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Generalized (φ, ψ)-weak contractions involving (f, g)-reciprocally continuous maps in fuzzy metric spaces

Penumarthy Parvateesam Murthy, Urmila Mishra, Rashmi, Calogero Vetro

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Abstract. We introduce the notion of (f, g)-reciprocal continuity in fuzzy metric spaces and prove a common fixed point theorem for a pair of sub-compatible maps by employing a generalized (φ, ψ)-weak contraction. As an application of our result, we prove a theorem for a (φ, ψ)-weak cyclic contraction in fuzzy metric spaces.

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Corresponding Author: Calogero Vetro (cvetro@math.unipa.it)

1. Introduction

The concept of fuzzy set was introduced by Zadeh [36] in 1965 and since then an extensive work was done by many researchers in the area of Fuzzy Logic, Artificial Intelligence and Fuzzy Analysis (see, for example, [19] and references therein). Particularly, it is our interest to underline that there are many viewpoints of the notion of metric space in fuzzy topology. More precisely, we are interested in the following two. The first viewpoint focuses on those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to [0, \infty)$, where X represents the totality of all fuzzy points of a set and satisfies some axioms which are analogous to the ordinary metric axioms. In such an approach numerical distances are set up between fuzzy objects. The second one focuses on those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Subsequently, in the last decades, there have appeared a large number of research papers devoted to the development of fixed point theorems and their applications in fuzzy metric spaces (see [8, 18, 22, 33, 34]). In 1988, Grabiec [8] proved an important fixed point theorem in fuzzy metric spaces, that is the fuzzy version of the contraction principle in ordinary metric spaces. Subramanyam [31] generalized Grabiec's result for a pair of commuting maps in the pattern of Jungck [14], or equivalently, we can say that he extended Jungck's theorem for ordinary metric spaces to fuzzy metric spaces. George and Veeramani [9] modified the concept of fuzzy metric spaces [22] and showed that every metric induces a fuzzy metric in Hausdorff topology. For more details, the reader can refer to [5, 35]. In this paper, first we introduce the new notion of (f, g)-reciprocal continuity, and then prove a common fixed point theorem for a pair of sub-compatible maps by employing a generalized (φ, ψ)-weak contraction in fuzzy metric spaces. In order to apply our result, we prove a theorem for (φ, ψ)-weak cyclic contractions.

2. Preliminaries

In this section, we recall useful definitions and give some examples.

Definition 2.1. ([36]) A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 2.2. ([30]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for every $a \in [0, 1]$,
(4) $a * b \le c * d$ if $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. ([9]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y \in X$ and $t, s > 0$:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$, for all $z \in X$,
(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In view of (1) and (2), it is worth pointing out that $0 < M(x, y, t) < 1$ for all $t > 0$, provided $x \ne y$. In view of Definition 2.3, George and Veermani [9] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric space induces a fuzzy metric space. In fact, we can fuzzify metric spaces into fuzzy metric spaces in a natural way, as is shown by the following example. In other words, every metric induces a fuzzy metric.

Example 2.4. Let (X, d) be a metric space and define $a * b = ab$ for all $a, b \in [0, 1]$. Also define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, usually called standard fuzzy metric space induced by (X, d).

For more properties and examples of fuzzy metric spaces, the reader can refer to [3, 4, 13, 18, 22, 24, 27].

Definition 2.5. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $0 < \varepsilon < 1$ and for every $t > 0$, there is $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for every $n, m \geq n_0$.

Definition 2.6. A sequence $\{x_n\}$ in X is said to be a G-Cauchy sequence, that is Cauchy sequence in the sense of Grabiec [8], if $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$, for every $p \in \mathbb{N}$ and for every $t > 0$.

Definition 2.7. A fuzzy metric space $(X, M, *)$ is said to be complete (respectively G-complete) if every Cauchy sequence (respectively G-Cauchy sequence) is convergent.

Vasuki and Veeramani [32] suggested that the definition of G-Cauchy sequence is weaker than the definition of Cauchy sequence.

In 1984, Khan et al. [21] introduced the concept of an altering distance function as follows:

Definition 2.8. A function $\varphi : [0, \infty) \to [0, \infty)$ is an altering distance function if $\varphi(t)$ is monotone non-decreasing, continuous and $\varphi(t) = 0$ iff $t = 0$.

Thus, an altering distance function is a control function used for altering the metric distance between two points and then for dealing with a new class of fixed point problems.

