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Common fixed point theorems under contractive condition in fuzzy symmetric spaces

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Abstract. In this paper, our target is to generalize the common fixed point theorems in fuzzy symmetric space. Using weakly compatible, Occasionally weakly compatible maps and property(E.A.), we will establish the unique common fixed point theorems in fuzzy symmetric space.

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1. Introduction

With the advent of notion of compatible maps introduced by Jungck [9], the study of common fixed point theorems for contractive maps has centered around the study of compatible maps and its weaker forms. On the other hand, the study of noncompatible maps is also equally interesting. Pant [12], Aamri and Moutawkil [2] and others have initiated wonderful works in this field. In [3], the authors gave a notion of property(E.A.) which generalizes the concept of noncompatible mappings in metric spaces, and they proved some common fixed point theorems for noncompatible mappings under strict contractive conditions. Recently, in [8] the authors extended the results of [3, 12] to symmetric(semi-metric) spaces under tight conditions, that is, nither the triangular inequality nor $d(x, x) = 0$, for all, x, are required for the proofs.

Symmetric spaces were introduced in 1931 by Wilson [17], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained, for example, see [4, 8, 18]. Hicks and Rhoades [7] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metrics are not required in the proofs of certain metric theorems.

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. But, in real life situation, the problem in economics, engineering, environment, social science, medical science, etc., do not always involve crisp data. Consequently, we can not successfully use the traditional classical methods because of various type of uncertainties presented in the problem. To deal with the uncertainties, fuzzy set theory [19] can be considered as one of the mathematical tool. That is why, so many researchers are trying to fuzzify different classical mathematical concepts. Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces (briefly , FM-spaces) in 1975 , which opened an avenue for further development of analysis in such spaces. Later on it is modified that a few concepts of mathematical analysis have been generalized by George and Veeramani [5, 6] and also they have generalized the fixed point theorem in fuzzy metric space [16]. In fuzzy metric space, the notion of compatible maps under the name of asymptotically commuting maps was introduced in the paper [11] and then in the paper [15], the notion of weak compatibility has been studied in fuzzy metric space. Later on Pant and Pant[13] studied the common fixed point theorems for a pair of non-compatible maps in fuzzy metric space.

In this paper, we have studied the common fixed point theorems in fuzzy symmetric space. Here our target is to generalize the common fixed point theorems for a pair of weakly compatible self mappings, for four self mappings in fuzzy symmetric space. Using weakly compatibility, Occasionally weakly compatible maps and property(E.A.), we will establish the unique common fixed point for pair of self mappings and for four self mappings in fuzzy symmetric space.

2. Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 ([14])**.** A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called continuous t - norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a \forall a \in [0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2. The pair (X, μ) is called a fuzzy symmetric space if X is an arbitrary non-empty set, μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $\mu(x, y, t) > 0$;
(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;
(iii) $\mu(x, y, t) = \mu(y, x, t)$;
(iv) $\mu(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous for all $x, y \in X$ and $t > 0$.

If (X, μ) be a fuzzy symmetric space then μ is called fuzzy symmetric for X.

Let (X, μ) be fuzzy symmetric space. (X, μ) is said to satisfy the following axioms:
(W_3) if for a sequence $\{x_n\}$ in X and $x, y \in X$,

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1$$

and

$$\lim_{n\to\infty} \mu(x_n, y, t) = 1 \Longrightarrow x = y.$$

(W_4) if for any two sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$,

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1$$

and

$$\lim_{n\to\infty} \mu(x_n, y_n, t) = 1 \Longrightarrow \lim_{n\to\infty} \mu(y_n, x, t) = 1.$$

(H_E) if for any two sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$,

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1$$

and

$$\lim_{n\to\infty} \mu(y_n, x, t) = 1 \Longrightarrow \lim_{n\to\infty} \mu(x_n, y_n, t) = 1.$$

(W^*) if for any two sequences $\{x_n\}$, $\{y_n\}$ in X and $x, y \in X$,

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1, \lim_{n\to\infty} \mu(y_n, y, t) = 1$$

and

$$\lim_{n\to\infty} \mu(x_n, y_n, t) = 1 \text{ imply } x = y.$$

Remark 2.3. (W_4) implies (W_3). So, (W_4) implies (W^*).

