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Coupled fixed point theorems for weak compatible mappings in fuzzy metric spaces

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Abstract. In this paper, first we introduce the notion of common coupled coincidence point for pairs of mappings. Secondly, we introduce the concept of weakly commuting and variants of weakly commuting mappings in coupled fixed point theory (R-weakly commuting, R-weakly commuting of type (A_f), type (A_g), type (P) mappings). Thirdly, we introduce the notion of weakly f-compatible maps and weakly g-compatible maps to obtain common coupled coincidence points in fuzzy metric spaces. At the end, we prove common fixed point theorems for pairs of weakly compatible mappings and their variants, which generalize the results of various authors present in fixed point theory literature.

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Keywords: Weakly commuting mappings, R-weakly commuting mapping, R-weakly commuting of type (A_f) mappings, R-weakly commuting of type (A_g), R-weakly commuting of type (P) mappings, Weakly f-compatible and Weakly g-compatible.

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1. Introduction

The notion of fuzzy sets introduced by Zadeh [18] proved a turning point in the development of mathematics. This notion laid the foundation of fuzzy mathematics. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. Kramosil and Michalek [10] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [3] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [10]. There are many view points of the notion of the metric space in fuzzy topology for instance one can refer to Kaleva and Seikkala [9], Kramosil and Michalek [10] and George and Veeramani [3].

We can divide them into following two groups as follows:

The first group involves those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to R^+$ where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. This proved a milestone in fixed point point theory of fuzzy metric space and afterwards a flood of papers appeared for fixed point theorems in fuzzy metric space.

Bhaskar and Lakshmikantham [1], Lakshmikantham and Ćirić [11] gave some coupled fixed point theorems. Coupled fixed point theorem under contraction conditions given by Sedghi et al. [15] are of great importance in the theory of fixed points in fuzzy metric spaces. Fang [2] proved a result for compatible and weakly compatible mappings under ϕ-contractive conditions in fuzzy metric spaces which provide a tool to Xin-Qi Hu [6] to prove a result, which is actually a generalization of the result of Sedghi [15].

In this paper, first we introduce the notion of common coupled coincidence point for pairs of mappings. Secondly, we introduce the concept of weakly commuting and variants of weakly commuting mappings in coupled fixed point theory (R-weakly commuting, R-weakly commuting of type (A_f), type (A_g), type (P) mappings). Thirdly, we introduce the notion of weakly f-compatible maps and weakly g-compatible maps to obtain common coupled coincidence points in fuzzy metric spaces. At the end, we prove common fixed point theorems for pairs of weakly compatible mappings and their variants, which generalize the results of various authors present in fixed point theory literature.

2. Preliminaries

Before we give our main result we need the following definitions:

Definition 2.1 ([18])**.** A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 2.2 ([16])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $([0, 1], *)$ is a topological abelian monoid with unit 1 s.t. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$. Some examples are below:

(1) $*(a, b) = ab$,
(2) $*(a, b) = \min(a, b)$.

Definition 2.3 ([5])**.** Let $\sup\limits_{0<t<1} \Delta(t, t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t, \quad \Delta^{m+1}(t) = t(\Delta^m(t)), \quad m = 1, 2, \ldots, \; t \in [0, 1].$$

A t-norm Δ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > (1 - \lambda)$ for all $m \in N$, when $t > (1 - \delta)$.
The t-norm $\Delta_M = \min.$ is an example of t-norm of H-type.

Definition 2.4 ([3]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1)$M(x, y, 0) > 0$,
(FM-2) $M(x, y, t) = 1$ iff $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous, for all $x, y, z \in X$ and $s, t > 0$.

In present paper, we consider $(X, M, *)$ to be fuzzy metric space with condition:
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.

Definition 2.5 ([3]). Let $(X, M, *)$ be fuzzy metric space. A sequence $\{x_n\}$ in X is said to be

(i) Convergent to a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$;
(ii) Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, for all $t > 0$ and $p > 0$.

Definition 2.6 ([3]). A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Lemma 2.1 ([4]). *$M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.2 ([4]). *Let $x_n \to x$ and $y_n \to y$, then*

(i) $\lim_{n\to\infty} M(x_n, y_n, t) \geq M(x, y, t)$, *for all* $t > 0$,
(ii) $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$, *for all* $t > 0$, *if* $M(x, y, t)$ *is continuous.*

Definition 2.7 ([6]). Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is non-decreasing;
(ϕ-2) ϕ is upper semicontinuous from the right;
(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in N$.

Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Definition 2.8 ([2]). An element $x \in X$ is called a common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$x = f(x, x) = g(x).$$

Definition 2.9 ([11]). An element $(x, y) \in X \times X$ is called

(i) a coupled fixed point of the mapping $f : X \times X \to X$ if

$$f(x, y) = x, \quad f(y, x) = y.$$

(ii) a coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$f(x, y) = g(x), \quad f(y, x) = g(y).$$

(iii) a common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$x = f(x, y) = g(x), \quad y = f(y, x) = g(y).$$

In 1994, Mishra [12] introduced the concept of compatible mappings in fuzzy metric spaces akin to the concept of compatible mapping in metric spaces, see [8].

In 1994, Pant [13] introduced the concept of R-weakly commuting maps in metric spaces. Later on, Vasuki [17] initiated the concept of non compatible of mapping in fuzzy metric spaces and introduced the notion of R-weakly commuting mappings in fuzzy metric spaces and proved some common fixed point theorems for R-weakly commuting maps in the fuzzy metric space. Further, Pathak et al. [14] generalized the concept of R-weakly commuting maps of type (Ag) and type (A_f) as follows.

Definition 2.10. A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be

(i) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$.

(ii) R-weakly commuting if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R).$$

(iii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that

$$M(fgx, ggx, t) \geq M(fx, gx, t/R).$$

(iv) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that

$$M(gfx, ffx, t) \geq M(fx, gx, t/R).$$

Notice that Definition 2.10(iii) and Definition 2.10(iv) are embarked by Imdad and Ali [7] with inspiration from Pathak et al. [14].

In 2006, Imdad and Javid Ali [7] introduced the definition of R-weakly commuting mappings of type (P) as follow.

A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M(fx, gx, t/R), \quad \text{for all } x \in X \text{ and } t > 0.$$

Definition 2.11 ([2]). The mappings $f : X \times X \to X$ and $g : X \to X$ are called commutative if

$$gf(x, y) = f(gx, gy), \quad \text{for all } x, y \in X.$$

Now we introduce the following notions in coupled fixed point theory of fuzzy metric spaces.

Definition 2.12. The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be weakly commuting if

$$M(f(gx, gy), gf(x, y), t) \geq M(f(x, y), gx, t),$$
$$M(f(gy, gx), gf(y, x), t) \geq M(f(y, x), gy, t)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

Definition 2.13. The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be

(i) R-weakly commuting if there exists some $R > 0$ such that

$$M(f(gx, gy), gf(x, y), t) \geq M(f(x, y), gx, t/R),$$
$$M(f(gy, gx), gf(y, x), t) \geq M(f(y, x), gy, t/R)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

(ii) R-weakly commuting maps of type (A_f) if there exists some $R > 0$ such that

$$M(f(gx, gy), ggx, t) \geq M(f(x, y), gx, t/R),$$
$$M(f(gy, gx), ggy, t) \geq M(f(y, x), gy, t/R)$$
$$\text{for all } x, y \in X \text{ and } t > 0.$$

(iii) R-weakly commuting maps of type (A_g) if there exists some $R > 0$ such that

$$M(gf(x, y), f(f(x, y), f(y, x)), t) \geq M(f(x, y), gx, t/R),$$
$$M(gf(y, x), f(f(y, x), f(x, y)), t) \geq M(f(y, x), gy, t/R)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

(iv) R-weakly commuting maps of type (P) if there exists some $R > 0$ such that

$$M(f(f(x, y), f(y, x)), ggx, t) \geq M(f(x, y), gx, t/R),$$
$$M(f(f(y, x), f(x, y)), ggy, t) \geq M(f(y, x), gy, t/R)$$
$$\text{for all } x, y \text{ in } X \text{ and } t > 0.$$

Definition 2.14 ([2])**.** The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1,$$
$$\lim_{n\to\infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1,$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x,$$
$$\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y, \quad \text{for all } x, y \in X.$$

Definition 2.15. The maps $f : X \times X \to X$ and $g : X \to X$ are called weakly compatible if

$$f(x, y) = g(x), \qquad f(y, x) = g(y)$$

implies

$$gf(x, y) = f(gx, gy), \quad gf(y, x) = f(gy, gx), \quad \text{for all } x, y \in X.$$