Now we introduce the notion of (f, g)-reciprocal continuity as follows:

Definition 2.9. Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be (f, g)-reciprocally continuous iff $\lim_{n\to\infty} ffx_n = fu$ and $\lim_{n\to\infty} ggx_n = gu$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

In 1994, Mishra et al. [24] defined the concept of compatible maps in fuzzy metric spaces as follows:

Definition 2.10. Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

In 1996 Pathak et al. [28] introduced the concept of compatible maps of type (P) in metric spaces and showed its relationship with compatible maps and compatible maps of type (A) introduced by Jungck [15] and Jungck, Murthy and Cho [16], respectively. According to this notion, we give the following definition:

Definition 2.11. Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be weakly compatible of type (P) iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and $\lim_{n\to\infty} M(ffx_n, ggx_n, t) = 1$, for all $t > 0$.

Observation 2.12. Let f and g be (f, g)-reciprocally continuous maps of a fuzzy metric space $(X, M, *)$. Then f and g have a coincidence point iff they are weakly compatible maps of type (P).

Proof. **(If part)** Suppose that f and g have a coincidence point, say u. If we consider a sequence $\{x_n\}$ in X such that $x_n = u$ for all $n \geq 0$, then we have

$$\lim_{n\to\infty} fx_n = fu = z = gu = \lim_{n\to\infty} gx_n,$$

for some $z \in X$.
Since f and g are (f, g)-reciprocally continuous maps, then $\lim_{n\to\infty} ffx_n = fu$ and $\lim_{n\to\infty} ggx_n = gu$. This implies that $\lim_{n\to\infty} M(ffx_n, ggx_n, t) = M(fu, gu, t) = 1$ and therefore, f and g are weakly compatible maps of type (P).

(Only if part) Now, assume that f and g are weakly compatible maps of type (P). Corresponding to a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and using (4) of Definition 2.3, we can write

$$M(fu, gu, t) \geq M(fu, ffx_n, \frac{t}{3}) * M(ffx_n, ggx_n, \frac{t}{3}) * M(ggx_n, gu, \frac{t}{3}).$$

Taking $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} M(fu, gu, t) \quad &\geq \quad \lim_{n\to\infty} M(fu, ffx_n, \frac{t}{3}) * \lim_{n\to\infty} M(ffx_n, ggx_n, \frac{t}{3}) \\ &\quad * \lim_{n\to\infty} M(ggx_n, gu, \frac{t}{3}),\end{aligned}$$

which implies, since f and g are (f, g)-reciprocally continuous and weakly compatible maps of type (P), that $M(fu, gu, t) = 1$. Therefore, f and g have a coincidence point. □

In 2011, Gopal and Imdad [10] studied the concept of sub-compatible maps in fuzzy metric spaces. Subsequently, Murthy and Tas discussed and utilized it in [25]. We also recall that this concept was initially introduced by Bouhadjera and Thobie [2] in metric spaces to weaken the notion of occasionally weakly compatible maps [1] and weak compatible maps [17]. On this topic, we ask the reader to see [7, 11, 23].

Definition 2.13. Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be sub-compatible iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1,$$

for all $t > 0$.

Now, we give some illustrative examples of sub-compatible and (f, g)-reciprocally continuous maps with coincidence point and without coincidence point.

Example 2.14. Let $X = [0, \infty)$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$fx = \begin{cases} x^2 & \text{if } x \in [0, 1) \\ 2x & \text{if } x \in [1, \infty) \end{cases} \text{ and } gx = \begin{cases} x & \text{if } x \in [0, 1) \\ x + 1 & \text{if } x \in [1, \infty) \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $x_n = 1 - \frac{1}{n+1}$, for all $n \geq 0$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = 1$$

and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore, the maps f and g are sub-compatible.
On the other hand, the maps f and g are not compatible. In fact, let $\{x_n\}$ be a sequence in X such that $x_n = 1 + \frac{1}{n+1}$, for all $n \geq 0$. Then, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2$, $\lim_{n\to\infty} fgx_n = 4$, $\lim_{n\to\infty} gfx_n = 3$ and so