Proof. Suppose (W_4) holds and for any two sequences $\{x_n\}$, $\{y_n\}$ in X and $x, y \in X$,

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1 \lim_{n\to\infty} \mu(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} \mu(x_n, y_n, t) = 1.$$

This imply $\lim_{n\to\infty} \mu(x_n, y, t) = 1$ and hence by (W_3), we have $x = y$. □

Definition 2.4. Let A and B be two self mappings of a fuzzy symmetric space (X, μ).

(i) The order pair (A, B) satisfy the condition (p) if $\lim_{n\to\infty} \mu(AAx_n, Az, t) = 1$ and $\lim_{n\to\infty} \mu(BAx_n, Bz, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mu(Ax_n, z, t) = 1$ and $\lim_{n\to\infty} \mu(Bx_n, z, t) = 1$ for some $z \in X$.

(ii) The order pair (A, B) is **strongly partially commuting**, if for any sequence $\{x_n\}$ in X ,

$$\lim_{n\to\infty} \mu(Ax_n, Bx_n, t) = 1 \Longrightarrow \lim_{n\to\infty} \mu(AAx_n, BAx_n, t) = 1.$$

(iii) A and B are said to be **compatible** (or asymptotically commuting) if for all $t > 0$

$$\lim_{n\to\infty} \mu(ABx_n, BAx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} \mu(Ax_n, z, t) = \lim_{n\to\infty} \mu(Bx_n, z, t) = 1,$$

for some $z \in X$.

(iv) [1] A point x in X is called a **coincidence point** of A and B iff $Ax = Bx$.

(v) A and B are said to be **weakly compatible** if they commute at their coincidence points, that is, $Az = Bz$ implies that $ABz = BAz$.

(vi) We say that A and B satisfy the property $(E.A.)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mu(Ax_n, z, t) = \lim_{n\to\infty} \mu(Bx_n, z, t) = 1,$$

for some $z \in X$.

Example 2.5. Let $X = [0, +\infty)$ with the fuzzy symmetric μ defined by

$$\mu(x, y, t) = \frac{t}{t + e^{|x-y|} - 1}$$

for all x, y in X. Define $A, B : X \longrightarrow X$ as follows

$$Ax = 2x + 1 \text{ and } Bx = x + 2$$

for all $x \in X$. Note that the function μ is not a fuzzy metric but a fuzzy symmetric. Consider the sequence $x_n = \frac{1}{n} + 1$, $n = 1, 2, \cdots$.
Clearly

$$\lim_{n\to\infty} \mu(Ax_n, 3, t) = \lim_{n\to\infty} \mu(Bx_n, 3, t) = 1.$$

Then A and B satisfy the property $(E.A.)$.

Example 2.6. Let $X = R_+$ with the fuzzy symmetric μ defined by

$$\mu(x, y, t) = \frac{t}{t + e^{|x-y|} - 1}$$

for all x, y in X. Define $A, B : X \longrightarrow X$ as follows

$$Ax = x^2 \text{ and } Bx = x + 2.$$

$Ax = Bx$ iff. $x = 2$.
Let $\{x_n\}$ be a sequence in X defined by: $x_n = 2 + \frac{1}{n}$, $n \geq 1$. Now,

$$\lim_{n\to\infty} \mu(Ax_n, 4, t) = \lim_{n\to\infty} \mu(Bx_n, 4, t) = 1.$$

Then A and B satisfy the property $(E.A.)$.
As $AB(2) = A(4) = 16, BA(2) = B(4) = 6$, therefore (A, B) is not weakly compatible .

