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# Intuitionistic fuzzy prime bi-Γ-ideals of Γ-semigroups

Muhammad Akram

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Abstract. In this paper, we introduce the notion of intuitionistic fuzzy prime, strongly prime, semiprime, irreducible and strongly irreducible bi-Γ-ideals in a Γ-semigroup. Also, we investigate some of the properties related to these Γ-ideals.

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Corresponding Author: Muhammad Akram (makram_69@yahoo.com)

## 1. Introduction

In 1965, Zadeh [16] introduced the concept of fuzzy sets. After that a lot of work has been done on various aspects of fuzzy sets by several researchers and it is still in progress. In 1986, Atanassov [1] introduced the notion of intuitionistic fuzzy sets, which is a generalization of fuzzy sets. Fuzzy sets give the degree of membership of an element in a given set, but the intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. Both, the degree of membership and degree of non-membership are real numbers between 0 and 1, having sum not greater than 1. For more details on intuitionistic fuzzy sets, we refer to [1, 2, 3]. Fuzzy sets are intuitionistic fuzzy sets but the converse is not necessarily true [14].

Kim and Jun [5, 6] introduced the concept of intuitionistic fuzzification of several ideals of semigroups. In [7] Kim and Lee gave the notion of intuitionistic fuzzy bi-ideals of semigroups. Sen and Saha [12] defined the Γ-semigroup and established a relation between regular Γ-semigroup and Γ-group (see also [9, 10]. In 2007, Mustafa et al. [15] introduced the notion of intuitionistic fuzzy Γ-ideals in Γ-semigroups. Sardar et al. [11] gave the concept of intuitionistic fuzzy prime ideals, semiprime ideals and also intuitionistic fuzzy ideal extension in a Γ-semigroup. The concept of intuitionistic fuzzy bi-Γ-ideals of Γ-semigroup was introduced by Lekkoksung [8].

In this paper, we introduce the notion of the prime, semiprime, strongly prime, irreducible and strongly irreducible intuitionistic fuzzy bi-Γ-ideals in Γ-semigroup.

## 2. Preliminaries

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then $S$ is called a Γ-semigroup if it satisfies

(i) $x\gamma y \in S$

(ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

A non-empty subset $A$ of a Γ-semigroup $S$ is called Γ-subsemigroup of $S$, if $A\Gamma A \subseteq A$. A left (right) Γ-ideal of a Γ-semigroup $S$ is a non-empty subset $I$ of $S$ such that $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). A two sided Γ-ideal or simply a Γ-ideal is that, which is both a left and a right Γ-ideal of $S$. A Γ-subsemigroup $B$ of a Γ-semigroup $S$ is called a bi-Γ-ideal of $S$ if $B\Gamma S\Gamma B \subseteq B$. Every left(right, two sided) Γ-ideal of a Γ-semigroup $S$ is a bi-Γ-ideal of $S$ but the converse is not true in general. Also the intersection of any number of bi-Γ-ideals of $S$ is a bi-Γ-ideal of $S$. For any $a \in S$, the intersection of all bi-Γ-ideals of $S$ which contains "$a$" is a bi-Γ-ideal of $S$. This is the smallest bi-Γ-ideal of $S$ containing $a$ and is called the bi-Γ-ideal generated by $a$. It is denoted by $B(a)$ clearly $B(a) = \{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a$. A bi-Γ-ideal $B$ of $S$ is called prime(strongly prime) if $B_1\Gamma B_2 \subseteq B$ ($B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$) implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-Γ-ideals $B_1$ and $B_2$ of $S$. A bi-Γ-ideal $B$ of $S$ is called semiprime if $B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ for any bi-Γ-ideal $B_1$ of $S$. A bi-Γ-ideal $B$ of $S$ is said to be irreducible(strongly irreducible) if $B_1 \cap B_2 = B$ ($B_1 \cap B_2 \subseteq B$) implies that $B_1 = B$ or $B_2 = B$ ($B_1 \subseteq B$ or $B_2 \subseteq B$) for any bi-Γ-ideals $B_1$ and $B_2$ of $S$. Above definitions are due to [4] and [13].

A fuzzy set $\mu$ in a non-empty set $X$ is a function, $\mu : X \to [0, 1]$ and the compliment of $\mu$, denoted by $\bar{\mu}$ is the fuzzy set in $X$ given by $\bar{\mu} = 1 - \mu(x)$ for all $x \in X$.

An intuitionistic fuzzy set (briefly, IFS), $A$ in a non-empty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denotes the degree of membership and the degree of non-membership respectively, and for all $x \in X$

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1.$$

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in $X$ can be identified by an ordered pair $(\mu_A, \gamma_A)$ in $I^X \times I^X$. For simplicity, we shall use IFS for intuitionistic fuzzy set and $A = (\mu_A, \gamma_A)$ for IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

For any two intuitionistic *fuzzy* sets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of a Γ-semigroup $S$, we define, $A\Gamma B = (\mu_{A\Gamma B}, \gamma_{A\Gamma B})$ where,

$$\mu_{A\Gamma B}(x) = \begin{cases} \bigvee\limits_{x=a\beta b} \min[\mu_A(a), \mu_B(b)] & \text{if } x = a\beta b, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\gamma_{A\Gamma B}(x) = \begin{cases} \bigwedge\limits_{x=a\beta b} \max[\gamma_A(a), \gamma_B(b)] & \text{if } x = a\beta b, \\ 1 & \text{otherwise.} \end{cases}$$

Also if $B \subseteq C$, then $A\Gamma B \subseteq A\Gamma C$ and $B\Gamma A \subseteq C\Gamma A$. If $\{A_i, i \in I\}$ be a collection of intuitionistic fuzzy subsets of a $\Gamma$-semigroup $S$ then their intersection $\underset{i\in I}{\cap} A_i = (\underset{i\in I}{\wedge}\mu_{A_i}, \underset{i\in I}{\vee}\gamma_{A_i})$ is an intuitionistic fuzzy subset of $S$, where $\underset{i\in I}{\wedge}\mu_{A_i}(x) = \inf\{\mu_{A_i}(x) \mid i \in I, x \in S\}$ and $\underset{i\in I}{\vee}\gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x) \mid i \in I, x \in S\}$.

## 3. Intuitionistic fuzzy bi-$\Gamma$-ideals

In what follows, let $S$ denote a $\Gamma$-semigroup unless otherwise specified.

