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T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroups

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Abstract. In this paper, we introduce the notions of an interval-valued $(\in, \in \vee q)$-fuzzy subgroups, T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroups and investigate some of their related properties.

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1. Introducoion

The theory of fuzzy set was first developed by Zadeh [23] and has been applied to many branches in Mathematics. Later fuzzification of the "group" concept into "fuzzy subgroup" was made by Rosenfeld [19]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. To study more about fuzzy subgroups, see [1, 2, 3, 7, 16, 17]. Also, Zadeh [22] have introduced the concept of fuzzy set by an interval-valued fuzzy set (ie., a fuzzy set with an interval-valued membership function) and he constructed a method of approximate inference using his interval-valued fuzzy sets. The interval-valued fuzzy subgroups were first defined and studied by Biswas [6] which are the subgroups of the same nature of the fuzzy subgroups defined by Rosenfeld. In [24] Zeng et.al, gave a kind of method to describe the entropy of interval-valued fuzzy set based on its similarity measure and discussed their relationship between the similarity measure and the entropy of the interval-valued fuzzy sets in detail. However, the results obtained can still be applied in many fields such as pattern recognition, image processing and fuzzy reasoning etc. To study more about interval-valued fuzzy subgroups, see [6, 20, 24]. Later Davvaz [10] have defined T_H and S_H - interval-valued fuzzy H_v - subgroups. A new type of fuzzy subgroups that is, the $(\in, \in \vee q)$-fuzzy subgroups were introduced in Bhakat et. al [5] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [18]. In fact, the $(\in, \in \vee q)$-fuzzy subgroups are an important generalization of Rosenfeld's fuzzy subgroups. With this objective in view, Davvaz [9] have applied this theory to near-rings and obtained some useful results. Using the relations between fuzzy points and fuzzy sets Osman et al.[13] have introduced the notions fuzzy bi-ideals with thresholds, $(\in, \in \vee q)$-fuzzy bi-ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy bi-ideals and have studied in detail. The notion of interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras was discussed by Jianming Zhan et.al [25]. To study more about $(\in, \in \vee q)$-fuzzy algebraic structures, see [4, 5, 8, 9, 13, 15, 21, 25]. Using the 'belongs to relation' $(\in)$ and 'quasi-coincidence with the relation' q between fuzzy points and fuzzy sets, the concept of (α, β) -fuzzy algebras where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced, and related properties were investigated in [12]. As a continuation of paper [12], relations between a fuzzy subalgebras with thresholds and an $(\in, \in \vee q)$-fuzzy subalgebras were discussed in [11].

The aim of this paper is to introduce the combined notion of "belongingness" and "quasi-coincidence" of a fuzzy interval value with an interval-valued fuzzy set. By using this new idea, we introduce the notion of an interval-valued $(\in, \in \vee q)$ -fuzzy subgroups and investigate some of their related properties. Also we characterize interval-valued $(\in, \in \vee q)$-fuzzy subgroups with thresholds by using their level subgroups. We also define a T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroups, the direct product and T_G-product of an interval-valued $(\in, \in \vee q)$-fuzzy subgroups are discussed. Finally, we define a T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroups with thresholds.

2. Preliminaries

In this section, we recall some basic definitions for the sake of completeness.

Definition 2.1 ([23])**.** Let G be a non-empty set. A fuzzy subset μ on G is defined by $\mu : G \to [0, 1]$, for all $x \in G$.

Definition 2.2 ([19])**.** Let μ be a fuzzy set in a group G. Then μ is called a fuzzy subgroup of G if

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$
(ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$

An interval number on $[0, 1]$, say $\hat{a}$, is a closed subinterval of $[0, 1]$, that is, $\hat{a} = [a^L, a^U]$ where $0 \leq a^L \leq a^U \leq 1$. Let $D[0, 1]$ denotes the family of all closed sub-intervals of $[0, 1]$, $\hat{0} = [0, 0]$ and $\hat{1} = [1, 1]$. Now we define " $\leq$ ", " $\geq$ ", " $=$ ", "rmin"; "rmax" in case of two elements in $D[0, 1]$. Consider two elements $\hat{a} = [a^L, a^U]$ and $\hat{b} = [b^L, b^U]$ in $D[0, 1]$. Then

i) $\hat{a} \leq \hat{b}$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$.
ii) $\hat{a} \geq \hat{b}$ if and only if $a^L \geq b^L$ and $a^U \geq b^U$.
iii) $\hat{a} = \hat{b}$ if and only if $a^L = b^L$ and $a^U = b^U$.
iv) $\mathrm{rmin}\{\hat{a}, \hat{b}\} = [\min\{a^L, b^L\}, \min\{a^U, b^U\}]$.
v) $\mathrm{rmax}\{\hat{a}, \hat{b}\} = [\max\{a^L, b^L\}, \max\{a^U, b^U\}]$.

Let G be a set. An interval-valued fuzzy set F defined on G is given by

$$F = \left\{ \left(x, \left[\mu_F^L(x), \mu_F^U(x)\right]\right) \right\}, \forall x \in G.$$

Briefly denote F by $F = \left[\mu_F^L, \mu_F^U\right]$ where μ_F^L and μ_F^U are any two fuzzy sets in G such that $\mu_F^L(x) \leq \mu_F^U(x)$ for all $x \in G$.

First we can extend the concept of fuzzy subgroup to the concept of interval-valued fuzzy subgroup of G as follows:

Definition 2.3 ([14]). An interval-valued fuzzy set F in G is called an interval-valued fuzzy subgroup of G if

(i) $\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$ for all $x, y \in G$
(ii) $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x)$ for all $x \in G$.

Definition 2.4 ([10]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if for every $x, y, z \in [0,1]$, it satisfies the following conditions:

(i) $T(x, 1) = x$,
(ii) $T(x, y) = T(y, x)$,
(iii) $T(T(x, y), z) = T(x, T(y, z))$,
(iv) $T(x, y) \leq T(x, z)$, if $y \leq z$.

Let T be a t-norm, if for all arbitrary $x \in [0,1]$, it satisfies $T(x, x) = x$, then T is called an idempotent t-norm.

Definition 2.5 ([10]). Let T be an idempotent t-norm. Define the mapping $T_G : D[0,1] \times D[0,1] \to D[0,1]$ by $\left(\hat{a}, \hat{b}\right) \to T_G\left(\hat{a}, \hat{b}\right) = \left[T\left(a^L, b^L\right), T\left(a^U, b^U\right)\right]$, then T_G is called an idempotent interval t-norm.

Definition 2.6 ([10]). Let G be a group and T_G be an idempotent interval t-norm. An interval-valued fuzzy set F in G is called an T_G - interval-valued - fuzzy subgroup of G if the following conditions hold:

(i) $\hat{\mu}_F(xy) \geq T_G\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$ for all $x, y \in G$
(ii) $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x)$ for all $x \in G$.

Definition 2.7 ([10]). A mapping $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if for every $x, y, z \in [0,1]$, it satisfies the following conditions:

(i) $S(x, 0) = x$,
(ii) $S(x, y) = S(y, x)$,
(iii) $S(S(x, y), z) = S(x, S(y, z))$,
(iv) $S(x, y) \leq S(x, z)$, if $y \leq z$.

Let S be a t-conorm, if for arbitrary $x \in [0,1]$, it satisfies $S(x, x) = x$, then S is called an idempotent t-conorm.

Definition 2.8 ([10]). Let S be an idempotent t-conorm. Define the mapping $S_G : D[0,1] \times D[0,1] \to D[0,1]$ by $\left(\hat{a}, \hat{b}\right) \to S_G\left(\hat{a}, \hat{b}\right) = \left[S\left(a^L, b^L\right), S\left(a^U, b^U\right)\right]$, then S_G is called an idempotent interval t-conorm.

Definition 2.9 ([10]). Let G be a group and S_G be an idempotent interval t-conorm. An interval-valued fuzzy set F in G is called an S_G - interval-valued - fuzzy subgroup of G if the following conditions hold:

(i) $\hat{\mu}_F(xy) \leq S_G\left\{\hat{\mu}_F(x), \hat{\mu}_F(y)\right\}$ for all $x, y \in G$
(ii) $\hat{\mu}_F(x^{-1}) \leq \hat{\mu}_F(x)$ for all $x \in G$.