Definition 2.7. The mapping $A, B, S, T : X \longrightarrow X$ of a fuzzy symmetric space (X, μ) satisfy a common property $(E.A.)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Example 2.8. Let $X = [0, +\infty)$ with the fuzzy symmetric μ defined by

$$\mu(x, y, t) = \frac{t}{t + e^{|x-y|} - 1}$$

for all x, y in X. Define $A, B, S, T : X \longrightarrow X$ as follows

$$Ax = 2x + 5, Bx = 3x + 2, Sx = x + 5 \text{ and } Tx = 2x + 3$$

for all $x \in X$. Consider the sequence

$$x_n = \frac{1}{n} \text{ and } y_n = \frac{1}{n} + 1, \quad n = 1, 2, \cdots .$$

Clearly

$$\lim_{n \to \infty} \mu(Ax_n, 5, t) = \lim_{n \to \infty} \mu(Sx_n, 5, t) =$$

$$\lim_{n \to \infty} \mu(By_n, 5, t) = \lim_{n \to \infty} \mu(Ty_n, 5, t) = 1.$$

So, A, B, S, T satisfy the property ($E.A.$).

Definition 2.9. Two self maps f and g of a set X are occasionally weakly compatible (***owc***) iff there is a point $x \in X$ which is a coincidence point of f and g at which f and g commute.

3. Common fixed point

In the sequel, we need a function $\phi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the condition $\phi(t) > t$ for each $t \geq 0$. For example, we could let $\phi(t) = \alpha t + k$ for some $\alpha > 1$ and $k > 0$.

Theorem 3.1. *Let μ be a fuzzy symmetric for X that satisfies (W_3) and (H_E). Let A and B be weakly compatible self-mappings of fuzzy symmetric space (X, μ) such that for all $x, y \in X$ and $Ax \neq Ay$,*
(i) $\mu(Ax, Ay, t) \geq \phi(\min\{\mu(Bx, By, t), \mu(Bx, Ay, t), \mu(Ay, By, t)\})$;
(ii) $A\ and\ Bsatisfy\ the\ property(E.A.)$;
(iii) $AX \subset BX$. *If the range of $AorB$ is a complete subspace of X, then $AandB$ have a unique common fixed point .*

Proof. Since $AandB$ satisfy the property ($E.A.$), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} \mu(Ax_n, l, t) = \mu(Bx_n, l, t) = 1,$$

for some $l \in X$. Therefore, by (H_E), we have

$$\lim_{n \to \infty} \mu(Ax_n, Bx_n, t) = 1.$$

Suppose that BX is a complete subspace of X. Then $l = Bu$, for some $u \in X$. We claim that $Au = Bu$. Indeed by (i), we have

$$\begin{aligned} &\mu(Au, Ax_n, t) \\ \geq &\phi\{\min(\mu(Bu, Bx_n, t), \mu(Bu, Ax_n, t), \mu(Ax_n, Bx_n, t))\} \\ > &\min(\mu(Bu, Bx_n, t)\mu(Bu, Ax_n, t), \mu(Ax_n, Bx_n, t)). \end{aligned}$$

Taking limit as $n \longrightarrow \infty$, we have $\lim_{n \to \infty} \mu(Au, Ax_n, t) \geq 1$.
This implies that $\lim_{n \to \infty} \mu(Au, Ax_n, t) = 1$. Hence by (W_3), we have $Au = Bu$. The weak compatibility of A *and* B implies that $ABu = BAu$ and then $AAu = ABu = BAu = BBu$.

Let us show that Au is a common fixed point of A *and* B. Suppose that $AAu \neq Au$. In view of (i), it follows

$$\begin{aligned}
&\mu(Au, AAu, t)\\
\geq&\phi\{\min(\mu(Bu, BAu, t), \mu(Bu, AAu, t), \mu(AAu, BAu, t))\}\\
\geq&\phi\{\min(\mu(AAu, Au, t), \mu(AAu, Au, t), 1)\}\\
\geq&\phi\{\mu(AAu, Au, t)\}\\
>&\mu(AAu, Au, t)
\end{aligned}$$

which is a contradiction . Therefore $Au = AAu = BAu$ and Au is a common fixed point of A *and* B. The proof is similar when AX is assumed to be a complete subspace of X, since $AX \subset BX$. If $Au = Bu = u$ and $Av = Bv = v$ and $u \neq v$, then (i) gives

$$\begin{aligned}
\mu(u, v, t) &= \mu(Au, Av, t)\\
&\geq \phi\{\min(\mu(Bu, Bv, t), \mu(Bu, Av, t), \mu(Av, Bv, t))\}\\
&\geq \phi\{\mu(u, v, t)\}\\
&> \mu(u, v, t)
\end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique. □