**Definition 3.1.** Let $A$ be a non-empty subset of a $\Gamma$-semigroup $S$, the *intuitionistic characteristic function* of $A$ is defined as $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$, where,

$$\mu_{\chi_A}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \quad \text{and} \quad \gamma_{\chi_A}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A. \end{cases}$$

**Lemma 3.2.** *Let $A$ and $B$ be subsets of a $\Gamma$-semigroup $S$ then,*
*(i) $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$ (ii) $\chi_A\Gamma\chi_B = \chi_{A\Gamma B}$ (iii) $\chi_A \cap \chi_B = \chi_{A\cap B}$.*

**Definition 3.3.** An IFS $A = (\mu_A, \gamma_A)$ in $S$ is called an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$ if

$$\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x\alpha y) \leq \max\{\gamma_A(x), \gamma_A(y)\}, \text{ for all } x, y \in S, \ \alpha \in \Gamma.$$

**Definition 3.4.** An IFS $A = (\mu_A, \gamma_A)$ in $S$ is called an intuitionistic fuzzy left (right) $\Gamma$-ideal of $S$ if

$$\mu_A(x\alpha y) \geq \mu_A(y) \qquad (\mu_A(x\alpha y) \geq \mu_A(x))$$

and

$$\gamma_A(x\alpha y) \leq \gamma_A(y) \qquad (\gamma_A(x\alpha y) \leq \gamma_A(x)), \text{ for all } x, y \in S, \alpha \in \Gamma.$$

**Definition 3.5.** An IFS $A = (\mu_A, \gamma_A)$ in a $\Gamma$- semigroup $S$ is called an intuitionistic fuzzy $\Gamma$- ideal of $S$ if it is both an intuitionistic fuzzy left $\Gamma$-ideal and an intuitionistic fuzzy right $\Gamma$- ideal.

It is clear that an intuitionistic fuzzy left (right) $\Gamma$-ideal of $S$ is an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$ but the converse is not true.

**Definition 3.6.** An intuitionistic fuzzy $\Gamma$-subsemigroup $A = (\mu_A, \gamma_A)$ of $S$ is called an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$ if

$$\mu_A(x\alpha z\beta y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\gamma_A(x\alpha z\beta y) \leq \max\{\gamma_A(x), \gamma_A(y)\}, \text{ for all } x, y, z \in S, \alpha, \beta \in \Gamma.$$

**Lemma 3.7** ([15]). *Every intuitionistic fuzzy $\Gamma$-ideal of a $\Gamma$-semigroup $S$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$ but the converse is not true.*

**Lemma 3.8.** *Let $A$ be a non-empty subset of a $\Gamma$-semigroup $S$, then $A$ is $\Gamma$-subsemigroup of $S$ if and only if, $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$.*

*Proof.* We suppose that $A$ is a $\Gamma$-subsemigroup of $S$ then $A\Gamma A \subseteq A$. Let $x, y \in S$ and $\alpha \in \Gamma$.

Case I. If $x, y \in A$ then $x\alpha y \in A$. Also $\mu_{\chi_A}(x) = 1$ and $\mu_{\chi_A}(y) = 1$ implies that, $\min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = 1 = \mu_{\chi_A}(x\alpha y)$. Also $\gamma_{\chi_A}(x) = 0$ and $\gamma_{\chi_A}(y) = 0$ implies that, $\max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y)\} = 0 = \gamma_{\chi_A}(x\alpha y)$.

Case II. If $x \notin A$ or $y \notin A$ then $\mu_{\chi_A}(x) = 0$ or $\mu_{\chi_A}(y) = 0$ implies that, $\min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = 0 \leq \mu_{\chi_A}(x\alpha y)$. Also $\gamma_{\chi_A}(x) = 1$ or $\gamma_{\chi_A}(y) = 1$ implies that $\max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y)\} = 1 \geq \gamma_{\chi_A}(x\alpha y)$. Hence $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$.

Conversely, we suppose that $\chi_A = (\mu_{\chi_A},\ \gamma_{\chi_A})$ is an intuitionistic fuzzy $\Gamma$ - subsemigroup of $S$. Let $x, y \in A$ and $\alpha \in \Gamma$ then $\mu_{\chi_A}(x\alpha y) \geq \min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = 1$ but $\mu_{\chi_A}(x\alpha y) \leq 1$. This gives $\mu_{\chi_A}(x\alpha y) = 1$ that is $x\alpha y \in A$ implies that $A\Gamma A \subseteq A$. Hence $A$ is a $\Gamma$-subsemigroup of $S$. □

**Lemma 3.9.** *Let $A$ be a non-empty subset of a $\Gamma$-semigroup $S$, then $A$ is a left (right,two sided) $\Gamma$-ideal of $S$ if and only if $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic fuzzy left (right,two sided) $\Gamma$-ideal of $S$.*

*Proof.* Straightforward. □

**Lemma 3.10.** *Let $B$ be a non-empty subset of a $\Gamma$-semigroup $S$, then $B$ is a bi-$\Gamma$-ideal of $S$ if and only if $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$.*

*Proof.* Let $B$ be a bi-$\Gamma$-ideal of $S$ then by *lemma* 3.8, $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ is a $\Gamma$-subsemigroup of $S$. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. If $x, y \in B$ then $x\alpha z\beta y \in B$ and $\mu_{\chi_B}(x) = 1, \gamma_{\chi_B}(x) = 0$ and $\mu_{\chi_B}(y) = 1, \gamma_{\chi_B}(y) = 0$. Also $\mu_{\chi_B}(x\alpha z\beta y) = 1$ and $\gamma_{\chi_B}(x\alpha z\beta y) = 0$. Hence $\mu_{\chi_B}(x\alpha z\beta y) = 1 = \min\{\mu_{\chi_B}(x), \mu_{\chi_B}(y)\}$ and $\gamma_{\chi_B}(x\alpha z\beta y) = 0 = \max\{\gamma_{\chi_B}(x), \gamma_{\chi_B}(y)\}$. If $x \notin B$ or $y \notin B$ then $\mu_{\chi_B}(x) = 0$ or $\mu_{\chi_B}(y) = 0$ and $\gamma_{\chi_B}(x) = 1$ or $\gamma_{\chi_B}(y) = 1$. Then $\min\{\mu_{\chi_B}(x), \mu_{\chi_B}(y)\} = 0 \leq \mu_{\chi_B}(x\alpha z\beta y)$ and $\max\{\gamma_{\chi_B}(x), \gamma_{\chi_B}(y)\} = 1 \geq \gamma_{\chi_B}(x\alpha z\beta y)$. Hence, $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$.