Definition 2.16. The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be weakly f-compatible if
either

$$\lim_{n\to\infty} gf(x_n, y_n) = f(x, y), \qquad \lim_{n\to\infty} gf(y_n, x_n) = f(y, x)$$

or

$$\lim_{n\to\infty} ggx_n = f(x, y), \qquad \lim_{n\to\infty} ggy_n = f(y, x)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x, \qquad \lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$$

and

$$\lim_{n\to\infty} f(gx_n, gy_n) = \lim_{n\to\infty} f(f(x_n, y_n), f(y_n, x_n)) = f(x, y),$$
$$\lim_{n\to\infty} f(gy_n, gx_n) = \lim_{n\to\infty} f(f(y_n, x_n)) = f(y, x) \quad \text{for some } x, y \in X.$$

Definition 2.17. The mappings $f : X \times X \to X$ and $g : X \to X$ are said to be weakly g-compatible if
either

$$\lim_{n\to\infty} f(gx_n, gy_n) = gx, \qquad \lim_{n\to\infty} f(gy_n, gx_n) = gy$$

or

$$\lim_{n\to\infty} f(f(x_n, y_n), f(y_n, x_n)) = gx, \qquad \lim_{n\to\infty} f(f(y_n, x_n), f(x_n, y_n)) = gy$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x, \qquad \lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$$

and

$$\lim_{n\to\infty} gf(x_n, y_n) = \lim_{n\to\infty} gg(x_n) = gx,$$
$$\lim_{n\to\infty} gf(y_n, x_n) = \lim_{n\to\infty} gg(y_n) = gy \quad \text{for some } x, y \in X.$$

Definition 2.18. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings. Then, the pair of maps (B, S) and (A, T) are said to have common coupled coincidence point if there exists a, b in X such that

$$B(a, b) = S(a) = T(a) = A(a, b) \quad \text{and} \quad B(b, a) = S(b) = T(b) = A(b, a).$$

Now we show that the various kinds of above mentioned 'R-weakly commutativity' notions are independent of each other. We demonstrate this in the following discussion.

Example 2.1. Let $X = (0, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$.
Then $(X, M, *)$ is a FM-space. Define $f : X \times X \to X$ as $f(x, y) = x + y$ for all x, y in X and $g : X \to X$ as $g(x) = x$ for all x in X. Then, clearly for all x, y in X and $t > 0$,

$$M(f(gx, gy), gf(x, y), t) = 1 > \frac{t}{t+y} = M(f(x, y), gx, t),$$
$$M(f(gy, gx), gf(y, x), t) = 1 > \frac{t}{t+x} = M(f(y, x), gy, t),$$

which shows that the pair (f, g) is weakly commuting.
Moreover, for all x, y in X and $t > 0$,

$$M(f(gx, gy), gf(x, y), t) = 1 > \frac{t}{t + Ry} = M\left(f(x, y), gx, \frac{t}{R}\right),$$
$$M(f(gy, gx), gf(y, x), t) = 1 > \frac{t}{t + Rx} = M\left(f(y, x), gy, \frac{t}{R}\right) \quad \text{for each } R > 0,$$

which shows that the pair (f, g) is R-weakly commuting for each $R > 0$.
We note the followings:
For $R \geq 1$, the pair (f, g) are R-weakly commuting of type (A_f)

$$M(f(gx, gy), ggx, t) = \frac{t}{t + y} \geq \frac{t}{t + Ry} = M\left(f(x, y), gx, \frac{t}{R}\right),$$
$$M(f(gy, gx), ggy, t) = \frac{t}{t + x} \geq \frac{t}{t + Rx} = M\left(f(y, x), gy, \frac{t}{R}\right).$$

For $R \geq 2$, and $x = y$, shows that the pair (f, g) is R-weakly commuting of type (A_g).

$$M(gf(x, y), f(f(x, y), f(y, x)), t) = \frac{t}{t + (x + y)} \geq \frac{t}{t + Ry} = M\left(f(x, y), gx, \frac{t}{R}\right),$$
$$M(gf(y, x), f(f(y, x), f(x, y)), t) = \frac{t}{t + (x + y)} \geq \frac{t}{t + Rx} = M\left(f(y, x), gy, \frac{t}{R}\right).$$

Finally, we proceed towards R-commutativity of type(P).
For $R \geq 3$, $x = y$, shows that the pair (f, g) is R-weakly commuting of type (P)

$$M(f(f(x, y), f(y, x)), ggx, t) = \frac{t}{t + (x + 2y)} \geq \frac{t}{t + Ry} = M\left(f(x, y), gx, \frac{t}{R}\right),$$
$$M(f(f(y, x), f(x, y)), ggy, t) = \frac{t}{t + (2x + y)} \geq \frac{t}{t + Rx} = M\left(f(y, x), gy, \frac{t}{R}\right).$$

In the above example, the maps f and g are weakly f-compatible and weakly g-compatible.