Theorem 3.2. *Let μ be a fuzzy symmetric for X that satisfies (W_4) and (H_E). Let A, B, T and S be self-mappings of (X, μ) such that for all $x, y \in X$ and $Ax \neq By$,*
(i) $\mu(Ax, By, t) \geq \phi(\min\{\mu(Sx, Ty, t), \mu(Sx, By, t)\mu(Ty, By, t)\})$;
(ii) (A, T) *and* (B, S) *are weakly compatibles*;
(iii) (A, S) *and* (B, T) *satisfy the property* $(E.A.)$;
(iv) $AX \subset TX$ *and* $BX \subset SX$. *If the range of any one of the mappings A, B, T or S is a complete subspace of X then A, B, T and S have a unique common fixed point.*

Proof. Suppose that (B, T) satisfies the property $(E.A.)$. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mu(Bx_n, l, t) = \lim_{n\to\infty} \mu(Tx_n, l, t) = 1,$$

for some $l \in X$. Since $BX \subset SX$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence

$$\lim_{n\to\infty} \mu(Sy_n, l, t) = 1.$$

Let us show that $\lim\limits_{n\to\infty} \mu(Ay_n, l, t) = 1$. Indeed, in view of (i), we have

$$\begin{aligned}
&\mu(Ay_n, Bx_n, t)\\
\geq&\phi(\min\{\mu(Sy_n, Tx_n, t), \mu(Sy_n, Bx_n, t), \mu(Tx_n, Bx_n, t)\})\\
\geq&\phi(\min\{\mu(Bx_n, Tx_n, t)1\mu(Tx_n, Bx_n, t)\})\\
=&\phi(\mu(Tx_n, Bx_n, t))\\
>&\mu(Tx_n, Bx_n, t).
\end{aligned}$$

Therefore, by (H_E), one has $\lim_{n\to\infty} \mu(Ay_n, Bx_n, t) = 1$.
By (W_4), we deduce that $\lim_{n\to\infty} \mu(Ay_n, l, t) = 1$. Suppose that SX is a complete subspace of X. Then $l = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} \mu(Ay_n, Su, t) = \lim_{n\to\infty} \mu(Bx_n, Su, t) = \lim_{n\to\infty} \mu(Tx_n, Su, t)$$
$$= \lim_{n\to\infty} \mu(Sy_n, Su, t) = 1.$$

Using (i) it follows

$$\mu(Au, Bx_n, t) \geq \phi(\min\{\mu(Su, Tx_n, t), \mu(Su, Bx_n, t), \mu(Tx_n, Bx_n, t)\}).$$

Taking limit as $n \longrightarrow \infty$, we have, $\lim_{n\to\infty} \mu(Au, Bx_n, t) = 1$.
By (W_3), we have $Au = Su$.
The weak compatibility of A *and* S implies that $ASu = SAu$ and then

$$AAu = ASu = SAu = SSu.$$

On the other hand, since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$. We claim that $Bv = Au$. If not, condition (i) gives

$$\mu(Au, Bv, t) \geq \phi(\min(\mu(Su, Tv, t), \mu(Su, Bv, t), \mu(Tv, Bv, t)\})$$
$$= \phi(\min(1, \mu(Au, Bv, t), \mu(Au, Bv, t)\})$$
$$= \phi(\mu(Au, Bv, t)) > \mu(Au, Bv, t),$$

which is a contradiction . Hence

$$Au = Su = Tv = Bv.$$

The weak compatibility of B *and* T implies that $BTv = TBv$ and

$$TTv = TBv = BTv = BBv.$$

Suppose that $AAu \neq Au$. Then we have

$$\begin{aligned}
&\mu(Au, AAu, t) = \mu(AAu, Bv, t)\\
\geq&\phi(\min\{\mu(SAu, Tv, t), \mu(SAu, Bv, t), \mu(Tv, Bv, t)\})\\
\geq&\phi(\min\{\mu(AAu, Au, t), \mu(AAu, Au, t), 1\})\\
\geq&\phi(\mu(Au, AAu, t))\\
>&\mu(Au, AAu, t)
\end{aligned}$$