Conversely, we suppose that $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$ then by *lemma* 3.8, $B$ is an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$. For all $x, y \in B$, $z \in S$ and $\alpha, \beta \in \Gamma$, we have $\mu_{\chi_B}(x\alpha z\beta y) \geq \min\{\mu_{\chi_B}(x), \mu_{\chi_B}(y)\}$, because $\chi_B = (\mu_{\chi_B}, \gamma_{\chi_B})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$. Since, $x, y \in B$ then $\mu_{\chi_B}(x) = \mu_{\chi_B}(y) = 1$, and

$$\mu_{\chi_B}(x\alpha z\beta y) \geq \min\{\mu_{\chi_B}(x), \mu_{\chi_B}(y)\} = 1 \text{ but } \mu_{\chi_B}(x\alpha z\beta y) \leq 1.$$

This gives $\mu_{\chi_B}(x\alpha z\beta y) = 1$, which implies that $x\alpha z\beta y \in B$, for all $x, y \in B, z \in S$ and $\alpha, \beta \in \Gamma$. Hence $B\Gamma S\Gamma B \subseteq B$ implies that $B$ is a bi-$\Gamma$-ideal of $S$. □

**Lemma 3.11.** *Let $\{A_i, i \in I\}$ be a collection of intuitionistic fuzzy bi-$\Gamma$-ideals of $S$, then their intersection $\underset{i\in I}{\cap} A_i = (\underset{i\in I}{\wedge}\mu_{A_i}, \underset{i\in I}{\vee}\gamma_{A_i})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$, where*

$$\underset{i\in I}{\wedge}\mu_{A_i}(x) = \inf\{\mu_{A_i}(x) \mid i \in I,\ x \in S\}$$

*and*

$$\underset{i\in I}{\vee}\gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x) \mid i \in I, x \in S\}.$$

*Proof.* As $\underset{i\in I}{\cap}A_i = (\underset{i\in I}{\wedge}\mu_{A_i}, \underset{i\in I}{\vee}\gamma_{A_i})$, where $\underset{i\in I}{\wedge}\mu_{A_i}(x) = \inf\{\mu_{A_i}(x) \mid i \in I, x \in S\}$ and $\underset{i\in I}{\vee}\gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x) \mid i \in I, x \in S\}$. For $x, y \in S, \alpha \in \Gamma$, we have

$$\begin{aligned}
\underset{i\in I}{\wedge}\mu_{A_i}(x\alpha y) &= \inf\{\mu_{A_i}(x\alpha y) \mid i \in I, x\alpha y \in S\}\\
&\geq \inf\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\geq \min\{\inf\{\mu_{A_i}(x) \mid i \in I, x \in S\}, \inf\{\mu_{A_i}(y) \mid i \in I, y \in S\}\\
\underset{i\in I}{\wedge}\mu_{A_i}(x\alpha y) &\geq \min\{\underset{i\in I}{\wedge}\mu_{A_i}(x), \underset{i\in I}{\wedge}\mu_{A_i}(y)\}
\end{aligned}$$

Also

$$\begin{aligned}
\underset{i\in I}{\vee}\gamma_{A_i}(x\alpha y) &= \sup\{\gamma_{A_i}(x\alpha y) \mid i \in I, x\alpha y \in S\}\\
&\leq \sup\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\leq \max\{\sup\{\gamma_{A_i}(x) \mid i \in I, x \in S\}, \sup\{\gamma_{A_i}(y) \mid i \in I, y \in S\}\\
\underset{i\in I}{\vee}\gamma_{A_i}(x\alpha y) &\leq \max\{\underset{i\in I}{\vee}\gamma_{A_i}(x), \underset{i\in I}{\vee}\gamma_{A_i}(y)\}.
\end{aligned}$$

So $\underset{i\in I}{\cap}A_i = (\underset{i\in I}{\wedge}\mu_{A_i}, \underset{i\in I}{\vee}\gamma_{A_i})$ is a $\Gamma$-subsemigroup. Now for $x, y, z \in S, \alpha, \beta \in \Gamma$,

$$\begin{aligned}
\underset{i\in I}{\wedge}\mu_{A_i}(x\alpha z\beta y) &= \inf\{\mu_{A_i}(x\alpha z\beta y) \mid i \in I, x\alpha z\beta y \in S\}\\
&\geq \inf\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\geq \min\{\inf\{\mu_{A_i}(x) \mid i \in I, x \in S\}, \inf\{\mu_{A_i}(y) \mid i \in I, y \in S\}\\
\underset{i\in I}{\wedge}\mu_{A_i}(x\alpha z\beta y) &\geq \min\{\underset{i\in I}{\wedge}\mu_{A_i}(x), \underset{i\in I}{\wedge}\mu_{A_i}(y)\}.
\end{aligned}$$

Also

$$\begin{aligned}
\underset{i\in I}{\vee}\gamma_{A_i}(x\alpha z\beta y) &= \sup\{\gamma_{A_i}(x\alpha z\beta y) \mid i \in I, x\alpha z\beta y \in S\}\\
&\leq \sup\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\leq \max\{\sup\{\gamma_{A_i}(x) \mid i \in I, x \in S\}, \sup\{\gamma_{A_i}(y) \mid i \in I, y \in S\}\\
\underset{i\in I}{\vee}\gamma_{A_i}(x\alpha z\beta y) &\leq \max\{\underset{i\in I}{\vee}\gamma_{A_i}(x), \underset{i\in I}{\vee}\gamma_{A_i}(y)\}.
\end{aligned}$$

Hence $\underset{i\in I}{\cap}A_i = (\underset{i\in I}{\wedge}\mu_{A_i}, \underset{i\in I}{\vee}\gamma_{A_i})$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$. □

## 4. Intuitionistic fuzzy prime bi-Γ-ideals

**Definition 4.1.** An intuitionistic fuzzy bi-$\Gamma$-ideal $A = (\mu_A, \gamma_A)$ of a $\Gamma$-semigroup $S$ is called an intuitionistic fuzzy prime bi-$\Gamma$-ideal of $S$, if for any intuitionistic fuzzy bi-$\Gamma$-ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of $S$, $B\Gamma C \subseteq A$ implies that $B \subseteq A$ or $C \subseteq A$.