For this, the consideration of sequences $\{x_n\}$ and $\{y_n\}$ in X with $\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ implies $x = 0, y = 0$ and $\lim_{n\to\infty} = 0$, $\lim_{n\to\infty} y_n = 0$. Then, it can be easily seen that the maps f and g are weakly f-compatible and weakly g-compatible.

Now, we shall give our main results.

3. Main results.

Let $(X, M, *)$ be a Complete Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying the following conditions:

(3.1) $\quad A(X \times X) \subseteq T(X), B(X \times X) \subseteq S(X)$,

(3.2) $\quad$ there exists $\phi \in \Phi$ such that

$M(A(x,y), B(u,v), \phi(t)) \geq M(Sx, Tu, t) * M(Sy, Tv, t)$, for all $x, y, u, v \in X$ and $t > 0$.

Then for arbitrary points x_0, y_0 in X, by (3.1), we can choose x_1, y_1 in X such that $T(x_1) = A(x_0, y_0), T(y_1) = A(y_0, x_0)$.
Again, by (3.1), we can choose x_2, y_2 in X such that $S(x_2) = B(x_1, y_1)$ and $S(y_2) = B(y_1, x_1)$.
Continuing in this way, we can construct two sequences $\{z_n\}$ and $\{z'_n\}$ in X such that

$$z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}), \quad z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \tag{3.3}$$

and

$$z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}), \quad z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}), \quad \text{for all } n \geq 0. \tag{3.4}$$

Xin-Qi Hu [6] proved the following result.

Theorem 3.1. *Let $(X, M, *)$ be a complete FM-Space, $*$ being continuous t-norm of H-type. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$M(F(x,y), F(u,v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t)$, *for all x, y, u, v in X and $t > 0$.*

Suppose that $F(X \times X) \subseteq g(X)$ and g is continuous, F and g are compatible. Then there exits a unique x in X such that $x = g(x) = F(x, x)$.

Now we prove the following result which generalizes Theorem 3.1 for two pairs of weakly compatible maps:

Theorem 3.2. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying* (3.1), (3.2) *and the following conditions:*

(3.5) *the pairs (A, S) and (B, T) are weakly compatible,*
(3.6) *one of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.*

Then there exists a unique point a in X such that

$$A(a, a) = S(a) = a = T(a) = B(a, a).$$

We need the following Lemma to prove our main result.

Lemma 3.1. *The sequences $\{z_n\}$ and $\{z'_n\}$ defined by* (3.3) *and* (3.4) *respectively are Cauchy in X.*

Proof. Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\epsilon), \quad \text{for all } p \in N. \tag{3.7}$$

Since $\lim_{t \to \infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(Sx_0, Tx_1, t_0) \geq (1-\delta) \quad \text{and} \quad M(Sy_0, Ty_1, t_0) \geq (1-\delta). \tag{3.8}$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.9}$$

Using condition (3.2), we have

$$\begin{aligned} M(z_1, z_2, \phi(t_0)) &= M(A(x_0, y_0), B(x_1, y_1), \phi(t_0)) \\ &\geq M(Sx_0, Tx_1, t_0) * M(Sy_0, Ty_1, t_0), \\ M(z_1', z_2', \phi(t_0)) &= M(A(y_0, x_0), B(y_1, x_1), \phi(t_0)) \\ &\geq M(Sy_0, Ty_1, t_0) * M(Sx_0, Tx_1, t_0). \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} M(z_2, z_3, \phi^2(t_0)) &= M(B(x_1, y_1), A(x_2, y_2), \phi^2(t_0)) \\ &= M(A(x_2, y_2), B(x_1, y_1), \phi^2(t_0)) \\ &\geq M(Sx_2, Tx_1, \phi(t_0)) * M(Sy_2, Ty_1, \phi(t_0)) \\ &= M(z_2, z_1, \phi(t_0)) * M(z_2', z_1', \phi(t_0)) \\ &\geq [M(Sx_0, Tx_1, t_0)]^2 * [M(Sy_0, Ty_1, t_0)]^2, \\ M(z_2', z_3', \phi^2(t_0)) &\geq [M(Sy_0, Ty_1, t_0)]^2 * [M(Sx_0, Tx_1, t_0)]^2. \end{aligned}$$