which is a contradiction. Therefore $Au = AAu = SAu$ and Au is a common fixed point of A *and* S. Similarly, we can prove that Bv is a common fixed point of B *and* T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, T and S. The proof is similar when TX is assumed to be a complete subspace of X. The cases in which AX or BX is a complete subspace of X are similar to the cases in which TX or SX respectively is complete, since $AX \subset TX$ and $BX \subset SX$. If

$$Au = Bu = Tu = Su = u$$

and

$$Av = Bv = Tv = Sv = v$$

and $u \neq v$, then (i) gives

$$\begin{aligned}\mu(u,v,t) =&\mu(Au,Bv,t)\\ \geq&\phi(\min\{\mu(Su,Tv,t),\mu(Su,Bv,t),\mu(Tv,Bv,t)\})\\ \geq&\phi(\mu(u,v,t))\\ >&\mu(u,v,t)\end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique. □

Theorem 3.3. *Let X be a set endowed with a symmetric μ. Suppose f, g, h, and k are four self-mappings of (X, μ) satisfying the condition :*

(i) $\mu^2(fx,gy,t) \geq \min\{\phi(\mu(hx,ky,t))\phi(\mu(hx,fy,t)),$
$\phi(\mu(hx,ky,t))\phi(\mu(ky,gy,t)),\phi(\mu(hx,fx,t))$
$\phi(\mu(ky,gy,t)),\phi(\mu(hy,gy,t))\phi(\mu(ky,fx,t))\}$

for all $x,y \in X$ and $fx \neq gy$,

(ii) *the pairs (f,h) and (g,k) are* **owc**.

Then f,g,h and k have a unique common fixed point .

Proof. Since pairs of mappings (f,h) and (g,k) are ***owc***, there exist two elements u and v in X such that $fu = hu$ and $fhu = hfu$, $gv = kv$ and $gkv = kgv$. First, we prove that $fu = gv$. Indeed, by (i) we get

$$\begin{aligned}&\mu^2(fu,gv,t)\\ &\geq \min\{\phi(\mu(hu,kv,t))\phi(\mu(hu,fu,t)),\phi(\mu(hu,kv,t))\phi(\mu(kv,gv,t)),\\ &\qquad \phi(\mu(hu,fu,t))\phi(\mu(kv,gv,t)),\phi(\mu(hu,gv,t))\phi(\mu(kv,fu,t))\}\\ &= \min\{\phi(\mu(fu,gv,t)),\phi(\mu(fu,gv,t)),1,\phi(\mu(fu,gv,t))\phi(\mu(fu,gv,t))\}\\ &= \phi^2(\mu(fu,gv,t)),\\ &\Longrightarrow \mu^2(fu,gv,t) \geq \phi^2(\mu(fu,gv,t)) > \mu^2(fu,gv,t),\end{aligned}$$

which is a contradiction, hence $fu = gv = hu = kv$.

Now, suppose that $ffu \neq fu$. By using inequality (i) we obtain

$$\begin{aligned}&\mu^2(ffu,gv,t)\\ &\geq \min\{\phi(\mu(hfu,kv,t))\phi(\mu(hfu,ffu,t)),\\ &\qquad \phi(\mu(hfu,kv,t))\phi(\mu(kv,gv,t)),\phi(\mu(hfu,ffu,t))\\ &\qquad \phi(\mu(kv,gv,t)),\phi(\mu(hfu,gv,t))\phi(\mu(kv,ffu,t))\}\\ &= \min\{\phi(\mu(ffu,gv,t)),\phi(\mu(ffu,gv,t)),1,\phi(\mu(ffu,gv,t))\phi(\mu(ffu,gv,t))\}\\ &= \phi^2(\mu(ffu,gv,t)),\\ &\Longrightarrow \mu^2(ffu,gv,t) \geq \phi^2(\mu(ffu,gv,t)) > \mu^2(ffu,gv,t),\end{aligned}$$

which is a contradiction, hence $ffu = fu = hfu$.

Similarly $gfu = kfu = fu$. Therefore $fu = gv = hu = kv$ is a common fixed point of mappings f,g,h and k.