**Definition 4.2.** An intuitionistic fuzzy bi-$\Gamma$-ideal $A = (\mu_A, \gamma_A)$ of a $\Gamma$-semigroup $S$ is called an intuitionistic fuzzy strongly prime bi-$\Gamma$-ideal of $S$ if for any intuitionistic fuzzy bi-$\Gamma$-ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of $S$, $B\Gamma C \cap C\Gamma B \subseteq A$ implies that $B \subseteq A$ or $C \subseteq A$.

**Definition 4.3.** An intuitionistic fuzzy bi-$\Gamma$-ideal $A = (\mu_A, \gamma_A)$ of a $\Gamma$-semigroup $S$ is called an intuitionistic fuzzy semiprime bi-$\Gamma$-ideal of $S$ if for any intuitionistic fuzzy bi-$\Gamma$-ideal $B = (\mu_B, \gamma_B)$ of $S$, $B\Gamma B \subseteq A$ implies that $B \subseteq A$.

**Definition 4.4.** An intuitionistic fuzzy bi-$\Gamma$-ideal $A = (\mu_A, \gamma_A)$ of a $\Gamma$-semigroup $S$ is called an idempotent if $A = A\Gamma A$ that is $\mu_{A\Gamma A} = \mu_A \Gamma \mu_A = \mu_A$ and $\gamma_{A\Gamma A} = \gamma_A \Gamma \gamma_A = \gamma_A$.

**Theorem 4.5.** *A non-empty subset $P$ of $S$ is a prime bi-$\Gamma$-ideal of $S$ if and only if the intuitionistic characteristic function $\chi_P = (\mu_{\chi_p}, \gamma_{\chi_p})$ of $P$ is an intuitionistic fuzzy prime bi-$\Gamma$-ideal of $S$.*

*Proof.* Suppose $P$ is a prime bi-$\Gamma$-ideal of $S$. Then by *Lemma* 3.10, $\chi_B$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$. Let $A = (\mu_A, \gamma_A)$ and $C = (\mu_C, \gamma_C)$ be any intuitionistic fuzzy bi-$\Gamma$-ideals of $S$ such that $A\Gamma C \subseteq \chi_B$ but $A \nsubseteq \chi_B$ and $C \nsubseteq \chi_B$. Then there exist $x, y \in S$ such that

$$\mu_A(x) \neq 0, \gamma_A(x) \neq 1 \text{ and } \mu_C(y) \neq 0, \gamma_C(y) \neq 1$$

$$\text{but } \mu_{\chi_P}(x) = 0, \gamma_{\chi_P}(x) = 1 \text{ and } \mu_{\chi_P}(y) = 0, \gamma_{\chi_P}(y) = 1.$$

Hence $x \notin P$ and $y \notin P$. Since $P$ is a prime bi-$\Gamma$-ideal of $S$, then for bi-$\Gamma$-ideals $B(x)$ and $B(y)$, we have $B(x)\Gamma B(y) \nsubseteq P$. Since $\mu_A(x) \neq 0$, $\gamma_A(x) \neq 1$ and $\mu_C(y) \neq 0$, $\gamma_C(y) \neq 1$ therefore, $\min\{\mu_A(x), \mu_C(y)\} \neq 0$ and $\max\{\gamma_A(x), \gamma_C(y)\} \neq 1$.

As $B(x)\Gamma B(y) \nsubseteq P$, therefore there exists $s \in S$ such that $s \in B(x)\Gamma B(y)$ but $s \notin P$ then $\mu_{\chi_P}(s) = 0$, $\gamma_{\chi_P}(s) = 1$. Hence $\mu_{A\Gamma C}(s) = 0$ and $\gamma_{A\Gamma C}(s) = 1$. Since $s \in B(x)\Gamma B(y)$ then $s = b_1\alpha b_2$ for some $b_1 \in B(x)$ and $b_2 \in B(y)$ and $\alpha \in \Gamma$. So we have

$$\mu_{A\Gamma C}(s) = \bigvee_{s=b_1\alpha b_2} \min\{\mu_A(b_1), \mu_C(b_2)\} \geq \min\{\mu_A(b_1), \mu_C(b_2)\}$$

and

$$\gamma_{A\Gamma C}(s) = \bigwedge_{s=b_1\alpha b_2} \max\{\gamma_A(b_1), \gamma_C(b_2)\} \leq \max\{\gamma_A(b_1), \gamma_C(b_2)\}.$$

Since $b_1 \in B(x)$, which is a bi-$\Gamma$-ideal generated by $x$ and $B(x) = \{x\} \cup x\Gamma x \cup x\Gamma S\Gamma x$, we have $b_1 = x$ or $b_1 = x\beta x$ or $b_1 = x\theta r\phi x$ for some $r \in S$ and $\beta, \theta, \phi \in \Gamma$.

If $b_1 = x$ then $\mu_A(b_1) = \mu_A(x)$ and $\gamma_A(b_1) = \gamma_A(x)$.
If $b_1 = x\beta x$ then $\mu_A(b_1) = \mu_A(x\beta x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$
and $\gamma_A(b_1) = \gamma_A(x\beta x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$.
If $b_1 = x\theta r\phi x$ then $\mu_A(b_1) = \mu_A(x\theta r\phi x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$
and $\gamma_A(b_1) = \gamma_A(x\theta r\phi x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$.

Also as $b_2 \in B(y) = \{y\} \cup y\Gamma y \cup y\Gamma S\Gamma y$, then $b_2 = y$ or $b_2 = y\delta y$ or $b_2 = y\eta t\pi y$ for some $t \in S$, and $\delta, \eta, \pi \in \Gamma$.

If $b_2 = y$ then $\mu_C(b_2) = \mu_C(y)$ and $\gamma_C(b_2) = \gamma_C(y)$.
If $b_2 = y\delta y$ then $\mu_C(b_2) = \mu_C(y\delta y) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$
and $\gamma_C(b_2) = \gamma_C(y\delta y) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$.
If $b_2 = y\eta t\pi y$ then $\mu_C(b_2) = \mu_C(y\eta t\pi y) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$
and $\gamma_C(b_2) = \gamma_C(y\eta t\pi y) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$.

