(x) > 1$. Therefore, we have $f_A(e)(x) + g_B(e)(x) > 1$ for an $e \in E$ and a $x \in X$. This shows that $f_A q g_B$.

(7) It is obvious from (1).

(8) Let $e_x^\alpha, f_A \in FS(X,E)$ and $e_x^\alpha q f_A$. Then $\alpha + f_A(e)(x) > 1$. Since $f_A \sqsubseteq g_B$, $\alpha + g_B(e)(x) > 1$. Hence we have $e_x^\alpha q g_B$.

Conversely, suppose that f_A is not fuzzy soft subset g_B. Then there exists an $e \in E$ and a $x \in X$ such that $f_A(e)(x) > g_B(e)(x)$. If we choose $g_B(e)(x) = \alpha$, then $e_x^{1-\alpha} q f_A$ and $e_x^{1-\alpha} \overline{q} g_B$. This is contradiction. □

Proposition 3.8. *Let $\{f_{A_i} : i \in J\}$ be a family of fuzzy soft sets in $FS(X,E)$ where J is an index set. Then e_x^α is soft quasi-coincident with $\sqcup_{i \in J} f_{A_i}$ if and only if there exists some $f_{A_i} \in \{f_{A_i} : i \in J\}$ such that $e_x^\alpha q f_{A_i}$.*

Proof. Obvious. □

Theorem 3.9. *Let $f_A \in FS(X,E)$, $g_B \in FS(Y,K)$ and $f_{up} : FS(X,E) \to FS(Y,K)$ be fuzzy soft mapping. Then*
(1) $g_B \overline{q} f_{up}(f_A) \Rightarrow f_{up}^{-1}(g_B) \overline{q} f_A$
(2) $g_B q\ f_{up}(f_A) \Rightarrow f_{up}^{-1}(g_B) q f_A$

Proof.

(1) $g_B \overline{q} f_{up}(f_A) \Rightarrow f_{up}(f_A) \sqsubseteq g_B^c$
$\Rightarrow f_A \sqsubseteq f_{up}^{-1}(f_{up}(f_A)) \sqsubseteq f_{up}^{-1}(g_B^c)$
$\Rightarrow f_A \sqsubseteq (f_{up}^{-1}(g_B))^c$
$\Rightarrow f_A \overline{q} f_{up}^{-1}(g_B)$

(2) Let $f_{up}(f_A) q g_B$ and $f_A \overline{q} f_{up}^{-1}(g_B)$. Then $f_A \sqsubseteq (f_{up}^{-1}(g_B))^c = (f_{up}^{-1}(g_B^c))$. It follows that $f_{up}(f_A) \sqsubseteq f_{up}(f_{up}^{-1}(g_B^c)) \sqsubseteq g_B^c$. This shows that $f_{up}(f_A) \overline{q} g_B$. This is contradiction. □

4. Some applications of soft quasi-coincidence

Definition 4.1. (see [23, 21]) A fuzzy soft topological space is a pair (X, τ) where X is a nonempty set and τ is a family of fuzzy soft sets over X satisfying the following properties:
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau$
(2) If f_A, $g_B \in \tau$, then $f_A \sqcap g_B \in \tau$
(3) If $f_{Ai} \in \tau$, $\forall i \in J$, then $\sqcup_{i \in J} f_{Ai} \in \tau$.
Then τ is called a topology of fuzzy soft sets on X. Every member of τ is called fuzzy soft open. g_B is called fuzzy soft closed in (X, τ) if $(g_B)^c \in \tau$.

Theorem 4.2. *Let (X, τ) be a fuzzy soft topological space and τ' denotes the collection of all fuzzy soft closed sets. Then*
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau'$
(2) *If* f_A, $g_B \in \tau'$, *then* $f_A \sqcup g_B \in \tau'$
(3) *If* $f_{Ai} \in \tau'$, $\forall i \in J$, *then* $\sqcap_{i \in J} f_{Ai} \in \tau$.

Definition 4.3 ([23]). Let (X, τ) be a fuzzy soft topological space and $f_A \in FS(X, E)$. The fuzzy soft closure of f_A denoted by $\overline{f_A}$ is the intersection of all fuzzy soft closed supersets of f_A.