Definition 2.10 ([19])**.** A fuzzy relation F on any set G is a fuzzy subset F with a membership function $\Omega_F : G \times G \to [0,1]$.

Definition 2.11 ([10])**.** Let $F_1 = \left[\mu^L_{F_1},\ v^U_{F_1}\right]$ and $F_2 = \left[\mu^L_{F_2},\ v^U_{F_2}\right]$ be any two interval-valued fuzzy subgroup in G. The cartesian product of F_1 and F_2 is defined by $F_1 \times F_2 = \left\{((x,y), \hat{\mu}_{F_1} \times \hat{\mu}_{F_2})\,;\ \forall (x,y) \in G \times G\right\}$, where $\hat{\mu}_{F_1} \times \hat{\mu}_{F_2} : G \times G \to D[0,1]$.

Definition 2.12 ([10])**.** Let $\hat{\mu}_B$ be an interval-valued membership function of each element $x \in G$ to the set B. Then the strongest interval-valued fuzzy relation on G, that is a fuzzy relation $\hat{\mu}_A$ on $\hat{\mu}_B$(denoted by $\hat{\mu}_{A_B}$) whose interval-valued membership function, of each element $(x,y) \in G \times G$ and is defined by

$$\hat{\mu}_{A_B}(x,y) = r\min\left\{\hat{\mu}_B(x), \hat{\mu}_B(y)\right\}.$$

Definition 2.13 ([10])**.** Let $B = \left[\mu^L_B, \mu^U_B\right]$ be an interval-valued set in G. Then the strongest interval-valued fuzzy relation on G, that is an interval-valued A on B is defined by $A_B = \left[\mu^L_{A_B}, \mu^U_{A_B}\right]$.

The following example provides clear understanding of definitions 2.12 and 2.13.

Example 2.14. Let G be a set given by: $G = \{1,2,3,4\}$. Let $\mu_B : G \to D[0,1]$ be defined by $\mu_B(1) = [0.1, 0.4]$, $\mu_B(2) = [0.2, 0.4]$, $\mu_B(3) = \mu_B(4) = [0.3, 0.5]$. and $B = \{[0.1, 0.4], [0.2, 0.4], [0.3, 0.5]\}$. Now, a strongest fuzzy relation μ_A on μ_B is

$$\mu_{AB} : G \times G \to B$$

defined by

$$\begin{aligned}
\mu_{AB}(1,2) &= \text{rmin}\{\mu_B(1), \mu_B(2)\} \\
&= \text{rmin}\{[0.1, 0.4], [0.2, 0.4]\} \\
&= \text{rmin}\{\min\{0.1, 0.2\}, \min\{0.4, 0.4\}\} \\
&= [0.1, 0.4].
\end{aligned}$$

3. Interval-valued $(\in, \in\vee q)$-fuzzy subgroups

Now we introduce the combined notion of "belongingness" and "quasi - coincidence" of a fuzzy interval value with an interval-valued fuzzy set. By using this new idea, we introduce the notion of an interval-valued $(\in, \in\vee q)$ -fuzzy subgroups. An interval-valued fuzzy set $F = \{x, \hat{\mu}_F(x) | x \in G\}$ of the form

$$\hat{\mu}_F(x) = \begin{cases} \hat{t}(\neq [0,0]) & \text{if} \quad x = y, \\ [0,0] & \text{if} \quad x \neq y. \end{cases}$$

is said to be a fuzzy interval value with support x and the interval value $\hat{t}$ and is denoted by $U(x, \hat{t})$. We now say that a fuzzy interval value $U(x, \hat{t})$ belongs to (or *resp.* quasi-coincident with) an interval-valued fuzzy set F, written by $U(x : \hat{t}) \in F$ $(resp.\, U(x : \hat{t})qF)$ if $\hat{\mu}_F(x) \geq \hat{t}\,(resp.\, \hat{\mu}_F(x) + \hat{t} > [1,1])$. If $U(x : \hat{t}) \in F$ or $U(x : \hat{t})qF$, then we write $U(x : \hat{t}) \in \vee q$. If $U(x : \hat{t}) \in F$ and $U(x : \hat{t})qF$ then we write $U(x : \hat{t}) \in \wedge qF$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

In what follows, G is a group unless otherwise specified.

We emphasis that $\hat{\mu}_F(x) = \left[\mu_F^L(x), \mu_F^U(x)\right]$ must satisfy the following properties: $\left[\mu_F^L(x), \mu_F^U(x)\right] < [0.5, 0.5]$ or $[0.5, 0.5] \leq \left[\mu_F^L(x), \mu_F^U(x)\right]$, for all $x \in G$.

Now, we extend the concept of interval-valued fuzzy subgroups to the concept interval-valued $(\in, \in \vee q)$-fuzzy subgroups of G as follows:

Definition 3.1. An interval-valued fuzzy set F of G is said to be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if for all $\hat{t}, \hat{r} \in (0, 1]$ and $x, y \in G$:
(IVFSG1) $U(x : \hat{t}) \in F$ and $U(y : \hat{t}) \in F$ imply $U\left(xy : r\min\left\{\hat{t}, \hat{r}\right\}\right) \in \vee qF$,
(IVFSG2) $U(x : \hat{t}) \in F$, imply that $U\left(x^{-1} : \hat{t}\right) \in \vee qF$.

Example 3.2. Let $G = \{0, 1, 2, 3\}$ be a set with the following table :

•	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then G is a group. Let F be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_F(x) = \begin{cases} [0.7, 0.8] & x = 0, \\ [0.3, 0.4] & x \in [1, 3]. \end{cases}$$

It is easy to verify that F is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G.

Example 3.3. Let I be a subgroup of G and let F be an interval-valued fuzzy set in G defined by

$$\hat{\mu}_F(x) = \begin{cases} [0.6, 0.7] & x \in I, \\ [0.2, 0.3] & \text{otherwise}. \end{cases}$$

It is easy to verify that F is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G.

Theorem 3.4. *An interval-valued fuzzy set F of G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if and only if for all $x, y \in G$ the following two conditions are satisfied:*
(IVFSG3) $\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$, *for all $x, y \in G$,*
(IVFSG4) $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x)$ *for all $x \in G$.*

Proof. At first, we prove that the conditions (IVFSG1) and (IVFSG3) are equivalent. Suppose that (IVFSG1) do not implies (IVFSG3) that is , (IVFSG1) holds but (IVFSG3) is not satisfied. In this case there $x, y \in G$ such that $\hat{\mu}_F(xy) < r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$. If $r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} < [0.5, 0.5]$, then $\hat{\mu}_F(xy) < r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$. This means that for some $\hat{t} \in (0, 1]$ satisfying the condition $\hat{\mu}_F(xy) < \hat{t} < r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$, we have $U(x : \hat{t}) \in F$ and $U(y : \hat{t}) \in F$ but $U(xy : \hat{t})\overline{\in \vee q}F$, but, which contradicts to (IVSFG1). So this case is impossible. Thus $r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} \geq [0.5, 0.5]$. In this case $\hat{\mu}_F(xy) < [0.5, 0.5], U(x : [0.5, 0.5]) \in F, U(y : [0.5, 0.5]) \in F$ and $U(xy : [0.5, 0.5])\overline{\in \vee q}F$, which is also impossible. Hence (IVSFG1) implies (IVSFG3).

Conversely, let (IVSFG3)⇒(IVSFG1). Let $U(x : \hat{t}) \in F$ and $U(y : \hat{r}) \in F$, then $\hat{\mu}_F(x) \geq \hat{t}$ and $\hat{\mu}_F(y) \geq \hat{r}$.We have $\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$. That is, $\hat{\mu}_F(xy) \geq r\min\left\{\hat{t}, \hat{r}, [0.5, 0.5]\right\}$. If $r\min\left\{\hat{t}, \hat{r}\right\} > [0.5, 0.5]$, then $\hat{\mu}_F(xy) \geq [0.5, 0.5]$ which implies $\hat{\mu}_F(xy) + r\min\{\hat{t}, \hat{r}\} > [1, 1]$, that is, $U\left(xy : r\min\left\{\hat{t}, \hat{r}\right\}\right) qF$. If $r\min\left\{\hat{t}, \hat{r}\right\} \leq [0.5, 0.5]$, then $\hat{\mu}_F(xy) \geq r\min\left\{\hat{t}, \hat{r}\right\}$. Thus $U\left(xy : r\min\left\{\hat{t}, \hat{r}\right\}\right) \in F$. Therefore, $U\left(xy : r\min\left\{\hat{t}, \hat{r}\right\}\right) \in \vee qF$. So (IVFSG3) ⇒ (IVFSG1) .