From above we can write

$$\begin{aligned}\mu_{A\Gamma C}(s) &\geq \min\{\mu_A(b_1), \mu_C(b_2)\} \geq \min\{\mu_A(x), \mu_C(y)\} \neq 0,\\ \text{because } \mu_A(x) &\neq 0, \mu_C(y) \neq 0,\end{aligned}$$

and

$$\begin{aligned}\gamma_{A\Gamma C}(s) &\leq \max\{\gamma_A(b_1), \gamma_C(b_2)\} \leq \max\{\gamma_A(x), \gamma_C(y)\} \neq 1,\\ \text{because } \gamma_A(x) &\neq 1, \gamma_C(y) \neq 1.\end{aligned}$$

Which is a contradiction that $\mu_{A\Gamma C}(s) = 0$ and $\gamma_{A\Gamma C}(s) = 1$. Thus for any intuitionistic fuzzy bi-Γ-ideals $A$ and $C$ of $S$, $A\Gamma C \subseteq \chi_P$ implies that $A \subseteq \chi_P$ or $C \subseteq \chi_P$. Hence $\chi_P = (\mu_{\chi_p}, \gamma_{\chi_p})$ is an intuitionistic fuzzy prime bi-Γ-ideal of $S$.

Conversely, we suppose that $\chi_P = (\mu_{\chi_p}, \gamma_{\chi_p})$ is an intuitionistic fuzzy prime bi-Γ-ideal of $S$ and let $B_1$and $B_2$ be any bi-Γ-ideals of $S$ such that $B_1\Gamma B_2 \subseteq P$, then by *Lemma* 3.10, $\chi_{B_1}$ and $\chi_{B_2}$ are intuitionistic fuzzy bi-Γ-ideals of $S$ also $\chi_{B_1\Gamma B_2} \subseteq \chi_P$ but we know $\chi_{B_1\Gamma B_2} = \chi_{B_1}\Gamma\chi_{B_2}$ implies $\chi_{B_1}\Gamma\chi_{B_2} \subseteq \chi_P$. As by supposition $\chi_P$ is prime so we have, $\chi_{B_1} \subseteq \chi_P$ or $\chi_{B_2} \subseteq \chi_P$. Hence by Lemma 3.2, $B_1 \subseteq P$ or $B_2 \subseteq P$ implies that $P$ is a prime bi-Γ-ideal of $S$. □

**Theorem 4.6.** *A non-empty subset $P$ of $S$ is a strongly prime bi-Γ-ideal of $S$ if and only if the intuitionistic characteristic function $\chi_P = (\mu_{\chi_p}, \gamma_{\chi_p})$ of $P$ is an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$.*

*Proof.* Let $P$ be a bi-Γ-ideal of $S$ then by *Lemma* 3.10, $\chi_P$ is an intuitionistic fuzzy bi-Γ-ideal of $S$. We suppose that $P$ is a strongly prime. Let $A = (\mu_A, \gamma_A)$ and $C = (\mu_C, \gamma_C)$ be any intuitionistic fuzzy bi-Γ-ideal of $S$ such that $A\Gamma C \cap C\Gamma A \subseteq \chi_P$ but $A \nsubseteq \chi_p$ and $C \nsubseteq \chi_P$. Then there exist $x, y \in S$ such that $\mu_A(x) \neq 0$, $\gamma_A(x) \neq 1$ and $\mu_C(y) \neq 0$, $\gamma_C(y) \neq 1$ but $\mu_{\chi_P}(x) = 0$, $\gamma_{\chi_P}(x) = 1$ and $\mu_{\chi_P}(y) = 0$, $\gamma_{\chi_P}(y) = 1$. Hence $x \notin P$ and $y \notin P$. Since $P$ is a strongly prime bi-Γ-ideal of $S$, then for bi-Γ-ideals $B(x)$ and $B(y)$, we have $B(x)\Gamma B(y) \cap B(y)\Gamma B(x) \nsubseteq P$. Since $\mu_A(x) \neq 0$, $\gamma_A(x) \neq 1$ and $\mu_C(y) \neq 0$, $\gamma_C(y) \neq 1$ therefore, $\min\{\mu_A(x), \mu_C(y)\} \neq 0$ and $\max\{\gamma_A(x), \gamma_C(y)\} \neq 1$.

As $B(x)\Gamma B(y) \cap B(y)\Gamma B(x) \nsubseteq P$, then there exists $s \in S$ such that $s \in B(x)\Gamma B(y) \cap B(y)\Gamma B(x)$ but $s \notin P$ then $\mu_{\chi_P}(s) = 0$ and $\gamma_{\chi_P}(s) = 1$ and hence $\mu_{A\Gamma C}(s) \wedge \mu_{C\Gamma A}(s) = 0$ and $\gamma_{A\Gamma C}(s) \vee \gamma_{C\Gamma A}(s) = 1$. Since $s \in B(x)\Gamma B(y)$ and $s \in B(y)\Gamma B(x)$.

When $s \in B(x)\Gamma B(y)$, we can write $s = b_1\alpha b_2$, for some $b_1 \in B(x)$, $b_2 \in B(y)$ and $\alpha \in \Gamma$. So we have

$$\mu_{A\Gamma C}(s) = \bigvee_{s=b_1\alpha b_2} \min\{\mu_A(b_1), \mu_C(b_2)\} \geq \min\{\mu_A(b_1), \mu_C(b_2)\}$$

and

$$\gamma_{A\Gamma C}(s) = \bigwedge_{s=b_1\alpha b_2} \max\{\gamma_A(b_1), \gamma_C(b_2)\} \leq \max\{\gamma_A(b_1), \gamma_C(b_2)\}.$$

Since $b_1 \in B(x)$, which is a bi-Γ-ideal generated by $x$ and $B(x) = \{x\} \cup x\Gamma x \cup x\Gamma S\Gamma x$, then $b_1 = x$ or $b_1 = x\beta x$ or $b_1 = x\theta r\phi x$, for some $r \in S$ and $\beta, \theta, \phi \in \Gamma$.

If $b_1 = x$ then $\mu_A(b_1) = \mu_A(x)$ and $\gamma_A(b_1) = \gamma_A(x)$.