Clearly, $\overline{f_A}$ is the smallest fuzzy soft closed set over X which contains f_A.

Definition 4.4 ([23]). Let (X, τ) be a fuzzy soft topological space and $f_A \in FS(X, E)$. The fuzzy soft interior of f_A denoted by f_A° is the union of all fuzzy soft open subsets of f_A.

Clearly, f_A° is the largest fuzzy soft open set over X which contained in f_A.

Theorem 4.5. *Let (X, τ) be a fuzzy soft topological space and $f_A, g_B \in FS(X, E)$. Then,*
(1) $(\overline{f_A})^c \sqsubseteq (f_A^c)^{\circ}$.
(2) $(f_A^{\circ})^c \sqsubseteq \overline{(f_A^c)}$.

Proof.

$$
\begin{aligned}
(1)\quad (\overline{f_A})^c &= (\sqcap \{g_B \mid g_B \text{ is fuzzy soft closed set and } f_A \sqsubseteq g_B\})^c \\
&= (\sqcup \{g_B^c \mid g_B \text{ is fuzzy soft closed set and } f_A \sqsubseteq g_B\}) \\
&= (\sqcup \{g_B^c \mid g_B^c \text{ is fuzzy soft open set and } g_B^c \sqsubseteq f_A^c\}) \\
&= (f_A^c)^{\circ} \\
(2)\quad (f_A^{\circ})^c &= (\sqcup \{g_B \mid g_B \text{ is fuzzy soft open set and } g_B \sqsubseteq f_A\})^c \\
&= (\sqcap \{g_B^c \mid g_B \text{ is fuzzy soft open set and } g_B \sqsubseteq f_A\}) \\
&= (\sqcap \{g_B^c \mid g_B^c \text{ is fuzzy soft closed set and } f_A^c \sqsubseteq g_B^c\}) \\
&= \overline{(f_A^c)}
\end{aligned}
$$
□

Definition 4.6. A fuzzy soft set f_A in $FS(X, E)$ is called Q-neighborhood (briefly, Q-nbd) of g_B if and only if there exists a fuzzy soft open set h_C in τ such that $g_B q h_C \sqsubseteq f_A$.

Theorem 4.7. *Let e_x^{α}, $f_A \in FS(X, E)$. Then $e_x^{\alpha} \widetilde{\in} \overline{f_A}$ if and only if each Q-nbd of e_x^{α} is soft quasi-coincident with f_A.*

Proof. Let $e_x^\alpha \widetilde{\in} \overline{f_A}$. For every fuzzy soft closed set g_B which is containing f_A, $e_x^\alpha \widetilde{\in} g_B$ or $g_B(e)(x) \geq \alpha$. Suppose that h_C be a Q-nbd of e_x^α and $h_C \overline{q} f_A$. Then for every $e \in E$ and every $x \in X$, $h_C(e)(x) + f_A(e)(x) \leq 1$ and so $f_A \sqsubseteq h_C^c$. Since h_C is Q-nbd of e_x^α, by Proposition 3.7(7), e_x^α does not belong to h_C^c. Therefore, we have that e_x^α does not belong to $\overline{f_A}$. This is a contradiction.

Conversely, Let each Q-nbd of e_x^α be soft quasi-coincident with f_A. Suppose that e_x^α does not belong to $\overline{f_A}$. Then there exists a fuzzy soft closed set g_B which is containing f_A such that e_x^α does not belong to g_B. By proposition 3.7(7), we have $e_x^\alpha q g_B^c$. Then g_B^c is a Q-nbd of e_x^α and by Proposition 3.7(1), $f_A \overline{q} g_B^c$. This is a contradiction with the hypothesis. □

Definition 4.8 ([23])**.** Let (X, τ) be a fuzzy soft topological space and β be a subfamily of τ. If every element of τ can be written as the arbitrary fuzzy soft union of some elements of β, then β is called a fuzzy soft basis for the fuzzy soft topology τ.