Hence (IVFSG1) ⇔ (IVFSG3) are equivalent and obviously (IVFSG2) ⇔ (IVFSG4) are also equivalent. This completes the proof. □

Definition 3.5. Let F be an interval-valued fuzzy set. Then, for every $\hat{t} \in (0, 1]$, the set $F_{\hat{t}} = \left\{x \in G | \hat{\mu}_F(x) \geq \hat{t}\right\}$ is called the level subset of F.

Now, we characterize the interval-valued $(\in, \in \vee q)$-fuzzy subgroups by using their level subgroups.

Theorem 3.6. *An interval-valued fuzzy set F of G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup if and only if for all $[0, 0] < \hat{t} \leq [0.5, 0.5]$, non-empty level subset $F_{\hat{t}}$ are subgroup of G.*

Proof. Let F be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup and $[0, 0] < \hat{t} \leq [0.5, 0.5]$. If $x, y \in F_{\hat{t}}$, then $\hat{\mu}_F(x) \geq \hat{t}$ and $\hat{\mu}_F(y) \geq \hat{t}$. Now we have,

$$\begin{aligned}
\hat{\mu}_F(xy) &\geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} \\
&\geq r\min\left\{\hat{t}, \hat{t}, [0.5, 0.5]\right\} \\
&= \hat{t}.
\end{aligned}$$

Thus $xy \in F_{\hat{t}}$.

Conversely, let F be an interval-valued fuzzy set of G such that all non empty $F_{\hat{t}}$, where $[0, 0] < \hat{t} \leq [0.5, 0.5]$, are subgroup of G.

Define for all $x, y \in G$:

$\hat{t_0} = r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$, for all$[0, 0] < \hat{t_0} \leq [0.5, 0.5]$.

Thus

$$\hat{\mu}_F(x) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} = \hat{t_0}$$

and

$$\hat{\mu}_F(y) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} = \hat{t_0}.$$

This implies that $x, y \in F_{\hat{t_0}}$ and so $xy \in F_{\hat{t_0}}$, that is,

$$\hat{\mu}_F(xy) \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}.$$

Also (IVFSG4) is obvious. Therefore, F is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G. □

Naturally, we can establish a similar result when each non-empty level subset $F_{\hat{t}}$ is a subgroup of G for $[0.5, 0.5] < \hat{t} \leq [1, 1]$.

Theorem 3.7. *For $[0.5, 0.5] < \hat{t} \leq [1, 1]$, each non-empty level subset $F_{\hat{t}}$ of an interval-valued fuzzy set F of G is a subgroup if and only if for all $x, y \in G$ the following two conditions are satisfied:*

(IVFSG5) $r\max\{\hat{\mu}_F\{(xy), [0.5, 0.5]\}\} \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$,

(IVFSG6) $r\max\left\{\hat{\mu}_F(x^{-1}), [0.5, 0.5]\right\} \geq \hat{\mu}_F(x)$.

Proof. Let $F_{\hat{t}}$ be an nonempty level subset of G. Assume that $F_{\hat{t}}$ is a subgroup of G. Define $\hat{t} = r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\}$, for some $x, y \in G$. Then

$$r\max\{\hat{\mu}_F\{(xy), [0.5, 0.5]\}\} < r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} = \hat{t}.$$

Then $[0.5, 0.5] < \hat{t} \leq [1, 1], \hat{\mu}_F(xy) < \hat{t}$ and $x, y \in F_{\hat{t}}$. Thus $xy \in F_{\hat{t}}$ and $\hat{\mu}_F(xy) \geq \hat{t}$, which contradicts to $\hat{\mu}_F(xy) < \hat{t}$. Hence $(IVFSG5)$ is satisfied.

Define $\hat{\mu}_F(x) = \hat{t}$, for some $x \in G$. Then

$$r\max\left\{\hat{\mu}_F\{(x^{-1}), [0.5, 0.5]\}\right\} < \hat{\mu}_F(x) = \hat{t}.$$

Then $[0.5, 0.5] < \hat{t} \leq [1, 1]$, $\hat{\mu}_F\{(x^{-1}) < \hat{t}$, and $x \in F_{\hat{t}}$. We also have $x^{-1} \in F_{\hat{t}}$. Thus $\hat{\mu}_F\{(x^{-1}) \geq \hat{t}$, which is impossible. This implies that $r\max\left\{\hat{\mu}_F\{(x^{-1}), [0.5, 0.5]\}\right\} \geq \hat{\mu}_F(x)$.

Conversely, suppose (IVFSG5) and (IVFSG6) are satisfied. In order to prove that for all $[0.5, 0.5] < \hat{t} \leq [1, 1]$, each nonempty level subset $F_{\hat{t}}$ is a subgroup of G. Assume that $x, y \in F_{\hat{t}}$, for all $x, y \in G$. In this case

$$[0.5, 0.5] < \hat{t} \leq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y)\} \leq r\max\{\hat{\mu}_F\{(xy), [0.5, 0.5]\}\} = \hat{\mu}_F(xy),$$

which proves $xy \in F_{\hat{t}}$. If $x \in F_{\hat{t}}$, then

$$[0.5, 0.5] < \hat{t} \leq \hat{\mu}_F(x) \leq r\max\left\{\hat{\mu}_F\{(x^{-1}), [0.5, 0.5]\}\right\} = \hat{\mu}_F(x^{-1}),$$

and so $x^{-1} \in F_{\hat{t}}$. This completes the proof. □

For an interval-valued fuzzy set F on G and a map $\theta : G \to G$, we define a mapping $\hat{\mu}_F[\theta] : G \to [0, 1]$ by $\hat{\mu}_F[\theta](x) = \hat{\mu}_F(\theta(x))$ for all $x \in G$.

Theorem 3.8. *If F is an interval-valued $(\in, \in \vee q)$-fuzzy set in G and θ is an epimorphism of G, then $\hat{\mu}_F[\theta]$ is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G.*

Proof. Let $x, y \in G$

$$\begin{aligned}
\hat{\mu}_F[\theta](xy) &= \hat{\mu}_F(\theta(xy)) \\
&= \{\mu_F^L(\theta(xy)), \mu_F^U(\theta(xy)), [0.5, 0.5]\} \\
&\geq r\min\left[\min\{\mu_F^L(\theta(x)), \mu_F^L(\theta(y))\}, \min\{\mu_F^U(\theta(x)), \mu_F^U(\theta(y))\}, [0.5, 0.5]\right] \\
&= r\min\left\{\min\left[\mu_F^L(\theta(x)), \mu_F^U(\theta(x))\right], \min[\mu_F^L(\theta(y)), \mu_F^U(\theta(y))], [0.5, 0.5]\right\} \\
&= r\min\{\hat{\mu}_F(\theta(x)), \hat{\mu}_F(\theta(y)), [0.5, 0.5]\} \\
&= r\min\{\hat{\mu}_F[\theta](x), \hat{\mu}_F[\theta](y), [0.5, 0.5]\}
\end{aligned}$$

This completes the proof. □

Let f be a mapping defined on G. If $\hat{\nu}_F$ is an interval-valued fuzzy set in $f(G)$, then the interval-valued fuzzy set is defined by $\hat{\mu}_F(x) = \hat{\nu}_F(f(x))$ is called the preimage of $\hat{\nu}_F$ under f.