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \frac{t}{t+1} \neq 1.$$

Example 2.15. Let $X = \mathbb{R}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & \text{if } x \in (-\infty, 1) \\ x & \text{if } x \in [1, \infty) \end{cases} \text{ and } gx = \begin{cases} x+1 & \text{if } x \in (-\infty, 1) \\ 2x-1 & \text{if } x \in [1, \infty) \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $x_n = 1 + \frac{1}{n}$, for all $n \geq 1$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = 1$$

and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore, the maps f and g are sub-compatible.
On the other hand, the maps f and g are not compatible. In fact, let $\{x_n\}$ be a sequence in X such that $x_n = \frac{1}{n} - 2$, for all $n \geq 1$. Then, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = -1$, $\lim_{n\to\infty} fgx_n = -\frac{1}{2}$, $\lim_{n\to\infty} gfx_n = 0$ and so

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \frac{t}{t+1/2} \neq 1.$$

It is easy to show that f and g are (f,g)-reciprocally continuous maps with coincidence point $x = 1$ (that is also a common fixed point of the pair (f,g)) and this implies that f and g are weakly compatible maps of type (P).

Example 2.16. Let $X = [0, \infty)$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$fx = \begin{cases} 1+x & \text{if } x \in [0, 1) \\ x & \text{if } x \in [1, \infty) \end{cases} \text{ and } gx = \begin{cases} 1-x & \text{if } x \in [0, 1] \\ 2x-1 & \text{if } x \in (1, \infty) \end{cases}.$$

Notice that f and g are discontinuous. Let $\{x_n\}$ be a sequence in X such that $x_n = 1 + \frac{1}{n+1}$, for all $n \geq 0$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = 1$$

and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore, the maps f and g are sub-compatible.
On the other hand, the maps f and g are not compatible. In fact, let $\{x_n\}$ be a

sequence in X such that $x_n = \frac{1}{2n}$, for all $n \geq 1$. Then, we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1$, $\lim_{n\to\infty} fgx_n = 2$, $\lim_{n\to\infty} gfx_n = 0$ and so

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \frac{t}{t+2} \neq 1.$$

It is easy to show that f and g are (f, g)-reciprocally continuous maps without a coincidence point. Therefore, f and g are not weakly compatible maps of type (P).

3. Main Result

Before proving our main theorem, we introduce the following definition.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space and $f, g : X \to X$ be given maps. The map g is called a generalized (φ, ψ)-weak contraction with respect to f if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi\left(\frac{1}{M(gx, gy, t)} - 1\right) \leq \varphi\left(\frac{1}{m(f, g)} - 1\right) - \psi\left(\frac{1}{m(f, g)} - 1\right) \tag{3.1}$$

holds for all $x, y \in X$ and each $t > 0$ with

$$m(f, g) = \min\{M(fx, fy, t), M(gx, fx, t), M(gy, fy, t)\}.$$

If $f = I_X$, where I_X is the identity map, then g is called a generalized (φ, ψ)-weak contraction.

Theorem 3.2. *Let $f, g : X \to X$ be (f, g)-reciprocally continuous self-maps of a fuzzy metric space $(X, M, *)$ such that*

1. *$g(X) \subseteq f(X)$,*
2. *one of $f(X)$ and $g(X)$ is a G-complete subset of X,*
3. *g is a generalized (φ, ψ)-weak contraction with respect to f.*

If f and g are sub-compatible maps and ψ is a continuous function, then f and g have a unique common fixed point in X, that is, there exists $u \in X$ such that $u = fu = gu$.

Proof. Let x_0 be an arbitrary point in X. Since $g(X) \subseteq f(X)$, we can define, for each $n \geq 0$, a sequence of points $x_0, x_1, x_2, \ldots, x_n, \ldots$, such that x_{n+1} is in the pre-image under f of $\{gx_n\}$, that is, $gx_0 = fx_1, gx_1 = fx_2, \ldots, gx_n = fx_{n+1}, \ldots$. Moreover, we can define a sequence y_n in X by

$$y_n = gx_n = fx_{n+1}, \text{ for all } n. \tag{3.2}$$

Suppose that $y_n = y_{n+1}$ for some n. Then by condition (3.1) we have easily $y_{n+1} = y_{n+2}$ and so $y_m = y_n$ for every $m > n$. Thus the sequence $\{y_n\}$ is Cauchy. Assume that $y_{n+1} \neq y_n$, for all n. Then, for $x = x_{n+1}$ and $y = x_n$, we have

$$\begin{aligned} m(f, g) &= \min\{M(fx_{n+1}, fx_n, t), M(gx_{n+1}, fx_{n+1}, t), M(gx_n, fx_n, t)\} \\ &= \min\{M(y_n, y_{n-1}, t), M(y_{n+1}, y_n, t), M(y_n, y_{n-1}, t)\}. \end{aligned}$$