Continuing in this way, we can get

$$\begin{aligned} M(z_n, z_{n+1}, \phi^n(t_0)) &\geq [M(Sx_0, Tx_1, t_0)]^{2^{n-1}} * [M(Sy_0, Ty_1, t_0)]^{2^{n-1}}, \\ M(z_n', z_{n+1}', \phi^n(t_0)) &\geq [M(Sy_0, Ty_1, t_0)]^{2^{n-1}} * [M(Sx_0, Tx_1, t_0)]^{2^{n-1}}. \end{aligned}$$

So, from (3.7), (3.8) and (3.9), for $m > n \geq n_0$, we have

$$\begin{aligned} & M(z_n, z_m, t) \\ &\geq M\left(z_n, z_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(z_n, z_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\ &\geq M(z_n, z_{n+1}, \phi^n(t_0)) * M(z_{n+1}, z_{n+2}, \phi^{n+1}(t_0)) * \ldots * M(z_{m-1}, z_m, \phi^{m-1}(t_0)) \\ &\geq \{[M(Sx_0, Tx_1, t_0)]^{2^{n-1}} * [M(Sy_0, Ty_1, t_0)]^{2^{n-1}}\} * \\ &\quad * \{[M(Sx_0, Tx_1, t_0)]^{2^n} * [M(Sy_0, Ty_1, t_0)]^{2^n}\} * \\ &\quad * \{[M(Sx_0, Tx_1, t_0)]^{2^{m-2}} * [M(Sy_0, Ty_1, t_0)]^{2^{m-2}}\} \\ &= [M(Sx_0, Tx_1, t_0)]^{2^{n-1}(2^{m-n}-1)} * [M(Sy_0, Ty_1, t_0)]^{2^{n-1}(2^{m-n}-1)} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{2^n(2^{m-n}-1)} \geq (1-\epsilon), \quad \text{which implies that} \end{aligned}$$

$$M(z_n, z_m, t) \geq (1-\epsilon), \quad \text{for all } m, n \in N \text{ with } m > n > n_0 \text{ and } t > 0.$$

So $\{z_n\}$ is a Cauchy sequence. Similarly, we can get that $\{z'_n\}$ is a Cauchy sequence. □

Now we come to the main proof of Theorem 3.2.

Proof. By Lemma 3.1, the sequences $\{z_n\}$ and $\{z'_n\}$ defined by (3.3) and (3.4) respectively are both Cauchy.
Step 1: We show that $T(a) = B(a,b), T(b) = B(b,a)$ and $S(a) = A(a,b), S(b) = A(b,a)$.
Without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now, $\{z_{2n+1}\}$, $\{z_{2n+2}\}$ and $\{z'_{2n+1}\}, \{z'_{2n+2}\}$ being respective subsequences of Cauchy sequences $\{z_n\}$ and $\{z'_n\}$ are also Cauchy.
Since $T(X)$ is complete, so there exists a, b in X such that

$$\{z_{2n+1}\} \to \quad \text{a and} \quad \{z'_{2n+1}\} \to b.$$

Again convergence of the subsequences $\{z_{2n+1}\}$ and $\{z'_{2n+1}\}$ implies the convergence of original Cauchy sequences $\{z_n\}$ and $\{z'_n\}$ respectively such that $\{z_n\}$ a and $\{z'_n\} \to b$.
It follows that the sequences $\{z_n\}$, $\{z_{2n+1}\}$, $\{z_{2n+2}\}$ converges to a and $\{z'_n\}$, $\{z'_{2n+1}\}$, $\{z'_{2n+2}\}$ converges to b.
Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a, T(q) = b$, so that we have

$$\lim_{n\to\infty} z_{2n+1} = \lim_{n\to\infty} A(x_{2n}, y_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = a = T(p),$$
$$\lim_{n\to\infty} z_{2n+2} = \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} S(x_{2n+2}) = a = T(p)$$

and

$$\lim_{n\to\infty} z'_{2n+1} = \lim_{n\to\infty} A(y_{2n}, x_{2n}) = \lim_{n\to\infty} T(y_{2n+1}) = b = T(q),$$
$$\lim_{n\to\infty} z'_{2n+2} = \lim_{n\to\infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n\to\infty} S(y_{2n+2}) = b = T(q).$$

From condition (3.2),

$$M(A(x_{2n}, y_{2n}), B(p,q), \phi(t)) \geq M(Sx_{2n}, T(p), t) * M(Sy_{2n}, T(q), t)$$

Letting $n \to \infty$, we get

$$M(T(p), B(p,q), \phi(t)) \geq 1, \quad \text{this implies } T(p) = B(p,q) = a.$$