Put $fu = gv = hu = kv = l$, then $fl = gl = hl = kl = l$.

Now, let l and z be two common fixed point of mappings f,g,h and k such that $z \neq l$, so

$$fl = gl = hl = kl = l \text{ and } fz = gz = hz = kz = z.$$

From (i), we have

$$\begin{aligned}\mu^2(l,z,t) &= \mu^2(fl,gz,t)\\ &\geq \min\{\phi(\mu(hl,kz,t))\phi(\mu(hl,fl,t)),\phi(\mu(hl,kz,t))\phi(\mu(kz,gz,t)),\\ &\qquad \phi(\mu(hl,fl,t))\phi(\mu(kz,gz,t)),\phi(\mu(hl,gz,t))\phi(\mu(kz,fl,t))\}\end{aligned}$$

$$= \min\{\phi(\mu(l,z,t)), \phi(\mu(l,z,t)), 1, \phi(\mu(l,z,t))\phi(\mu(l,z,t))\}$$
$$= \phi^2(\mu(l,z,t)),$$
$$\Longrightarrow \mu^2(l,z,t) \geq \phi^2(\mu(l,z,t)) > \mu^2(l,z,t),$$

which is a contradiction, hence $l = z$. □

Theorem 3.4. *Let (X, μ) be a fuzzy symmetric space with the axiom (W^*) and property (H_E). Let A, B, S and T be self maps on X satisfying*

$$\mu(Ax, By, t) > \min\{\mu(Sx, Ty, t) * \mu(Ax, Sx, t) * \mu(By, Ty, t), \mu(Sx, By, t), \mu(Ty, Ax, t)\} \tag{3.1}$$

for all $x, y \in X$ and for all $t > 0$. Then A, B, S and T have a unique common fixed point if one set of the following conditions are true .

(I) (a) *A and S satisfy the property (E.A.);*
(b) *The order pair (A, S) satisfy the condition (p) and is strongly partially commuting;*
(c) *$A(X) \subseteq T(X)$;*
(d) *B and T are weakly compatible.*

or

(II) (a) *B and T satisfy the property (E.A.);*
(b) *The order pair (B, T) satisfy the condition (p) and is strongly partially commuting;*
(c) *$B(X) \subseteq S(X)$;*
(d) *A and S are weakly compatible.*

Proof. Suppose the set of conditions in (I) are true. From $I(a)$, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mu(Ax_n, z, t) = \lim_{n\to\infty} \mu(Sx_n, z, t) = 1, \tag{3.2}$$

for some $z \in X$.
Since (A, S) satisfy the condition (p), from (3.2), we have

$$\lim_{n\to\infty} \mu(AAx_n, Az, t) = 1 = \lim_{n\to\infty} \mu(SAx_n, Sz, t). \tag{3.3}$$

From (3.2) and the property (H_E), we have

$$\lim_{n\to\infty} \mu(Ax_n, Sx_n, t) = 1. \tag{3.4}$$

Since the order pair (A, S) is strongly partially commuting, from (3.4), we have

$$\lim_{n\to\infty} \mu(AAx_n, SAx_n, t) = 1. \tag{3.5}$$

From (3.3), (3.5) and the axiom (W^*) it follows that

$$Az = Sz. \tag{3.6}$$

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that

$$Az = Tv. \tag{3.7}$$

If possible, let $Az \neq Bv$. Then $\mu(Az, Bv, t) \in [0, 1)$. From (3.1) we have

$$\mu(Az, Bv, t) > \min\{\mu(Sz, Tv, t) * \mu(Az, Sz, t) * \mu(Bv, Tv, t),$$

$$\mu(Sz, Bv, t), \mu(Tv, Az, t)\}$$

$$= \mu(Az, Bv, t)$$

by (3.6), (3.7) and (ii) of definition(2.1).
This contradiction proves that

$$Az = Bv. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$Az = Sz = Bv = Tv = u(say). \tag{3.9}$$