Proposition 4.9. *Let (X, τ) be a fuzzy soft topological space and β is subfamily of τ. β is a base for τ if and only if for each e_x^α in $FS(X, E)$ and for each fuzzy soft open Q-nbd f_A of e_x^α, there exists a $g_B \in \beta$ such that $e_x^\alpha q g_B \sqsubseteq f_A$.*

Proof. Let β be a base for τ, $e_x^\alpha \in FS(X, E)$ and f_A be fuzzy soft open Q-nbd of e_x^α. Then there exists a subfamily β' of β such that $f_A = \sqcup \{g_B | \, g_B \in \beta'\}$. Suppose that $e_x^\alpha \overline{q} g_B$ for all $g_B \in \beta'$. Then $\alpha + g_B(e)(x) \leq 1$ for every $e \in E$. Therefore, we have $\alpha + f_A(e)(x) \leq 1$ since $f_A(e)(x) = \sup \{g_B(e)(x) | \, g_B \in \beta'\}$. This is contradiction.

Conversely, If β is not a base for τ, then there exists a $f_A \in \tau$ such that $h_C = \sqcup \{g_B \in \beta | \, g_B \sqsubseteq f_A\} \neq f_A$. Since $h_C \neq f_A$, there exist $e \in E$ and $x \in X$ such that $h_C(e)(x) < f_A(e)(x)$. Put $\alpha = 1 - h_C(e)(x)$. Since $1 = h_C(e)(x) + \alpha < f_A(e)(x) + \alpha$, $e_x^\alpha q f_A$. But since any member $g_B \in \beta$ which is contained in f_A is contained in h_C, we have $g_B(e)(x) + \alpha \leq h_C(e)(x) + \alpha = 1$; that is, $e_x^\alpha \overline{q} g_B$. This is a contradiction. □

Definition 4.10 ([23])**.** A fuzzy soft set g_B in a fuzzy soft topological space (X, τ) is called a fuzzy soft neighborhood (briefly: nbd) of the fuzzy soft set f_A if there exists a fuzzy soft open set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$.

Theorem 4.11. *g_B is fuzzy soft open if and only if for each fuzzy soft set f_A contained in g_B, g_B is a fuzzy soft neighborhood of f_A.*

Proof.

($\Rightarrow$) Obvious.

($\Leftarrow$) Since $g_B \sqsubseteq g_B$, there exists a fuzzy soft open set h_C such that $g_B \sqsubseteq h_C \sqsubseteq g_B$. Hence $h_C = g_B$ and g_B is fuzzy soft open. □

Definition 4.12. Let (X, τ_1) and (Y, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{up} : (X, \tau_1) \to (Y, \tau_2)$ is called fuzzy soft continuous if $f_{up}^{-1}(g_B) \in \tau_1$ for all $g_B \in \tau_2$.

Theorem 4.13. *Let (X, τ_1) and (Y, τ_2) be fuzzy soft topological spaces. For a function $f_{up} : FS(X, E) \longrightarrow FS(Y, K)$, the following statements are equivalent:*

(a) *f_{up} is fuzzy soft continuous;*

(b) *for each fuzzy soft set* f_A *in* $FS(X,E)$*, the inverse image of every nbd of* $f_{up}(f_A)$ *is a nbd of* f_A*;*

(c) *for each soft set* f_A *in* $FS(X,E)$ *and each nbd* h_C *of* $f_{up}(f_A)$*, there is a nbd* g_B *of* f_A *such that* $f_{up}(g_B) \sqsubseteq h_C$*;*

Proof.

(a)⇒(b). Let f_{up} be fuzzy soft continuous. If h_C is a nbd of $f_{up}(f_A)$, then h_C contains a fuzzy soft open nbd g_B of $f_{up}(f_A)$. Since $f_{up}(f_A) \sqsubseteq g_B \sqsubseteq h_C$, $f_{up}^{-1}(f_{up}(f_A)) \sqsubseteq f_{up}^{-1}(g_B) \sqsubseteq f_{up}^{-1}(h_C)$. But $f_A \sqsubseteq f_{up}^{-1}(f_{up}(f_A))$ and $f_{up}^{-1}(g_B)$ is fuzzy soft open. Consequently,$f_{up}^{-1}(h_C)$ is a nbd of f_A.