If $b_1 = x\beta x$ then $\mu_A(b_1) = \mu_A(x\beta x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$

and $\gamma_A(b_1) = \gamma_A(x\beta x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$.
If $b_1 = x\theta r\phi x$ then $\mu_A(b_1) = \mu_A(x\theta r\phi x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and $\gamma_A(b_1) = \gamma_A(x\theta r\phi x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$.
Also as $b_2 \in B(y) = \{y\} \cup y\Gamma y \cup y\Gamma S\Gamma y$, then $b_2 = y$ or $b_2 = y\delta y$ or $b_2 = y\eta t\pi y$, for some $t \in S$, $\delta, \eta, \pi \in \Gamma$.
If $b_2 = y$ then $\mu_C(b_2) = \mu_C(y)$ and $\gamma_C(b_2) = \gamma_C(y)$.
If $b_2 = y\delta y$ then $\mu_C(b_2) = \mu_C(y\delta y) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$ and $\gamma_C(b_2) = \gamma_C(y\delta y) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$.
If $b_2 = y\eta t\pi y$ then $\mu_C(b_2) = \mu_C(y\eta t\pi y) \geq \min\{\mu_C(y), \mu_C(y)\} = \mu_C(y)$ and $\gamma_C(b_2) = \gamma_C(y\eta t\pi y) \leq \max\{\gamma_C(y), \gamma_C(y)\} = \gamma_C(y)$.
From above we can write,

$$\begin{aligned} \mu_{A\Gamma C}(s) &\geq \min\{\mu_A(b_1), \mu_C(b_2)\} \geq \min\{\mu_A(x), \mu_C(y)\} \neq 0, \\ \text{since } \mu_A(x) &\neq 0 \text{ and } \mu_C(y) \neq 0, \end{aligned}$$

and

$$\begin{aligned} \gamma_{A\Gamma C}(t) &\leq \max\{\gamma_A(b_1), \gamma_C(b_2)\} \leq \max\{\gamma_A(x), \gamma_C(y)\} \neq 1, \\ \text{since } \gamma_A(x) &\neq 1 \text{ and } \gamma_C(y) \neq 1. \end{aligned}$$

Similarly, by taking $s \in B(y)\Gamma B(x)$, we can prove that $\mu_{C\Gamma A}(s) \neq 0$ and $\gamma_{C\Gamma A}(s) \neq 1$. This implies that $\mu_{A\Gamma C}(s) \wedge \mu_{C\Gamma A}(s) \neq 0$ and $\gamma_{A\Gamma C}(s) \vee \gamma_{C\Gamma A}(s) \neq 1$, which is a contradiction as we have $\mu_{A\Gamma C}(s) \wedge \mu_{C\Gamma A}(s) = 0$ and $\gamma_{A\Gamma C}(s) \vee \gamma_{C\Gamma A}(s) = 1$. Thus for any intuitionistic fuzzy bi-Γ-ideals $A$ and $C$ of $S$, $A\Gamma C \cap C\Gamma A \subseteq \chi_P$ implies that $A \subseteq \chi_P$ or $C \subseteq \chi_P$. Hence $\chi_P$ is an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$.

Conversely, we suppose that $\chi_P$ is an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$ and let $B_1$ and $B_2$, be any bi-Γ-ideals of $S$ such that $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq P$, then by *Lemma* 3.10, $\chi_{B_1}$ and $\chi_{B_2}$ are intuitionistic fuzzy bi-Γ-ideal of $S$ and also $\chi_{B_1\Gamma B_2} \cap \chi_{B_2\Gamma B_1} \subseteq \chi_P$ but we know $\chi_{B_1\Gamma B_2} = \chi_{B_1}\Gamma\chi_{B_2}$. Which implies that $\chi_{B_1}\Gamma\chi_{B_2} \cap \chi_{B_2}\Gamma\chi_{B_1} \subseteq \chi_P$. As by supposition $\chi_P$ is strongly prime so we have, $\chi_{B_1} \subseteq \chi_P$ or $\chi_{B_2} \subseteq \chi_P$. Hence by *Lemma* 3.2, $B_1 \subseteq P$ or $B_2 \subseteq P$ implies that $P$ is a strongly prime bi-Γ-ideal of $S$. □

Similarly, we can prove the following:

**Theorem 4.7.** *A non-empty subset $P$ of $S$ is a semiprime bi-Γ-ideal of $S$ if and only if the intuitionistic characteristic function $\chi_P = (\mu_{\chi_p}, \gamma_{\chi_p})$ of $P$ is an intuitionistic fuzzy semiprime bi-Γ-ideal of $S$.*

**Remark 4.8.** Obviously every intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$ is an intuitionistic fuzzy prime bi-Γ-ideal of $S$ and every intuitionistic fuzzy prime bi-Γ-ideal of $S$ is an intuitionistic fuzzy semiprime bi-Γ-ideal of $S$.

Note that the converse of above is not true.

**Example 4.9.** Consider $S = \{a, b, c\}$ be a semigroup with following multiplication table and Γ be a non-empty set. Define $S \times \Gamma \times S \to S$ by $x\gamma y = x * y$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a Γ-semigroup.

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ |
| $c$ | $a$ | $c$ | $c$ |

The bi-Γ-ideals of $S$ are $\{a\}, \{a,b\}, \{a,c\}$ and $S$. Here $\{a\}$ is a prime bi-Γ-ideal of $S$ but it is not strongly prime as $(\{a,b\}\Gamma\{a,c\}) \cap (\{a,c\}\Gamma\{a,b\}) = \{a\} \subseteq \{a\}$ but $\{a,b\} \nsubseteq \{a\}$, and $\{a,c\} \nsubseteq \{a\}$. Hence by Theorem 4.5 and 4.6, $\chi_{\{a\}}$ is an intuitionistic fuzzy prime bi-Γ-ideal of $S$ but not an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$.

**Example 4.10.** Let $0 \in S$ and $|S| > 3$ and $\Gamma$ be any set. Then $S$ is a Γ-semigroup with zero, under the operation defined by

$$x\gamma y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases} \text{ for all } x, y \in S, \alpha \in \Gamma.$$

Since for all subsets $A$ and $B$ of $S$ containing $\{0\}$, we have $A\Gamma S\Gamma A = A$ and $A\Gamma B = A \cap B$. All these subsets are semiprime bi-Γ-ideals of $S$. Let $C$ be a semiprime bi-Γ-ideal of $S$ such that $|S \backslash C| \geq 3$. Then for distinct $a, b \in S\backslash C$, we have $(C \cup \{a\})\Gamma(C \cup \{b\}) = (C \cup \{a\}) \cap (C \cup \{b\}) = C$, but $(C \cup \{a\}) \nsubseteq C$ and $(C \cup \{b\}) \nsubseteq C$, implies that $C$ is not a prime bi-Γ-ideal of $S$. Hence by Theorem 4.5 and 4.7, $\chi_C$ is an intuitionistic fuzzy semiprime bi-Γ-ideal of $S$ but not an intuitionistic fuzzy prime bi-Γ-ideal of $S$.