Now, if $m(f,g) = M(y_{n+1}, y_n, t)$, we obtain

$$\begin{aligned}\varphi\left(\frac{1}{M(gx_{n+1}, gx_n, t)} - 1\right) &= \varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) - \psi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right)\end{aligned}$$

which implies that $M(y_{n+1}, y_n, t) = 1$, a contradiction as $y_{n+1} \neq y_n$ for all n.
Then, we must have $m(f,g) = M(y_n, y_{n-1}, t)$ and hence

$$\begin{aligned}\varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) &\leq \varphi\left(\frac{1}{M(y_{n-1}, y_n, t)} - 1\right) - \psi\left(\frac{1}{M(y_{n-1}, y_n, t)} - 1\right) \\ &< \varphi\left(\frac{1}{M(y_n, y_{n-1}, t)} - 1\right).\end{aligned}$$

Consequently, considering that the φ function is non-decreasing, we have that

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t), \text{ for all } n,$$

and hence $\{M(y_{n-1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $(0,1]$. Let $S(t) = \lim_{n\to\infty} M(y_{n-1}, y_n, t)$. Now, we show that $S(t) = 1$, for all $t > 0$. If not, there exists $t > 0$ such that $S(t) < 1$. Then from the above inequality, on taking $n \to \infty$, we obtain

$$\varphi\left(\frac{1}{S(t)} - 1\right) \leq \varphi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right),$$

that is a contradiction. Therefore $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$.
Now, for each positive integer p, we write

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p}) * \cdots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}).$$

It follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

and hence $\{y_n\}$ is a G-cauchy sequence.
If $f(X)$ is G-complete, then there exists $u \in f(X)$ such that $y_n \to u$ as $n \to \infty$.
Clearly,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1} = u.$$

Now, (f,g)-reciprocal continuity of f and g implies that $ffx_n \to fu$ and $ggx_n \to gu$, as $n \to \infty$. From $ffx_n \to fu$, by construction of the sequence (3.2), we have $fgx_n = ffx_{n+1} \to fu$, as $n \to \infty$. On the other hand, sub-compatibility of f and g yields $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, which implies that $gfx_n \to fu$. Now, taking $x = u$ and $y = fx_n$, we get

$$\begin{aligned}m(f,g) &= \min\{M(fu, ffx_n, t), M(gu, fu, t), M(gfx_n, ffx_n, t)\} \\ &\to M(gu, fu, t),\end{aligned}$$

as $n \to \infty$. Now, by

$$\varphi\left(\frac{1}{M(gu, gfx_n, t)} - 1\right) \leq \varphi\left(\frac{1}{m(f,g)} - 1\right) - \psi\left(\frac{1}{m(f,g)} - 1\right),$$

on taking $n \to \infty$, we get $M(gu, fu, t) = 1$, which implies that $gu = fu$. It means that u is a coincidence point of f and g.
Now, we shall show that u is also a common fixed point of f and g. For this we assert that $gu = u$, and so $gu = u = fu$. On the other hand, if $gu \neq u$, then taking $x = u$ and $y = x_n$, we have

$$m(f,g) = \min\{M(fu, fx_n, t), M(gu, fu, t), M(gx_n, fx_n, t)\}.$$

It follows that $m(f, g) \to M(gu, u, t)$, as $n \to \infty$.
Now, by

$$\varphi\left(\frac{1}{M(gu, gx_n, t)} - 1\right) \leq \varphi\left(\frac{1}{m(f,g)} - 1\right) - \psi\left(\frac{1}{m(f,g)} - 1\right),$$

on taking $n \to \infty$, we get

$$\varphi\left(\frac{1}{M(gu, u, t)} - 1\right) \leq \varphi\left(\frac{1}{M(gu, u, t)} - 1\right) - \psi\left(\frac{1}{M(gu, u, t)} - 1\right),$$

which implies $gu = u$. Therefore $fu = u = gu$ and hence u is a common fixed point of f and g.
Finally, to prove uniqueness of the fixed point, we suppose that z is another common fixed point of f and g. Then, taking $x = u$ and $y = z$, we have

$$\begin{aligned} m(f,g) &= \min\{M(fu, fz, t), M(gu, fu, t), M(gz, fz, t)\} \\ &= \min\{M(u, z, t), M(u, u, t), M(z, z, t)\} \\ &= M(u, z, t). \end{aligned}$$