(b)⇒(a). We will use Theorem 4.11. Let g_B be fuzzy soft open over Y. Then $f_{up}^{-1}(g_B)$ is a fuzzy soft set over X. Let f_A be any fuzzy soft subset of $f_{up}^{-1}(g_B)$. Then g_B is a fuzzy soft open nbd of $f_{up}(f_A)$, and by (b), $f_{up}^{-1}(g_B)$ is a fuzzy soft nbd of f_A. This shows that $f_{up}^{-1}(g_B)$ is a fuzzy soft open set.

(b)⇒(c). Let f_A be any fuzzy soft set over X and let h_C be any nbd of $f_{up}(f_A)$. By (b), $f_{up}^{-1}(h_C)$ is a nbd of f_A. Then there exists a fuzzy soft open set g_B in (X,τ_1) such that $f_A \sqsubseteq g_B \sqsubseteq f_{up}^{-1}(h_C)$. Thus, we have a fuzzy soft open nbd g_B of f_A such that $f_{up}(f_A) \sqsubseteq f_{up}(g_B) \sqsubseteq h_C$.

(c)⇒(b). Let h_C be a nbd of $f_{up}(f_A)$. Then there is a nbd g_B of f_A such that $f_{up}(g_B) \sqsubseteq h_C$. Hence $f_{up}^{-1}(f_{up}(g_B)) \sqsubseteq f_{up}^{-1}(h_C)$. Furthermore, since $g_B \sqsubseteq f_{up}^{-1}(f_{up}(g_B))$, $f_{up}^{-1}(h_C)$ is a nbd of f_A. □

Theorem 4.14. *A mapping* $f_{up} : (X,E) \to (Y,K)$ *is fuzzy soft continuous if and only if corresponding fuzzy soft open Q-nbd* g_B *of* k_y^α *in* $FS(Y,K)$ *there exists a fuzzy soft open Q-nbd* f_A *of* e_x^α *in* $FS(X,E)$ *such that* $f_{up}(f_A) \sqsubseteq g_B$*, where* $f_{up}(e_x^\alpha) = k_y^\alpha$

Proof. Let f_{up} be continuous and let g_B be a fuzzy soft open Q-nbd of k_y^α in $FS(Y,K)$. Then $\alpha + g_B(k)(y) > 1$ and hence there exists a positive real number β such that $g_B(k)(y) > \beta > 1-\alpha$ so that g_B is a fuzzy soft open nbd of k_y^β. Since f_{up} is continuous, there exists a fuzzy soft open nbd f_A of e_x^β such that $f_{up}(f_A) \sqsubseteq g_B$. Now, $\beta \le f_A(e)(x)$ implies $1-\alpha < f_A(e)(x)$ and so f_A is a fuzzy soft open Q-nbd e_x^α.

Conversely, let the given condition hold. Let g_B be a fuzzy soft open set in $FS(Y,K)$. Let us $f_A = f_{up}^{-1}(g_B)$. If $f_A \neq \widetilde{0}_E$, then there exists $e \in A$ and $x \in X$, $f_A(e)(x) \neq 0$. Put $p(e) = k$, $u(x) = y$ so that $f_A(e)(x) = g_B(k)(y)$, there exists a positive integer m such that $\dfrac{1}{m} \le f_A(e)(x)$. Put $\alpha_n = 1 - f_A(e)(x) + \dfrac{1}{n}$ for $n \ge m$. Then $0 < \alpha_n \le 1$ for all $n \ge m$. Thus $g_B(k)(y) + \alpha_n = 1 + \dfrac{1}{n} > 1$ for each $n \ge m$ and by the given condition, there exists a fuzzy soft open h_{C_n} in $FS(X,E)$ such that $e_x^{\alpha_n} q h_{C_n}$ and $f_{up}(h_{Cn}) \sqsubseteq g_B$ for all $n \ge m$. Let us set $h_C = \sqcup\{h_{Cn} \mid n \ge m\}$ then $f_{up}(h_C) \sqsubseteq g_B$. Also, $n \ge m$ implies $h_{Cn}(e)(x) + \alpha_n > 1$ so that $h_{Cn}(e)(x) > f_A(e)(x) - \dfrac{1}{n}$ and hence $h_C(e)(x) \ge f_A(e)(x)$ which in turn implies that $f_A \sqsubseteq h_C$. Again from $f_{up}(h_C) \sqsubseteq g_B$ follows $h_C \sqsubseteq f_{up}^{-1}(g_B) = f_A$. Hence $h_C = f_A$ and f_A becomes fuzzy soft open set in $FS(X,E)$. Hence f_{up} is fuzzy soft continuous. □