**Proposition 4.11.** *The intersection of any family of intuitionistic fuzzy prime bi-Γ-ideals of $S$ is an intuitionistic fuzzy semiprime bi-Γ-ideal of $S$.*

*Proof.* Let $\{A_i : i \in I\}$ be a collection of intuitionistic fuzzy prime bi-Γ-ideals of $S$, where $A_i = (\mu_{A_i}, \gamma_{A_i})$ for $i \in I$. Then by *Lemma* 3.11, $\underset{i\in I}{\cap} A_i = (\underset{i\in I}{\wedge} \mu_{A_i}, \underset{i\in I}{\vee} \gamma_{A_i})$ is an intuitionistic fuzzy bi-Γ-ideal of $S$. Let $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy bi-Γ-ideal of $S$ such that $B\Gamma B \subseteq \underset{i\in I}{\cap} A_i$, which implies that $B\Gamma B \subseteq A_i$ for all $i \in I$. But each $A_i$ is an intuitionistic fuzzy prime bi-Γ-ideal of $S$, so $B \subseteq A_i$, for all $i \in I$ implies that $B \subseteq \underset{i\in I}{\cap} A_i$. Hence $\underset{i\in I}{\cap} A_i = (\underset{i\in I}{\wedge} \mu_{A_i}, \underset{i\in I}{\vee} \gamma_{A_i})$ is an intuitionistic fuzzy semiprime bi-Γ-ideal of $S$. □

**Definition 4.12.** An intuitionistic fuzzy bi-Γ-ideal $A = (\mu_A, \gamma_A)$ of a Γ-semigroup $S$ is called an intuitionistic fuzzy irreducible bi-Γ-ideal of $S$ if for any intuitionistic fuzzy bi-Γ-ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of $S$, $B \cap C = A$ implies that $B = A$ or $C = A$.

**Definition 4.13.** An intuitionistic fuzzy bi-Γ-ideal, $A = (\mu_A, \gamma_A)$ of a Γ-semigroup $S$ is called an intuitionistic fuzzy strongly irreducible bi-Γ-ideal of $S$ if for any intuitionistic fuzzy bi-Γ-ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of $S$, $B \cap C \subseteq A$ implies that $B \subseteq A$ or $C \subseteq A$.

**Lemma 4.14.** *A non-empty subset $Q$ of $S$ is an irreducible (strongly irreducible) bi-Γ-ideal of $S$ if and only if the intuitionistic characteristic function $\chi_Q = (\mu_{\chi_Q}, \gamma_{\chi_Q})$ of $Q$ is an intuitionistic fuzzy irreducible (strongly irreducible) bi-Γ-ideal of $S$.*

*Proof.* Straightforward. □

Every intuitionistic fuzzy strongly irreducible bi-Γ-ideal of $S$ is an intuitionistic fuzzy irreducible bi-Γ-ideal of $S$ but the converse is not true in general.

**Example 4.15.** Let $S = \{a, b, c, d, e, f\}$ be a semigroup with following multiplication table and Γ be a non-empty set. Define $S \times \Gamma \times S \to S$ by $x\gamma y = x * y$ for all $x, y \in S$ and $\gamma \in \Gamma$.Then $S$ is a Γ-semigroup.

| $*$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ | $d$ | $b$ | $b$ |
| $d$ | $a$ | $b$ | $b$ | $b$ | $c$ | $d$ |
| $e$ | $a$ | $b$ | $e$ | $f$ | $b$ | $b$ |
| $f$ | $a$ | $b$ | $b$ | $b$ | $e$ | $f$ |

All bi-Γ-ideals of $S$ are $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, b, f\}$, $\{a, b, c, e\}$, $\{a, b, d, f\}$, $\{a, b, c, d\}$, $\{a, b, e, f\}$ and $S$. Here $\{a\}$, $\{a, b, c, e\}$, $\{a, b, d, f\}$, $\{a, b, c, d\}$, $\{a, b, e, f\}$ and $S$ are irreducible but only $\{a\}$ and $S$ are strongly irreducible. If $A = \{a, b, c, e\}$, $B = \{a, b, d, f\}$, $C = \{a, b, c, d\}$ and $D = \{a, b, e, f\}$, then by Lemma 4.14, $\chi_A$, $\chi_B$, $\chi_C$, $\chi_D$ are intuitionistic fuzzy irreducible bi-Γ-ideals of $S$ which are not intuitionistic fuzzy strongly irreducible bi-Γ-ideals of $S$.

**Proposition 4.16.** *An intuitionistic fuzzy strongly irreducible semiprime bi-Γ-ideal of $S$ is an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$.*

*Proof.* Let $A$ be an intuitionistic fuzzy strongly irreducible semiprime bi-Γ-ideal of $S$. Let $B$ and $C$ be any two intuitionistic fuzzy bi-Γ-ideals of $S$ such that, $B\Gamma C \cap C\Gamma B \subseteq A$. Now as $B \cap C \subseteq B$ and $B \cap C \subseteq C$ implies that $(B \cap C)\Gamma(B \cap C) \subseteq B\Gamma C$, similarly $(B \cap C)\Gamma(B \cap C) \subseteq C\Gamma B$. Then we can write $(B \cap C)\Gamma(B \cap C) \subseteq B\Gamma C \cap C\Gamma B$ implies $(B \cap C)\Gamma(B \cap C) \subseteq A$. Since $A$ is an intuitionistic fuzzy semiprime bi-Γ-ideal, so $B \cap C \subseteq A$, as $A$ is a strongly irreducible so $B \subseteq A$ or $C \subseteq A$. Hence $A$ is an intuitionistic fuzzy strongly prime bi-Γ-ideal of $S$. □

**Proposition 4.17.** *Let $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy bi-Γ-ideal of $S$ with $\mu_B(a) = t$ and $\gamma_B(a) = 1-t$, for $a \in S$ and $t \in (0, 1]$, then there exist an intuitionistic fuzzy irreducible bi-Γ-ideal $C = (\mu_C, \gamma_C)$ of $S$ such that $B \subseteq C$ and $\mu_C(a) = t$ and $\gamma_C(a) = 1 - t$.*