Similarly $T(q) = B(q,p) = b$.
Since the pair (B,T) is weakly compatible, so that $T(p) = B(p,q) = a$ implies $T(a) = B(a,b)$. Similarly, $T(b) = B(b,a)$.
Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X so that $S(r) = a, S(s) = b$.
Now,

$$M(A(r,s), B(x_{2n+1}, y_{2n+1}, \phi(t)) \geq M(S(r), Tx_{2n+1}, t) * M(S(s), Ty_{2n+1}, t)$$

Letting $n \to \infty$, we get

$$M(A(r,s), a, \phi(t)) \geq 1, \quad \text{this implies } A(r,s) = a = S(r).$$

Similarly, $A(s,r) = b = S(s)$.
Since the pair (A,S) is weakly compatible, it follows that $A(a,b) = S(a)$ and $A(b,a) = S(b)$.

Step 2: Next we show that $S(a) = T(a)$ and $S(b) = T(b)$.
Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\epsilon), \quad \text{for all} \quad p \in N.$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$, for all x,y in X, there exists $t_0 > 0$ such that

$$M(S(a),T(a),t_0) \geq (1-\delta) \quad \text{and} \quad M(S(b),T(b),t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Form condition (3.2), we have

$$\begin{aligned} M(S(a),T(a),\phi(t_0)) &= M(A(a,b),B(a,b),\phi(t_0)) \\ &\geq M(S(a),T(a),t_0) * M(S(b),T(b),t_0) \end{aligned}$$

In general, we can get

$$M(S(a),T(a),\phi^n(t_0)) \geq [M(S(a),T(a),(t_0)]^{2^{n-1}} * [MS(b),T(b),(t_0)]^{2^{n-1}}, \quad \text{for all } n \geq 1.$$

Thus, we have

$$\begin{aligned} M(S(a),T(a),t) &\geq M\left(S(a),T(a),\sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(S(a),T(a),\phi^{n_0}(t_0)) \\ &\geq [M(S(a),T(a),t_0)]^{2^{n_0-1}} * [M(S(b),T(b),t_0)]^{2^{n_0-1}} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{2^{n_0}} \geq (1-\epsilon), \\ &\qquad \text{for all } t > 0 \text{ and for any } \epsilon > 0. \end{aligned}$$

Hence, $S(a) = T(a)$. Similarly, $S(b) = T(b)$.
Therefore, $S(a) = A(a,b) = B(a,b) = T(a)$ and $S(b) = A(b,a) = B(b,a) = T(b)$.

Step 3: We next show that $S(a) = a$ and $S(b) = b$.
Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\epsilon) \quad \text{for all} \quad p \in N.$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(a, S(a), t_0) \geq (1-\delta) \quad \text{and} \quad M(b, S(b), t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Form condition (3.2), we have

$M(A(a,b), B(x_{2n+1}, y_{2n+1}), \phi(t_0)) \geq M(S(a), T(x_{2n+1}), t_0) * M(S(b), T(y_{2n+1}), t_0)$

Letting $n \to \infty$, we get

$$M(A(a,b), a, \phi(t_0)) \geq M(S(a), a, t_0) * M(S(b), b, t_0)$$

or

$$M(S(a), a, \phi(t_0)) \geq M(S(a), a, t_0) * M(S(b), b, t_0)$$

Similarly,

$$M(S(b), b, \phi(t_0)) \geq M(S(b), b, t_0) * M(S(a), a, t_0).$$

Now,

$$\begin{aligned} M(S(a), a, t) &\geq M\left(S(a), a, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M(S(a), a, \phi^{n_0}(t_0)) \\ &\geq [M(S(a), a, t_0)]^{2^{n_0-1}} * [M(S(b), b, t_0)]^{2^{n_0-1}} \\ &\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{2^{n_0}} \geq (1-\epsilon), \quad \text{for all } t > 0 \text{ and for any } \epsilon > 0. \end{aligned}$$

Hence, $S(a) = a$. Similarly, $S(b) = b$.

Thus, $B(a,b) = S(a) = a = T(a) = A(a,b)$ and $B(b,a) = S(b) = b = T(b) = A(b,a)$.

Step 4: We shall that $a = b$.