Since the pair (A, S) is strongly partially commuting, it follows that A and S are weakly compatible. Hence,

$$Au = ASz = SAz = Su \quad (\text{from}(3.9))\,. \tag{3.10}$$

If possible, let $Au \neq u$. From (3.1) we have

$$\begin{aligned}
&\mu(Au, u, t) = \mu(Au, Bv, t) \\
&> \min\{\mu(Su, Tv, t) * \mu(Au, Su, t) * \mu(Bv, Tv, t), \mu(Su, Bv, t), \mu(Tv, Au, t)\} \\
&= \mu(Au, u, t)
\end{aligned}$$

by (3.9), (3.10) and (ii) of definition(2.1).
This contradiction proves that

$$Au = u = Su. \tag{3.11}$$

Since B and T are weakly compatible, similarly we have

$$Bu = u = Tu. \tag{3.12}$$

From (3.11) and (3.12) it follows that u is a common fixed point of A, B, S and T. Uniqueness of common fixed point follows from (3.1).
Similarly we can prove the result, if the second set of conditions are assumed to be true. □

Corollary 3.5. *Let (X, μ) be a fuzzy symmetric space with the axiom (W^*) and property (H_E). Let A and S be self maps on X satisfying*

$$\mu(Ax, Ay, t) > \min\{\mu(Sx, Sy, t) * \mu(Ax, Sx, t) * \mu(Ay, Sy, t),$$

$$\mu(Sx, Ay, t), \mu(Sy, Ax, t)\},$$

for all $x, y \in X$ and for all $t > 0$.
If A and S satisfy the property $(E.A.)$, the condition (p) and is strongly partially commuting, A and S have a unique common fixed point.

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References

[1] C. T. Aage and J. N. Salunke, On fixed point theorems in fuzzy metric spaces, J. Open Probl. Comput. Sci. Math. 3(2) (2010) 123–131.

[2] M. Aamri and D. El Moutawakil, Common fixed point under contractive copnditions in symmetric spaces, Applied Mathematics E-Notes 3 (2003) 156–162.

[3] M. Aamri and D. El Moutawakil, Some new common fixed pointtheorems under strict contractive conditions, J. Math. Anal. Appl. 270(1) (2002) 181–188.

[4] S. H. Cho, G. Y. Lee and J. S. Bae, On coincidence and fixed-point theorems in symmetric spaces, Fixed Point Theory Appl. 2008, Art. ID 562130, 9 pp.

[5] A. George and P. Veeramani, On some result in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.

[6] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.

[7] T. L. Hicks and B. E. Rhoades, Fixed point theory in symmetric spaces with application to probabilistic spaces, Nonlinear Anal. 36(3) (1999) Ser. A: Theory Methods 331–344.

[8] M. Imdad, J. Ali and L. Khan, Coincidence and fixed points in symmetric spaces under strict contractions, J. Math. Anal. Appl. 320(1) (2006) 352–360.

[9] G. Jungck, Common fixed points for noncontinuous, non-self maps on nonmetric spaces, Far East J. Math. Sci. 4 (1996) 199–225.

[10] O. Kramosil, J. Michalek, Fuzzy metric and statisticalmetric spaces, Kybernetica 11 (1975) 326–334.

[11] S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy metric spaces, Internat. J. Math. Sci. 17 (1994) 253–258.

[12] R. P. Pant and V. Pant, Common fixed points under strict contractive conditions, J. Math. Anal. Appl. 248(1) (2000) 327–332.

[13] V. Pant and R. P. Pant, Fixed points in fuzzy metric space for noncompatible maps, Soochow J. Math. 33(4) (2007) 647–655.

[14] B. Schweizer and A. Sklar, Statistical metric space, Pac. J. Math. 10 (1960) 314–334.

[15] B. Singh and S. Jain, Weak compatibility and fixed point theorems in fuzzy metric spaces, Ganita 56(2) (2005) 167–176.

[16] R. Vasuki, P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, Fuzzy Sets and Systems 135 (2003) 415–417.

[17] W. A. Wilson, On semi-metric spaces, Amer. J. Math. 53(2) (1931) 361–373

[18] J. Zhu, Y. J. Cho and S. M. Kang, Equivalent contractive conditions in symmetric spaces, Comput. Math. Appl. 50(10-12) (2005) 1621–1628.

[19] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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