Theorem 3.9. *An onto homomorphic preimage of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.*

Proof. Let $f : G \to G'$ be an onto homomorphism of G, $\hat{\nu}_F$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G', and $\hat{\mu}_F$ the preimage of $\hat{\nu}_F$ under f. Let $x, y \in G$, then

$$\begin{aligned}
\hat{\mu}_F(xy) &= \hat{\nu}_F(f(xy)) \\
&= \{\nu_F^L(f(xy)), \nu_F^U(f(xy)), [0.5, 0.5]\} \\
&= \{\nu_F^L(f(x)f(y)), \nu_F^U(f(x)f(y)), [0.5, 0.5]\} \\
&\geq r\min\left\{\min[\nu_F^L(f(x)), \nu_F^L(f(y))], \min[\nu_F^U(f(x)), \nu_F^U(f(y))], [0.5, 0.5]\right] \\
&= r\min\left\{\min\left[\nu_F^L(f(x)), \nu_F^U(f(x))\right], \min[\nu_F^L(f(y)), \nu_F^U(f(y))], [0.5, 0.5]\right\} \\
&= r\min\{\hat{\nu}_F(f(x)), \hat{\nu}_F(f(y)), [0.5, 0.5]\} \\
&= r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}
\end{aligned}$$

This completes the proof. □

If F is an interval-valued fuzzy set in G and f is a mapping defined on G, then the interval-valued fuzzy set $\hat{\mu}_F^f$ in $f(G)$ defined by

$$\hat{\mu}_F^f(x) = \sup_{x \in f^{-1}(y)} \hat{\mu}(x),$$

for all $y \in G$ is called the image of F under f. An interval-valued fuzzy set F in G is said to have sup property, if for every subset $T \subseteq G$, there exists $t_0 \in T$ such that $\hat{\mu}_F(t_0) = \sup_{t \in T} \hat{\mu}(t)$.

Theorem 3.10. *An onto homomorphic image of an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G with sup property is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.*

Proof. Let $f : G \to G'$ be an onto homomorphism of a group and let $\hat{\mu}_F$ be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G with sup property. Moreover, for all $x_1, y_1 \in G'$, let $x \in f^{-1}(x_1), y \in f^{-1}(y_1)$ such that

$$\begin{aligned}
\hat{\mu}_F^f(xy) &= \sup_{t \in f^{-1}(x_1 y_1)} \hat{\mu}(t) \\
\hat{\mu}_F^f(x) &= \sup_{t \in f^{-1}(x_1)} \hat{\mu}(t) \\
\hat{\mu}_F^f(y) &= \sup_{t \in f^{-1}(y_1)} \hat{\mu}(t).
\end{aligned}$$

$$\begin{aligned}
\text{Thus } \hat{\mu}_F^f(xy) &= \sup_{t \in f^{-1}(x_1 y_1)} \hat{\mu}(t) \\
&= \hat{\mu}_F(xy) \\
&\geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\} \\
&= r\min\left\{\sup_{t \in f^{-1}(x_1)} \hat{\mu}(t), \sup_{t \in f^{-1}(y_1)} \hat{\mu}(t), [0.5, 0.5]\right\} \\
&= r\min\left\{\hat{\mu}_F^f(x), \hat{\mu}_F^f(y), [0.5, 0.5]\right\}.
\end{aligned}$$

This completes the proof. □

Lemma 3.11 ([14]). *Let $\hat{\mu}_{F_1}$ and $\hat{\mu}_{F_2}$ be membership function of each $x \in G$ to the interval-valued fuzzy subgroups F_1 and F_2 respectively. Then $\mu_{F_1} \times \mu_{F_2}$ is a membership function of each element $(x, y) \in G \times G$ to the set $F_1 \times F_2$ and defined by*

$$(\hat{\mu}_{F_1} \times \hat{\mu}_{F_2})(x, y) \geq r\min\{\hat{\mu}_{F_1}(x), \hat{\mu}_{F_2}(y)\}.$$

Theorem 3.12. *Let $F_1 = \left[\mu^L_{F_1}, \mu^U_{F_1}\right]$ and $F_2 = \left[\mu^L_{F_2}, \mu^U_{F_2}\right]$ be two interval-valued $(\in, \in \vee q)$-fuzzy subgroups of G, then $F_1 \times F_2$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of $F_1 \times F_2$.*

Proof. Now for all $x, y \in G$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in G \times G$. Then

$$\begin{aligned}
&(\hat{\mu}_{F_1} \times \hat{\mu}_{F_2})((x_1, x_2)(y_1, y_2)) \\
&\quad = (\hat{\mu}_{F_1} \times \hat{\mu}_{F_2})((x_1y_1), (x_2y_2)) \\
&\quad = r\min\{\hat{\mu}_{F_1}(x_1y_1), \hat{\mu}_{F_2}(x_2y_2), [0.5, 0.5]\} \\
&\quad \geq r\min\{r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(y_1)\}, \\
&\qquad r\min\{\hat{\mu}_{F_2}(x_2), \hat{\mu}_{F_2}(y_2)\}, [0.5, 0.5]\} \\
&\quad = r\min\left\{\min[\mu^L_{F_1}(x_1), \mu^L_{F_1}(y_1)], \min[\mu^U_{F_1}(x_1), \mu^U_{F_1}(y_1)],\right. \\
&\qquad \left.\min[\mu^L_{F_2}(x_2), \mu^L_{F_2}(y_2)], \min[\mu^U_{F_2}(x_2), \mu^U_{F_2}(y_2)], [0.5, 0.5]\right\} \\
&\quad = r\min\{\min\{\mu^L_{F_1}(x_1), \mu^L_{F_2}(x_2)\}, \min\{\mu^L_{F_1}(y_1), \mu^L_{F_2}(y_2)\}, \\
&\qquad \min\{\mu^U_{F_2}(x_1), \mu^U_{F_2}(x_2)\}, \min\{\mu^U_{F_2}(y_1), \mu^U_{F_2}(y_2)\}, [0.5, 0.5]\} \\
&\quad = r\min\{(\hat{\mu}_{F_1} \times \hat{\mu}_{F_2})(x_1x_2), (\hat{\mu}_{F_1} \times \hat{\mu}_{F_2})(y_1y_2), [0.5, 0.5]\}.
\end{aligned}$$

Hence $F_1 \times F_2$ is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of $G \times G$. □

Theorem 3.13. *Let $F_1 = \left[\mu^L_{F_1}, \mu^U_{F_2}\right]$ be an interval-valued set in G and F_{F_1} be the strongest interval-valued fuzzy relation on G. Then F_1 is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if and only if F_{F_1} is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of $G \times G$.*

Proof. Let F_1 be an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G. Then for all $(x_1, x_2), (y_1, y_2) \in G \times G$ we have

$$\begin{aligned}
\hat{\mu}_{F_{F_1}}((x_1, x_2)(y_1, y_2)) &= \hat{\mu}_{F_{F_1}}((x_1y_1), (x_2y_2)) \\
&= r\min\{\hat{\mu}_{F_1}(x_1y_1), \hat{\mu}_{F_1}(x_2y_2), [0.5, 0.5]\} \\
&\geq r\min\{r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(y_1), [0.5, 0.5]\}, \\
&\quad r\min\{\hat{\mu}_{F_1}(x_2), \hat{\mu}_{F_1}(y_2), [0.5, 0.5]\}\} \\
&= r\min\{r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(x_2), [0.5, 0.5]\}, \\
&\quad r\min\{\hat{\mu}_{F_1}(y_1), \hat{\mu}_{F_1}(y_2), [0.5, 0.5]\}\} \\
&= r\min\left\{\hat{\mu}_{F_{F_1}}(x_1x_2), \hat{\mu}_{F_{F_1}}(y_1y_2), [0.5, 0.5]\right\}
\end{aligned}$$

Hence F_{F_1} is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of $G \times G$.
Conversely, let F_{F_1} be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of $G \times G$. Now

let $(x_1, x_2), (y_1, y_2) \in G \times G$ then,

$$\begin{aligned}
& r\min\{\hat{\mu}_{F_1}(x_1y_1), \hat{\mu}_{F_1}(x_2y_2)\} \\
&\qquad = \hat{\mu}_{F_{F_1}}((x_1y_1), (x_2y_2)) \\
&\qquad = \hat{\mu}_{F_{F_1}}((x_1x_2)(y_1y_2)) \\
&\qquad \geq r\min\left\{\hat{\mu}_{F_{F_1}}((x_1x_2),)\hat{\mu}_{F_{F_1}}((y_1y_2),), [0.5, 0.5]\right\} \\
&\qquad \geq r\min\{r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(x_2), [0.5, 0.5]\}\}, \\
&\qquad\quad \{r\min\{\hat{\mu}_{F_1}(y_1), \hat{\mu}_{F_1}(y_2), [0.5, 0.5]\}, [0.5, 0.5]\}
\end{aligned}$$