*Proof.* Let $\mathcal{F} = \{A \mid A$ is an intuitionistic fuzzy bi-Γ-ideal of $S$ such that $\mu_A(a) = t$ $, \gamma_A(a) = 1 - t$ and $B \subseteq A\}$, then $\mathcal{F} \neq \phi$ because $B \in \mathcal{F}$. Then the collection $\mathcal{F}$ is a partially ordered set under inclusion. If $\omega = \{A_i \mid i \in I\}$ is any totally ordered subcollection of $\mathcal{F}$ then $\underset{i\in I}{\cup} A_i = (\underset{i\in I}{\vee} \mu_{A_i}, \underset{i\in I}{\wedge} \gamma_{A_i})$, where, $\underset{i\in I}{\vee} \mu_{A_i}(x) = \sup\{\mu_{A_i}(x) \mid i \in I, x \in S\}$ and $\underset{i\in I}{\wedge} \gamma_{A_i}(x) = \inf\{\gamma_{A_i}(x) \mid i \in I, x \in S\}$. Now we will show that $\underset{i\in I}{\cup} A_i, i \in I$ is an intuitionistic fuzzy bi-Γ-ideal of $S$. For $x, y \in S$, $\alpha \in \Gamma$, we have,

$$\begin{aligned}
\underset{i\in I}{\vee}\mu_{A_i}(x\alpha y) &= \sup\{\mu_{A_i}(x\alpha y) \mid i \in I, x\alpha y \in S\}\\
&\geq \sup\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\geq \min\{\sup\{\mu_{A_i}(x) \mid i \in I, x \in S\}, \sup\{\mu_{A_i}(y)\} \mid i \in I, y \in S\}\\
\underset{i\in I}{\vee}\mu_{A_i}(x\alpha y) &\geq \min\{\underset{i\in I}{\vee}\mu_{A_i}(x), \underset{i\in I}{\vee}\mu_{A_i}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\underset{i\in I}{\wedge}\gamma_{A_i}(x\alpha y) &= \inf\{\gamma_{A_i}(x\alpha y) \mid i \in I, x\alpha y \in S\}\\
&\leq \inf\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\leq \max\{\inf\{\gamma_{A_i}(x) \mid i \in I, x \in S\}, \inf\{\gamma_{A_i}(y)\} \mid i \in I, y \in S\}\\
\underset{i\in I}{\wedge}\gamma_{A_i}(x\alpha y) &\leq \max\{\underset{i\in I}{\wedge}\gamma_{A_i}(x), \underset{i\in I}{\wedge}\gamma_{A_i}(y)\}.
\end{aligned}$$

Hence $\underset{i\in I}{\cup} A_i, i \in I$ is an intuitionistic fuzzy $\Gamma$-subsemigroup of $S$. Now for $x, y, z \in S, \alpha, \beta \in \Gamma$,

$$\begin{aligned}
\underset{i\in I}{\vee}\mu_{A_i}(x\alpha z\beta y) &= \sup\{\mu_{A_i}(x\alpha z\beta y) \mid i \in I, x\alpha z\beta y \in S\}\\
&\geq \sup\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\geq \min\{\sup\{\mu_{A_i}(x) \mid i \in I, x \in S\}, \sup\{\mu_{A_i}(y)\} \mid i \in I, y \in S\}\\
\underset{i\in I}{\vee}\mu_{A_i}(x\alpha z\beta y) &\geq \min\{\underset{i\in I}{\vee}\mu_{A_i}(x), \underset{i\in I}{\vee}\mu_{A_i}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\underset{i\in I}{\wedge}\gamma_{A_i}(x\alpha z\beta y) &= \inf\{\gamma_{A_i}(x\alpha z\beta y) \mid i \in I, x\alpha z\beta y \in S\}\\
&\leq \inf\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\} \mid i \in I, x, y \in S\}\\
&\leq \max\{\inf\{\gamma_{A_i}(x) \mid i \in I, x \in S\}, \inf\{\gamma_{A_i}(y)\} \mid i \in I, y \in S\}\\
\underset{i\in I}{\wedge}\gamma_{A_i}(x\alpha z\beta y) &\leq \max\{\underset{i\in I}{\wedge}\gamma_{A_i}(x), \underset{i\in I}{\wedge}\gamma_{A_i}(y)\}.
\end{aligned}$$

Thus $\underset{i\in I}{\cup} A_i$ is an intuitionistic fuzzy bi-$\Gamma$-ideal of $S$. Since $B \subseteq A_i$ for all $i \in I$, then $B \subseteq \underset{i\in I}{\cup} A_i$. Also $(\underset{i\in I}{\vee}\mu_{A_i})(a) = \underset{i\in I}{\vee}\mu_{A_i}(a) = t$ and $(\underset{i\in I}{\wedge}\gamma_{A_i})(a) = \underset{i\in I}{\wedge}\gamma_{A_i}(a) = 1 - t$. Hence $\underset{i\in I}{\cup} A_i$ is an upper bound of $\omega$ that is $\omega = \{A_i \mid i \in I\}$ is bounded above then by *Zorn's lemma*, there exists a maximal intuitionistic fuzzy bi-$\Gamma$-ideal, $C = (\mu_C, \gamma_C)$ of $S$ such that $B \subseteq C$ and $\mu_C(a) = t, \gamma_C(a) = 1 - t$. Now we will show that $C$ is irreducible. Let $B_1, B_2$ be any intuitionistic fuzzy bi-$\Gamma$-ideal of $S$, such that $B_1 \cap B_2 = C$ then $C \subseteq B_1$ and $C \subseteq B_2$. We suppose that $C \neq B_1$ and $C \neq B_2$. Since $C$ is the maximal intuitionistic fuzzy bi-$\Gamma$-ideal of $S$ with $\mu_C(a) = t, \gamma_C(a) = 1 - t$, then $\mu_{B_1}(a) \neq t$ or $\gamma_{B_1}(a) \neq 1 - t$ and $\mu_{B_2}(a) \neq t$ or $\gamma_{B_2}(a) \neq 1 - t$. Thus we have,

$$t = \mu_C(a) = \mu_{B_1 \cap B_2}(a) = \mu_{B_1}(a) \cap \mu_{B_2}(a) \neq t$$

or

$$1 - t = \gamma_C(a) = \gamma_{B_1 \cap B_2}(a) = \gamma_{B_1}(a) \cap \gamma_{B_2}(a) \neq 1 - t.$$

Which is a contradiction. Hence either $B_1 = C$ or $B_2 = C$ implies that $C$ is an irreducible intuitionistic fuzzy bi-$\Gamma$-ideal of $S$. □

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MUHAMMAD AKRAM (makram_69@yahoo.com)
Department of Mathematics, University of Gujrat, Gujrat, Pakistan