Since $*$ is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{p} \geq (1-\epsilon), \quad \text{for all } p \in N.$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(a, b, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.2), we have

$$\begin{aligned} M(Tx_{2n+1}, Sy_{2n+2}, \phi(t_0)) &= M(A(x_{2n}, y_{2n}), B(y_{2n+1}, x_{2n+1}), \phi(t_0)) \\ &\geq M(Sx_{2n}, Ty_{2n+1}, t_0) * M(Sy_{2n}, Tx_{2n+1}, t_0) \end{aligned}$$

Letting $n \to \infty$,

$$M(a, b, \phi(t_0)) \geq M(a, b, t_0) * M(b, a, t_0)$$

Then,

$$\begin{aligned} M(a, b, t) &\geq M\Bigg(a, b, \sum_{k=n_0}^{\infty} \phi^k(t_0)\Bigg) \\ &\geq M(a, b, \phi^{n_0}(t_0)) \\ &\geq [M(a, b, t_0)]^{2^{n_0-1}} * [M(b, a, t_0)]^{2^{n_0-1}} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \ldots * (1-\delta)}_{2^{n_0}} \geq (1-\epsilon), \end{aligned}$$

for all $t > 0$ and for any $\epsilon > 0$.

This implies $a = b$. Hence, there exists some point a in X such that $A(a, a) = T(a) = a = S(a) = B(a, a)$. Uniqueness of point a follows immediately from (3.2). □

On putting $\phi(t) = kt$, where $0 < k < 1$, we have the following result:

Corollary 3.1. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $A : X \times X \to X$, $B : X \times X \to X$, $S : X \to X$, $T : X \to X$ be four mappings satisfying* (3.1), (3.5), (3.6) *with the following condition:*

(3.10) *there exists $k, 0 < k < 1$, such that*

$$M(A(x, y), B(u, v), kt) \geq M(Sx, Tu, t) * M(Sy, Tv, t),$$

for all x, y, u, v in X and $t > 0$.

Then there exists a unique point a in X such that

$$A(a, a) = S(a) = a = T(a) = B(a, a).$$

Taking $A = B = F$ and $S = T = g$ in Theorem 3.2 we have the following result:

Corollary 3.2. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$$M(F(x, y), F(u, v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t),$$

for all x, y, u, v in X and $t > 0$.

Suppose that $F(X \times X) \subseteq g(X)$ and F and g are weakly compatible. If one of the range spaces of F or g is complete, then there exits a unique x in X such that $x = g(x) = F(x, x)$.

Taking $A = B = F$ and $S = T = I$ in Theorem 3.2 we have the following result:

Corollary 3.3. *Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type. Let $F : X \times X \to X$ and there exists $\phi \in \Phi$ such that*

$$M(F(x,y), F(u,v), \phi(t)) \geq M(x,u,t) * M(y,v,t), \text{ for all } x,y,u,v \text{ in } X \text{ and } t > 0.$$

If $F(X \times X)$ is complete, then F has a unique fixed point in X.

Theorem 3.3. *Theorem 3.2 remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypothesis) of the following:*
(i) *weakly commuting property,*
(ii) *R-weakly commuting property,*
(iii) *R-weakly commuting property of type (A_f),*
(iv) *R-weakly commuting property of type (A_g),*
(v) *R-weakly commuting property of type (P).*

Proof. Let a, b be two points in X so that $A(a,b) = Sa$ and $A(b,a) = Sb$. Taking $x_n = a, y_n = b$ it is easy to show that $A(Sa, Sb) = SA(a,b)$ and $A(Sb, Sa) = SA(b,a)$. Similarly, (B,T) commutes at all of its coincidence points. Now applying Theorem 3.2, we can conclude that A, B, S, T have a unique common fixed point.

In case if (A,S) satisfies R-weakly commuting property, then there exists some $R > 0$ such that

$$M(A(Sx, Sy), SA(x,y), t) \geq M(A(x,y), Sx, t/R),$$
$$M(A(Sy, Sx), SA(y,x), t) \geq M(A(y,x), Sy, t/R) \text{ for all } x,y \text{ in } X \text{ and } t > 0.$$

Let there exists a, b in X such that $A(a,b) = Sa$ and $A(b,a) = Sb$, then it is easy to see that A and S commutes at a and b.
Similarly, (B,T) commutes at all of its coincidence points. Now applying Theorem 3.2, we can conclude that A, B, S, T have a unique common fixed point.

Similarly, if pair (A,S) is weakly commuting, R-weakly commuting of type (A_f), type (A_g), type (P) then it commutes at their points of coincidence. Now, in view of Theorem 3.2, in all the cases A, B, S, T have a unique common fixed point in X. This completes our proof. □

In the following theorem, we shall see that Theorem 3.2 holds true under the variants of weakly compatible mappings.