Theorem 4.15. *Let* (X, τ_1) *and* (Y, τ_2) *be two fuzzy soft topological spaces and* $f_{up} : FS(X,E) \to FS(Y,K)$ *be a fuzzy soft mapping. Then the followings are equivalent:*

(1) f_{up} *is continuous;*
(2) $f_{up}^{-1}(h_C) \sqsubseteq (\ f_{up}^{-1}(h_C))^\circ$, $\forall h_C \in \tau_2$*;*
(3) $f_{up}(\overline{f_A}) \sqsubseteq \overline{f_{up}(f_A)}$, $\forall f_A \in FS(X,E)$*;*
(4) $\overline{f_{up}^{-1}(g_B)} \sqsubseteq f_{up}^{-1}(\overline{g_B})$, $\forall g_B \in FS(Y,K)$*;*
(5) $f_{up}^{-1}(g_B^\circ) \sqsubseteq (f_{up}^{-1}(g_B))^\circ$, $\forall g_B \in FS(Y,K)$.

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Let $f_A \in FS(X,E)$ and $f_{up}(e_x^\alpha)$ is not fuzzy soft subset of $\overline{f_{up}(f_A)}$. Then there exists a Q-nbd g_B of $f_{up}(e_x^\alpha)$ such that $g_B \overline{q} f_{up}(f_A)$ and hence $f_{up}^{-1}(g_B)\overline{q} f_A$ which implies $(f_{up}^{-1}(g_B))^\circ \overline{q} f_A$. Since $e_x^\alpha q f_{up}^{-1}(g_B)$, by (2), $e_x^\alpha q (f_{up}^{-1}(g_B))^\circ$. Choose $h_C = (f_{up}^{-1}(g_B))^\circ$. Then h_C is a Q-nbd of e_x^α and $h_C \overline{q} f_A$. This shows that e_x^α is not fuzzy soft subset of $\overline{f_A}$ which implies that $f_{up}(e_x^\alpha)$ is not fuzzy soft subset $f_{up}(\overline{f_A})$. Thus $f_{up}(\overline{f_A}) \sqsubseteq \overline{f_{up}(f_A)}$.

(3)⇒(4). Let $g_B \in FS(Y,K)$. Since $f_{up}(\ f_{up}^{-1}(g_B)) \sqsubseteq g_B$, we have $\overline{f_{up}(f_{up}^{-1}(g_B))} \sqsubseteq \overline{g_B}$. By (3), we obtain that $f_{up}(\overline{f_{up}^{-1}(g_B)}) \sqsubseteq \overline{g_B}$. Thus we have $\overline{f_{up}^{-1}(g_B)} \sqsubseteq f_{up}^{-1}(\overline{g_B})$.

(4)⇔(5). These follow from Theorems 2.15(3) and 4.5.

(5)⇒(1). Let $g_B \in \tau_2$. By (5), $f_{up}^{-1}(g_B) = f_{up}^{-1}(g_B^\circ) \sqsubseteq (f_{up}^{-1}(g_B))^\circ$ and so $f_{up}^{-1}(g_B) \in \tau_1$. This completes the proof. □

5. Conclusions

In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce soft quasi-coincidence and have established several interesting properties. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

Acknowledgements. The authors are highly grateful to referees and Professor Kul Hur , Editors-in-Chief, for their valuable comments and suggestion helpful in improving this paper.

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