If $x_2 = y_2 = 0$, then

$$\begin{aligned}
r\min\{\hat{\mu}_{F_1}(x_1y_1), \hat{\mu}_{F_1}(0)\} \quad \geq \quad & r\min\left\{r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(0), [0.5, 0.5]\},\right. \\
& \left. r\min\{\hat{\mu}_{F_1}(y_1), \hat{\mu}_{F_1}(0), [0.5, 0.5]\}, [0.5, 0.5]\right\}
\end{aligned}$$

or

$$\hat{\mu}_{F_1}(x_1y_1) \geq r\min\{\hat{\mu}_{F_1}(x_1), \hat{\mu}_{F_1}(y_1), [0.5, 0.5]\}.$$

Hence F_1 is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G. This completes the proof. □

4. Interval-valued $(\in, \in \vee q)$-fuzzy subgroups with thresholds

Let $IG = \{\hat{t} | \hat{t} \in (0, 1]$ and $F_{\hat{t}}$ is a subgroup of $G\}$. When $IG = (0, 1]$, F is an interval-valued fuzzy subgroup of G. When $IG = (0, 0.5]$, F is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. An obvious question is whether F is a kind of interval-valued fuzzy subgroup or not when $IG \neq \varnothing$ (For example $IG = (0.5, 1], (\lambda, \mu], \lambda < \mu$ and $\lambda, \mu \in (0, 1]$) ? Based on the above discussion, we introduce the following concept.

Definition 4.1. Let $[0, 0] \leq \hat{\alpha} \leq \hat{\beta} \leq [1, 1]$. An interval-valued fuzzy set F of G is called an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ if for all $x, y \in G$,the following two conditions are satisfied:
(IVFSG7) $r\max\{\hat{\mu}_F(xy), \hat{\alpha}\} \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\}$,
(IVFSG8) $r\max\{\hat{\mu}_F(x^{-1}), \hat{\alpha}\} \geq r\min\{\hat{\mu}_F(x), \hat{\beta}\}\}$

Example 4.2. Let $G = \{0, 1, 2, 3\}$ be a set with the following table :

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then G is a group.

(i) Let F be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_F(x) = \begin{cases} [0.6, 0.7] & x = 0 \\ [0.8, 0.9] & x = 1 \\ [0.2, 0.3] & x \in [2, 3] \end{cases}.$$

It is easy to verify that F is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha} = [0.4, 0.5]$ and $\hat{\beta} = [0.7, 0.8]$ of G.

(a) F is not an interval-valued $(\in, \in)$ - fuzzy subgroup of G, since $1_{[0.8,0.84]} \in F$ and $1_{[0.86,0.88]} \in F$, but $(1, 1)_{r\min\{[0.8,0.84],[0.86,0.88]\}} = 0_{[0.8,0.84]} \bar{\in} F$.

(b) F is not an interval-valued $(q, \in \vee q)$-fuzzy subgroup of G since $1_{[0.41,0.43]} qF$ and $2_{[0.77,0.86]} qF$. but $(1.2)_{r\min\{[0.4,0.43],[0.77.0.86]\}} = 3_{[0.77,0.86]} \overline{\in \vee q} F$.

(c) F is not an interval-valued $(\in \vee q, \in \vee q)$-fuzzy subgroup of G, since $1_{[0.51,0.55]} \in \vee qF$ and $3_{[0.67,0.77]} \in \vee qF$, but

$$(1.3)_{r\min\{[0.51,0.55],[0.67.0.77]\}} = 2_{[0.51,0.55]} \overline{\in \vee q} F.$$

(ii) Let F_1 be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_{F_1}(x) = \begin{cases} [0.5, 0.6] & x = 0 \\ [0.7, 0.8] & x = 1 \\ [0.3, 0.4] & x = 2 \\ [0.1, 0.2] & x = 3 \end{cases}.$$

Then F_1 is not an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha} = [0.63, 0.69]$ and $\hat{\beta} = [0.81, 0.84]$ of G, since

$$\begin{aligned} r\max\{\hat{\mu}_{F_1}(1.1), [0.63, 0.69]\} &\geq r\min\{\hat{\mu}_{F_1}(1), \hat{\mu}_1(1), [0.81, 0.84]\} \\ r\max\{\hat{\mu}_{F_1}(0), [0.63, 0.69]\} &\geq r\min\{[0.7, 0.8], [0.7, 0.8], [0.81, 0.84]\} \\ r\max\{[0.5, 0.6], [0.63, 0.69]\} &\geq r\min\{[0.7, 0.8], [0.7, 0.8], [0.81, 0.84]\} \end{aligned}$$

As, $[0.63, 0.69] < [0.7, 0.8]$. Moreover F_1 is not an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G because

$$\begin{aligned} \hat{\mu}_{F_1}(1.2) = \hat{\mu}_{F_1}(3) = [0.1, 0.2] &< r\min\{\hat{\mu}_{F_1}(1), \hat{\mu}_{F_1}(2), [0.5, 0.5]\} \\ &< r\min\{[0.7, 0.8], [0.3, 0.4], [0.5, 0.5]\} \\ &< [0.3, 0.4]. \end{aligned}$$

But we know that F_1 is an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup with thresholds $\hat{\alpha} = [0.73, 0.77]$ and $\hat{\beta} = [0.87, 0.89]$ of G.

In what follows let $\hat{\alpha}, \hat{\beta} \in [0, 1]$ and $\hat{\alpha} < \hat{\beta}$ unless otherwise specified.

Observations.

(i) An interval-valued fuzzy subgroup (*resp.*, interval-valued $(\in, \in \vee q)$-fuzzy subgroup) is an interval-valued fuzzy subgroup with thresholds(*resp.*, interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds).

(ii) Every interval-valued fuzzy subgroup with thresholds $\hat{\alpha} = [0, 0]$ and $\hat{\beta} = [1, 1]$ is an interval-valued fuzzy subgroup.

(iii) Every interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha} = [0, 0]$ and $\hat{\beta} = [0.5, 0.5]$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup.

(iv) Every interval-valued $(\in, \in \vee q)$-fuzzy subgroup F of G with thresholds $\hat{\alpha} < \hat{\mu}_F(x) \leq \hat{\beta}$ or $\hat{\alpha} \leq \hat{\mu}_F(x) < \hat{\beta}$ for all $x \in G$ is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

(v) If F is an interval-valued fuzzy set in G and $\hat{\alpha} \geq \hat{\mu}_F(x)$ for all $x \in G$, then F is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha}$ and $\hat{\beta}$ of G.

(vi) Let F be an interval-valued fuzzy set in G and let $\hat{\mu}_F(0) \leq \hat{\alpha} < \hat{\mu}_F(x) \leq \hat{\beta}$, $\hat{\mu}_F(0) \leq \hat{\alpha} < \hat{\beta} \leq \hat{\mu}_F(x)$, $\hat{\alpha} < \hat{\mu}_F(0) < \hat{\beta} \leq \hat{\mu}_F(x)$, or $\hat{\alpha} < \hat{\mu}_F(0) < \hat{\mu}_F(x) \leq \hat{\beta}$ for some $x(\neq 0) \in G$. Then F cannot be an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha}$ and $\hat{\beta}$ of G.