Theorem 3.4. *Theorem 3.2 remains true if the 'weakly compatible property' is replaced by any one (retaining the rest of the hypothesis) of the following:*
(i) *weakly f-compatible,*
(ii) *weakly g-compatible.*

Proof. In case if the pair (A,S) satisfies weakly f-compatible property, then either

$$\lim_{n\to\infty} SA(x_n, y_n) = A(x,y), \qquad \lim_{n\to\infty} SA(y_n, x_n) = A(y,x)$$

or

$$\lim_{n\to\infty} SSx_n = A(x,y), \qquad \lim_{n\to\infty} SSy_n = A(y,x)$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = x, \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = y$$

and

$$\lim_{n\to\infty} A(Sx_n, Sy_n) = \lim_{n\to\infty} A(A(x_n, y_n), A(y_n, x_n)) = A(x, y),$$
$$\lim_{n\to\infty} A(Sy_n, Sx_n) = \lim_{n\to\infty} A(A(y_n, x_n), A(x_n, y_n)) = A(y, x) \text{ for some } x, y \in X.$$

Let a, b be two points in X so that $A(a, b) = Sa$ and $A(b, a) = Sb$. Taking $x_n = a, y_n = b$ it is easy to show that $A(Sa, Sb) = SA(a, b)$ and $A(Sb, Sa) = SA(b, a)$. Similarly, (B, T) commutes at all of its coincidence points. Now applying Theorem 3.2, we can conclude that A, B, S, T have a unique common fixed point.

Similarly, the theorem holds good if the pairs (A, S) and (B, T) are weakly f-compatible. □

4. An application

Theorem 4.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm defined by $a * b = \min\{a, b\}$ for all a, b in X. Let f, g be weakly compatible self maps on X satisfying the following conditions:*

(4.1) *$f(X) \subseteq g(X)$,*
(4.2) *there exists $\phi \in \Phi$ such that*

$$M(fx, fy, \phi(t)) \geq M(gx, gy, t) \quad \textit{for all } x, y \textit{ in } X \textit{ and } t > 0.$$

If range space of any one of the maps f or g is complete, then f and g have a unique common fixed point in X.

Proof. By taking $A(x, y) = B(x, y) = f(x)$ and $S(x) = T(x) = g(x)$ for all $x, y \in X$ in Theorem 3.2, we get the desired result. Next, we give an example in support of Theorem 4.1. □

Example 4.1. Let $X = (0, 1]$. For each x, y in X and t in $[0, \infty)$ define

$$M(x, y, t) = \begin{cases} \left(\dfrac{t}{t + |x - y|}\right) & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space, with $*$ being a continuous t-norm defined by $a * b = ab$ for all a, b in X. Define self maps f and g on X by $f(x) = \frac{x}{3}, g(x) = \frac{x}{2}$. Then $f(X) = [0, \frac{1}{3}] \subseteq [0, \frac{1}{2}] = g(X)$ i.e., condition (4.1) holds, also for $k \in [\frac{2}{3}, 1)$ condition (4.2) is satisfied. However, maps are weakly compatible at $x = 0$ which is the unique common fixed of f and g.

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References

[1] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(7) (2006) 1379–1393.

[2] J. X. Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger Spaces, Nonlinear Anal. 71(5-6) (2009) 1833–1843.

[3] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64(3) (1994) 395–399.

[4] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27(3) (1988) 385–389.

[5] O. Hadžić and E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Vol.536 of Mathematics and its Applications, Kluwer Academic, Dordrecht, The Netherlands. 2001.

[6] Xin-Qi Hu, Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, Fixed Point Theory Appl. 2011, Art. ID 363716, 14 pp.

[7] M. Imdad and J. Ali, Some common fixed point theorems in fuzzy metric spaces, Math. Commun. 11(2) (2006) 153–163.

[8] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986) 771–779.

[9] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215–229.

[10] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 336–344.

[11] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70(12) (2009) 4341–4349.

[12] S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy metric spaces, Int. J. Math. Math. Sci. 17(2) (1994) 253–258.

[13] R. P. Pant, Common fixed point for non-commuting maps, J. Math. Anal. Appl. 188 (1994) 436–440.

[14] H. K. Pathak, Y. J. Cho and S. M. Kang, Remarks on R-weakly commuting maps and common fixed point theorems, Bull. Korean Math. Soc. 34 (1997) 247–257.

[15] S. Sedghi, I. Altun and N. Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlinear Anal. 72(3-4) (2010) 1298–1304.

[16] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland Series in Probability and Applied Math. 5 1983.

[17] R. Vasuki, Common fixed points for R-weakly commuting maps in fuzzy metric spaces, Indian J. Pure Appl. Math. 30 (1999) 419–423.

[18] L. A. Zadeh, Fuzzy sets, Information and Control 89 (1965) 338–353.

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