Theorem 4.3. *An interval-valued fuzzy set F of G is an interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ if and only if each non empty level subset $F_{\hat{t}}$, where $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ is a subgroup of G.*

Proof. Let F be an interval-valued $(\in, \in \vee q)$ - fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ of G, where $\hat{\alpha} < \hat{t} \leq \hat{\beta}$. If $x, y \in F_{\hat{t}}$ then $\hat{\mu}_F(x) \geq \hat{t}$ and $\hat{\mu}_F(y) \geq \hat{t}$. Now we have,

$$\begin{aligned} r\max\{\hat{\mu}_F(xy), \hat{\alpha}\} &\geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\} \\ &\geq r\min\{\hat{t}, \hat{t}, \hat{\beta}\} \geq \hat{t} \end{aligned}$$

and $xy \in F_{\hat{t}}$. If $x \in F_{\hat{t}}$ then $\hat{\mu}_F(x) \geq \hat{t}$ and so

$$\begin{aligned} r\max\{\hat{\mu}_F(x^{-1}), \hat{\alpha}\} &\geq r\min\{\hat{\mu}_F(x), \hat{\beta}\} \\ &\geq r\min\{\hat{t}, \hat{\beta}\} > \hat{t} \end{aligned}$$

So $\hat{\mu}_F(x^{-1}) \geq \hat{t}$ and $x^{-1} \in F_{\hat{t}}$. Hence $F_{\hat{t}}$ is a subgroup of G. Conversely, let $F_{\hat{t}}$ be a subgroup of G for all $\hat{\alpha} < \hat{t} \leq \hat{\beta}$. Define $\hat{t} = r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\}$, for all $x, y \in G$ and $(\hat{\alpha}, \hat{\beta})$ are thresholds of G. Suppose $r\max\{\hat{\mu}_F(xy), \hat{\alpha}\} < \hat{t} = r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\}$. Then $x \in F_{\hat{t}}$, $y \in F_{\hat{t}}$, $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ and $\hat{\mu}_F(xy) < \hat{t}$. Since $F_{\hat{t}}$ is a subgroup of G so $xy \in F_{\hat{t}}$ and $\hat{\mu}_F(xy) \geq \hat{t}$. This is a contradiction with $\hat{\mu}_F(xy) < \hat{t}$. Therefore, $r\max\{\hat{\mu}_F(xy), \hat{\alpha}\} \geq r\min\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\}$ for all $x, y \in G$. Define $\hat{t} = r\min\{\hat{\mu}_F(x), \hat{\beta}\}$, $x \in G$. Suppose $r\max\{\hat{\mu}_F(x^{-1}), \hat{\alpha}\} < \hat{t} = r\min\{\hat{\mu}_F(x), \hat{\beta}\}$. Then $x \in F_{\hat{t}}$, $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ and $\hat{\mu}_F(x^{-1}) < \hat{t}$. Since $F_{\hat{t}}$ is a subgroup of G so $x^{-1} \in F_{\hat{t}}$ and $\hat{\mu}_F(x^{-1}) \geq \hat{t}$. This is a contradiction with $\hat{\mu}_F(x^{-1}) < \hat{t}$. Hence $r\max\{\hat{\mu}_F(x^{-1}), \hat{\alpha}\} \geq r\min\{\hat{[\mu]}_F(x), \hat{\beta}\}$. This completes the proof. □

The following example shows that there is an interval-valued fuzzy subgroup with some thresholds which is neither an interval-valued fuzzy subgroup nor an interval-valued $(\in, \in \vee q)$-fuzzy subgroup.

Example 4.4. Let G be a group of integers. Then $(G, .)$ is a subgroup where the operation $\cdot$ is defined as $x.y = x - y$. Let F be a fuzzy set in G defined by

$$\hat{\mu}_F(x) = \begin{cases} [0, 0.1] & \text{if } x \in \{2k + 1/k \in G, k < 0\} \\ [0.2, 0.3] & \text{if } x \in \{2k - 1/k \in G, k > 0\} \\ [0.4, 0.5] & \text{if } x \in \{2k/k \in G\}/\{4k/k \in G\} \\ [0.6, 0.7] & \text{if } x \in \{4k/k \in G\}/\{8k/k \in G\} \\ [0.8, 0.9] & \text{if } x \in \{8k/k \in G, k < 0\} \\ [0.9, 1] & \text{if } x \in \{8k/k \in G, k \geq 0\} \end{cases}.$$

Then F is an interval-valued fuzzy subgroup with thresholds $\hat{\alpha} = [0.2, 0.4]$ and $\hat{\beta} = [0.7, 0.8]$ of G. But F is neither an interval-valued fuzzy subgroup nor an interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

5. T_G - Interval-valued $(\in, \in \vee q)$ -fuzzy subgroups

Definition 5.1. Let G be a group and T_G be an idempotent interval t-norm. An interval-valued fuzzy set F of G is said to be an T_G - interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G if the following conditions hold:

(IVFSG9) $U(x : \hat{t}) \in F$ and $U(y : \hat{t}) \in F$ imply $U\left(xy : T_G\{\hat{t}, \hat{r}\}\right) \in \vee qF$,

(IVFSG10) $U(x : \hat{t}) \in F$ imply that $U\left(x^{-1} : \hat{t}\right) \in \vee qF$.

for all $\hat{t}, \hat{r} \in (0, 1]$ and for all $x, y \in G$.

Theorem 5.2. *Let G be a group and T_G be an idempotent interval t-norm. An interval-valued fuzzy set F of G is said to be a T_G - interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G if the following conditions hold:*

(IVFSG11) $\hat{\mu}_F(xy) \geq T_G\{\hat{\mu}_F(x), \hat{\mu}_F(y), [0.5, 0.5]\}$,

(IVFSG12) $\hat{\mu}_F(x^{-1}) \geq \hat{\mu}_F(x)$.

for all $\hat{t}, \hat{r} \in (0, 1]$ and for all $x, y \in G$.

Proof. It is easy to show that

$$(\text{IVFSG9}) \leftrightarrow (\text{IVFSG11})$$
$$(\text{IVFSG10}) \leftrightarrow (\text{IVFSG12})$$

□

Example 5.3. Let $G = \{0, 1, 2, 3\}$ be a set with the following table:

•	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then G is a group. Let F be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_F(x) = \begin{cases} [0.6, 0.7] & x = 0 \\ [0.3, 0.4] & x \in [1, 3] \end{cases}$$

and $T_G : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_G(\hat{\alpha}, \hat{\beta}) = \max\{\hat{\alpha} + \hat{\beta} - [0.5, 0.5], [0,0]\}$ for all $\hat{\alpha}, \hat{\beta} \in [0,1]$. Then, T_G be an idempotent interval t-norm. It is easy to verify that F is an T_G - interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

Theorem 5.4. *Let G be a group and T_G be an idempotent interval t-norm and $G = G_1 \times G_2$ be a direct product of groups G_1 and G_2. If $\hat{\mu}_1$ (resp., $\hat{\mu}_2$) is an T_G-interval-valued $(\in, \in \vee q)$ - fuzzy subgroup of G_1 (resp., G_2), then $\hat{\mu} = \hat{\mu}_1 \times \hat{\mu}_2$ is an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G defined by*

$$\hat{\mu}((x_1, x_2)(y_1, y_2)) = (\hat{\mu}_1 \times \hat{\mu}_2)((x_1, x_2)(y_1, y_2)) = T_G\left(\hat{\mu}_1(x_1 x_2), \hat{\mu}_2(y_1 y_2), [0.5, 0.5]\right),$$

$\forall (x_1, x_2)(y_1, y_2) \in G_1 \times G_2$.

Proof. Let $(x_1, x_2), (y_1, y_2) \in G_1 \times G_2$

$$\begin{aligned}
& \hat{\mu}((x_1, x_2)(y_1, y_2)) \\
= \; & (\hat{\mu}_1 \times \hat{\mu}_2)\left((x_1, x_2)(y_1, y_2)\right) \\
= \; & (\hat{\mu}_1 \times \hat{\mu}_2)\left((x_1 y_1)(x_2 y_2)\right) \\
= \; & T_G\{\hat{\mu}_1(x_1 y_1), \hat{\mu}_2(x_2 y_2), [0.5, 0.5]\} \\
\geq \; & T_G\{T_G\{\hat{\mu}_1(x_1), \hat{\mu}_1(y_1), [0.5, 0.5]\}, T_G\{\hat{\mu}_2(x_2), \hat{\mu}_2(y_2), [0.5, 0.5]\}, [0.5, 0.5]\} \\
= \; & T_G\{T_G\{\hat{\mu}_1(x_1), \hat{\mu}_2(x_2), [0.5, 0.5]\}, T_G\{\hat{\mu}_1(y_1), \hat{\mu}_2(y_2), [0.5, 0.5]\}, [0.5, 0.5]\} \\
= \; & T_G\{(\hat{\mu}_1 \times \hat{\mu}_2)(x_1 x_2), (\hat{\mu}_1 \times \hat{\mu}_2)(y_1 y_2), [0.5, 0.5]\}.
\end{aligned}$$

This completes the proof. □

Now, We will generalize the idea of T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup over $n > 0$. We first need to generalize the domain of t-norm T_G to $\prod_{i=1}^{n}[0,1]$ as follows. The function $T_{G_n} : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by $T_{G_n}(\alpha_1, \alpha_2 \ldots, \alpha_n) = T_G(\alpha_i, T_{G_{n-1}}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))$ for all $1 \leq i \leq n$, where $n \geq 2$, $T_{G_2} = T_G$ and T_1 =id(identity). For a t-norm T_G and every $\alpha_i, \beta_i \in [0,1]$, where $1 \leq i \leq n$ and $n \geq 2$, we have

$$\begin{aligned}
& T_{G_n}(T_G(\alpha_1, \beta_1), T_G(\alpha_2, \beta_2), \ldots, T_G(\alpha_n, \beta_n)) \\
& \qquad = T_G(T_{G_n}(\alpha_1, \alpha_2, \ldots, \alpha_n), T_{G_n}(\beta_1, \beta_2, \ldots, \beta_n)).
\end{aligned}$$

Theorem 5.5. *Let T_G be an idempotent interval t-norm , $\{G_i\}_{i=1}^{n}$ the finite collection of subgroups of G, and $G = \prod_{i=1}^{n} G_i$ is the direct product of subgroup over $\{G_i\}$. Let $\hat{\mu}_i$ be an T_G-interval-valued $(\in, \in \vee q)$ -fuzzy subgroup of $\{G_i\}$, where $1 \leq i \leq n$. Then $\hat{\mu} = \prod_{i=1}^{n} \hat{\mu}_i$ defined by*

$$\begin{aligned}
\hat{\mu}((x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)) &= \left(\prod_{i=1}^{n} \hat{\mu}_i\right)(x_1 y_1, x_2 y_2, \ldots, x_n y_n) \\
&= T_{G_n}(\mu_1(x_1 y_1), \mu_2(x_2 y_2), \ldots, \mu_n(x_n y_n)
\end{aligned}$$

is an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

Proof. The proof goes by induction. Let $n = 2$. By Theorem 5.4 the result holds good. Let us assume that the result is true for $n = r - 1$. That is

$$\hat{\mu}((x_1, x_2, \ldots, x_{r-1})(y_1, y_2, \ldots, y_{r-1})) = \left(\prod_{i=1}^{n} \hat{\mu}_i\right)(x_1y_1, x_2y_2, \ldots, x_ny_n)$$
$$= T_n(\mu_1(x_1y_1), \mu_2(x_2y_2), \ldots, \mu_{r-1}(x_{r-1}y_{r-1}).$$

Let us prove the result for $n = r$.

$$\begin{aligned}
\hat{\mu}((x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)) &= \hat{\mu}(x_1y_1, \ldots, x_ry_r)\\
&= \left(\prod_{i=1}^{r-1} \hat{\mu}_i(x_iy_i)\right) \hat{\mu}_r(x_ry_r)\\
&= \left(\prod_{i=1}^{r-1} \hat{\mu}_i(x_iy_i) \times \hat{\mu}_r(x_ry_r)\right)\\
&\geq T_G\left(\prod_{i=1}^{r-1} \hat{\mu}_i(x_iy_i), \hat{\mu}_r(x_ry_r), [0.5, 0.5]\right)\\
&= T_G\Bigg\{T_G\left(\prod_{i=1}^{r-1} \hat{\mu}_i(x_i), \prod_{i=1}^{r-1} \hat{\mu}_i(y_i), [0.5, 0.5]\right),\\
&\qquad T_G\left(\hat{\mu}_r(x_r), \hat{\mu}_r(y_r), [0.5, 0.5]\right)\Bigg\}\\
&= T_G\Bigg\{T_G\left(\prod_{i=1}^{r-1} \hat{\mu}_i(x_i), \hat{\mu}_r(x_r)[0.5, 0.5]\right),\\
&\qquad T_G\left(\prod_{i=1}^{r-1} \hat{\mu}_i(y_i), \hat{\mu}_r(y_r), [0.5, 0.5]\right)\Bigg\}\\
&= T_G\left\{\prod_{i=1}^{r-1} (\hat{\mu}_i \times \hat{\mu}_r)(x_ix_r), \prod_{i=1}^{r-1} (\hat{\mu}_i \times \hat{\mu}_r)(y_iy_r), [0.5, 0.5]\right\}\\
&= T_G\left\{\prod_{i=1}^{r} \hat{\mu}_i(x_i), \prod_{i=1}^{r} \hat{\mu}_i(y_i), [0.5, 0.5]\right\}.
\end{aligned}$$

This completes the proof. □

Definition 5.6. Let T_G be an idempotent t-norm and let $\hat{\mu}$ and $\hat{\nu}$ be interval-valued fuzzy sets in G. Then the T_G-product of $\hat{\mu}$ and $\hat{\nu}$, written as $[\hat{\mu}, \hat{\nu}]_{T_G}$, is defined by

$$[\hat{\mu}, \hat{\nu}]_{T_G}(xy) = T_G\left(\hat{\mu}(x), \hat{\nu}(x), [0.5, 0.5]\right), \text{ for all } x \in G.$$

Theorem 5.7. *Let T_G be an idempotent t -norm and let $\hat{\mu}$ and $\hat{\nu}$ be an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G. If T_G^* be an idempotent t-norm which dominates T_G, that is,*

$$T_G^*(T(\hat{\alpha}, \hat{\beta}), T(\hat{\nu}, \hat{\delta}), [0.5, 0.5]) \geq T(T_G^*(\hat{\alpha}, \hat{\nu}), T_G^*(\hat{\beta}, \hat{\delta}), [0.5, 0.5]),$$

for all $\hat{\alpha}, \hat{\beta}, \hat{\nu}, \hat{\delta} \in [0, 1]$, then the T^*_G-product of $\hat{\mu}$ and $\hat{\nu}$, $[\hat{\mu}, \hat{\nu}]_{T^*_G}$ is an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup of G.

Proof. For all $x, y \in G$, then

$$\begin{aligned}
[\hat{\mu}, \hat{\nu}]_{T^*_G}(xy) &= T^*_G\left(\hat{\mu}(xy), \hat{\nu}(xy), [0.5, 0.5]\right) \\
&\geq T^*_G\left(T_G\left(\hat{\mu}(x), \hat{\mu}(y), [0.5, 0.5]\right), T_G\left(\hat{\nu}(x), \hat{\nu}(y), [0.5, 0.5]\right)\right) \\
&= T_G\left(T^*_G\left(\hat{\mu}(x), \hat{\nu}(x), [0.5, 0.5]\right), T^*_G\left(\hat{\mu}(y), \hat{\nu}(y), [0.5, 0.5]\right)\right) \\
&= T_G([\hat{\mu}, \hat{\nu}]_{T^*_G}(x), [\hat{\mu}, \hat{\nu}]_{T^*_G}(y), [0.5, 0.5])
\end{aligned}$$

This completes the proof. □

We now characterize the T_G-interval-valued $(\in, \in \vee q)$ - fuzzy subgroup with thresholds.

Definition 5.8. Let $[0, 0] \leq \hat{\alpha} \leq \hat{\beta} \leq [1, 1]$. An interval-valued fuzzy set F of G where T_G is an idempotent interval t-norm and S_G is an idempotent interval t-conorm is called an T_G-interval-valued $(\in, \in \vee q)$ -fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ if for all $x, y \in G$, the following two conditions are satisfied:

(IVFSG13) $T_G\left\{\hat{\mu}_F(xy), \hat{\alpha}\right\} \geq S_G\left\{\hat{\mu}_F(x), \hat{\mu}_F(y), \hat{\beta}\right\}$,

(IVFSG14) $T_G\left\{\hat{\mu}_F(x^{-1}), \hat{\alpha}\right\} \geq S_G\left\{\hat{\mu}_F(x), \hat{\beta}\right\}$.

Example 5.9. Let $G = \{0, 1, 2, 3\}$ be a set with the following table:

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then G is a group. Let F be an interval-valued fuzzy set in G defined as

$$\hat{\mu}_F(x) = \begin{cases} [0.6, 0.7] & x = 0 \\ [0.3, 0.4] & x \in [1, 2, 3] \end{cases}$$

and $T_G : [0, 1] \times [0, 1] \to [0, 1]$, and $S_G : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $T_G(\hat{\alpha}, \hat{\beta}) = \text{rmax}\left\{\hat{\alpha} + \hat{\beta} - [0.5, 0.5], [0, 0]\right\}$, $S_G(\hat{\alpha}, \hat{\beta}) = \text{rmin}\{\hat{\alpha} + \hat{\beta}, [0, 0]\}$ for all $\hat{\alpha}, \hat{\beta} \in [0, 1]$. Then, T_G is an idempotent interval t-norm and S_G is an idempotent interval t-conorm. It is easy to verify that F is an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $\hat{\alpha} = [0.4, 0.5]$ and $\hat{\beta} = [0.81, 0.88]$ of G.

Theorem 5.10. *An interval-valued fuzzy set F of G, where T_G is an idempotent interval t-norm and S_G is an idempotent interval t-conorm is an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ if and only if each non empty $F_{\hat{t}}$, where $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ is a subgroup of G.*

Proof. Let F be an T_G-interval-valued $(\in, \in \vee q)$-fuzzy subgroup with thresholds $(\hat{\alpha}, \hat{\beta})$ of G, where $\hat{\alpha} < \hat{t} \leq \hat{\beta}$. If $x, y \in F_{\hat{t}}$ then $\hat{\mu}_F(x) \geq \hat{t}$ and $\hat{\mu}_F(y) \geq \hat{t}$. Now we have,

$$\begin{aligned} T_G\{\hat{\mu}_F(xy),\hat{\alpha}\} &\geq S_G\left\{\hat{\mu}_F(x),\hat{\mu}_F(y),\hat{\beta}\right\} \\ &\geq S_G\left\{\hat{t},\hat{t},\hat{\beta}\right\} \geq \hat{t}. \end{aligned}$$

Thus $xy \in F_{\hat{t}}$. If $x \in F_{\hat{t}}$ then $\hat{\mu}_F(x) \geq \hat{t}$ and so

$$\begin{aligned} T_G\left\{\hat{\mu}_F(x^{-1}),\hat{\alpha}\right\} &\geq S_G\left\{\hat{\mu}_F(x),\hat{\beta}\right\} \\ &\geq S_G\left\{\hat{t},\hat{\beta}\right\} > \hat{t} \end{aligned}$$

So $\hat{\mu}_F(x^{-1}) \geq \hat{t}$ and $x^{-1} \in F_{\hat{t}}$. Therefore $F_{\hat{t}}$ is a subgroup of G. Conversely, let $F_{\hat{t}}$ be a subgroup of G for all $\hat{\alpha} < \hat{t} \leq \hat{\beta}$. Assume that

$$T_G\{\hat{\mu}_F(xy),\hat{\alpha}\} < S_G\left\{\hat{\mu}_F(x),\hat{\mu}_F(y),\hat{\beta}\right\},$$

for all $x, y \in G$ where $(\hat{\alpha},\hat{\beta})$ are thresholds of G. Then

$$T_G\{\hat{\mu}_F(xy),\hat{\alpha}\} < \hat{t} = S_G\left\{\hat{\mu}_F(x),\hat{\mu}_F(y),\hat{\beta}\right\}.$$

Then $x \in F_{\hat{t}}$, $y \in F_{\hat{t}}$, $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ and $\hat{\mu}_F(xy) < \hat{t}$. Since $F_{\hat{t}}$ is a subgroup of G so $xy \in F_{\hat{t}}$ and $\hat{\mu}_F(xy) \geq \hat{t}$. This is a contradiction with $\hat{\mu}_F(xy) < \hat{t}$. Hence, $T_G\{\hat{\mu}_F(xy),\hat{\alpha}\} \geq S_G\left\{\hat{\mu}_F(x),\hat{\mu}_F(y),\hat{\beta}\right\}$. Assume that $T_G\left\{\hat{\mu}_F(x^{-1}),\hat{\alpha}\right\} < S_G\left\{\hat{\mu}_F(x),\hat{\beta}\right\}$, for all $x \in G$. Then $x \in F_{\hat{t}}$, $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ and $\hat{\mu}_F(x^{-1}) < \hat{t}$. Since $F_{\hat{t}}$ is a subgroup of G so $x^{-1} \in F_{\hat{t}}$ and $\hat{\mu}_F(x^{-1}) < \hat{t}$. This is a contradiction with $\hat{\mu}_F(x^{-1}) < \hat{t}$. Hence $T_G\left\{\hat{\mu}_F(x^{-1},\hat{\alpha}\right\} \geq S_G\left\{\hat{\mu}_F(x),\hat{\beta}\right\}$. This completes the proof. □

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References

[1] M. T. Abu Osman, On some product of fuzzy subgroups, Fuzzy Sets and Systems 24 (1987) 79–86.
[2] M. Akgul, Some properties of fuzzy subgroups, J. Math. Anal. Appl. 133 (1988) 93–100.
[3] J. M. Anthony and H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets and Systems 7 (1982) 297–305.
[4] J. M. Anthony and H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl. 69 (1979) 124–130.
[5] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems 80 (1996) 359–368.
[6] R. Biswas, Rosenfeld's fuzzy subgroups with interval-valued membership functions, Fuzzy Sets and Systems 63 (1994) 87–90.
[7] P. S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1981) 264–269.
[8] B. Davvaz and P. Corsini, Redefined fuzzy H_v-submodules and many valued implications, Inform. Sci. 177 (2007) 865–875.
[9] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnear-rings and ideals, Soft Computing 10 (2006) 206–211.

[10] B. Davvaz, T_H and S_H-interval-valued fuzzy H_v-subgroups, Indian J. Pure Appl. Math. 35 (2004) 61–69.
[11] Y. B. Jun, Fuzzy subalgebras with thresholds in BCK/BCI-algrbras, Commun. Korean Math. Soc. 22 (2007) 173–181.
[12] Y. B. Jun, On (α, β) fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc. 42 (2005) 703–711.
[13] O. Kazanci and S. Yamak, Generalized fuzzy bi-ideals of semigroups, Soft Compting 12 (2008) 1119–1124.
[14] A. Kordi, A. Moussavi and A. Ahmadi, Fuzzy H-ideals of BCI-Algebras with intervalued membership functions, Int. Math. Forum 27 (2008) 1327–1388.
[15] X. Ma, J. Zhan and Y. B. Jun, Interval-valued $(\in, \in \vee q)$-fuzzy ideals of pseudo MV-algebras, Int. J. Fuzzy Syst. 10 (2008) 84–91.
[16] N. P. Mukerjee and P. Bhattacharya, Fuzzy groups: some group theoretic analogs, Inform. Sci. 39 (1986) 247–268.
[17] N. P. Mukerjee and P. Bhattacharya, Fuzzy relations and fuzzy groups, Inform. Sci. 36 (1985) 267–282.
[18] P. M. Pu and Y. M. Liu, Fuzzy topology I: Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599
[19] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[20] A. B. Saeid, Some results on interval-valued fuzzy BG-algebras, Proceedings of World Academy of Science, Engineering and Technology 5 (2005) 183–186.
[21] X. Yuan, C. Zhang and Y. Ren, Generalized fuzzy groups and many valued implications, Fuzzy Sets and Systems 138 (2003) 205–211.
[22] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reason I, Inform. Sci. 8 (1975) 199–249.
[23] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[24] W. Zeng and H. Li, Relationship between similarity measure and entropy of interval-valued fuzzy set, Fuzzy Sets and Systems 157 (2006) 1477–1484.
[25] J. Zhan, W. A. Dudek and Y. B. Jun, Interval-valued $(\in, \in \vee q)$-fuzzy filters of pseudo BL-algebras, Int. J. Fuzzy Syst. 10(2) (2008) 84–91.

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