Consequently

$$\begin{aligned} \varphi\left(\frac{1}{M(gu, gz, t)} - 1\right) &= \varphi\left(\frac{1}{M(u, z, t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{M(u, z, t)} - 1\right) - \psi\left(\frac{1}{M(u, z, t)} - 1\right), \end{aligned}$$

which implies $M(u, z, t) = 1$, that holds if and only if $u = z$. Therefore u is a unique common fixed point of f and g. □

Example 3.3. Let $X = [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X, t > 0$. Define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t$ and $\psi(t) = \frac{t}{2}$, for all $t > 0$. Define also $f, g : X \to X$ by

$$fx = \frac{x}{2} \text{ for all } x \in [0, 1] \text{ and } gx = \begin{cases} \frac{x}{16} & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $x_n = \frac{1}{2n}$, for all $n \geq 1$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = 0$$

and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

Therefore, the maps f and g are sub-compatible. It is easy to show that f and g are (f, g)-reciprocally continuous and satisfy the contractive condition (3.1). Thus, all the hypotheses of Theorem 3.2 hold and $x = 0$ is a unique common fixed point of f and g.

Remark 3.4. Theorem 3.2 extends and complements the related results of [34] and references therein.

4. Cyclic (φ, ψ)-weak contraction

In 2010, Pacurar and Rus [26] introduced the concept of cyclic ϕ-contraction and proved a fixed point theorem for cyclic ϕ-contraction in complete metric spaces. Later on, Gopal et al. [12] introduced the notion of cyclic weak ϕ-contraction in fuzzy metric spaces. For other results in partial metric spaces, the reader can refer to [6].

Definition 4.1. ([26]) Let X be a nonempty set, m a positive integer and $g : X \to X$ an operator. By definition, $X = \bigcup\limits_{i=1}^{m} X_i$ is a cyclic representation of X with respect to g if

(i) $X_i,\ i = 1, 2, \ldots, m$ are nonempty sets,
(ii) $g(X_1) \subset X_2, \ldots, g(X_{m-1}) \subset X_m, g(X_m) \subset X_1$.

Example 4.2. ([12]) Let $X = \mathbb{R}$. Assume $A_1 = A_3 = [-2, 0]$ and $A_2 = A_4 = [0, 2]$, so that $Y = \bigcup\limits_{i=1}^{4} A_i = [-2, 2]$. Define $g : Y \to Y$ such that $gx = -\frac{x}{2}$, for all $x \in Y$. It is clear that $Y = \bigcup\limits_{i=1}^{4} A_i$ is a cyclic representation of Y.

Here, following the idea of Gopal et al. [12], we present the notion of cyclic weak (φ, ψ)-contraction in fuzzy metric spaces.

Definition 4.3. Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup\limits_{i=1}^{m} A_i$. An operator $g : Y \to Y$ is called a cyclic weak (φ, ψ)-contraction if the following conditions hold:

(i) $Y = \bigcup\limits_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to g;
(ii) there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi\left(\frac{1}{M(gx, gy, t)} - 1\right) \le \varphi\left(\frac{1}{M(x, y, t)} - 1\right) - \psi\left(\frac{1}{M(x, y, t)} - 1\right), \tag{4.1}$$

for any $x \in A_i,\ y \in A_{i+1}$ $(i = 1, 2, \cdots, m,$ where $A_{m+1} = A_1)$ and each $t > 0$.

Now, we are ready to state and prove the following result.

Theorem 4.4. *Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup\limits_{i=1}^{m} A_i$ be G-complete. Suppose that there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering* *distance function φ. If $g : Y \to Y$ is a continuous cyclic weak (φ, ψ)-contraction, then g has a unique fixed point $u \in \bigcap_{i=1}^{m} A_i$.*

Proof. Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$ and set $x_n = gx_{n-1}$ $(n \geq 1)$. Clearly, we get $M(x_n, x_{n+1}, t) = M(gx_{n-1}, gx_n, t)$ for any $t > 0$. Besides for any $n \geq 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by (3.1), for $t > 0$ we have

$$\begin{aligned}
\varphi\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &= \varphi\left(\frac{1}{M(gx_{n-1}, gx_n, t)} - 1\right) \\
&\leq \varphi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) - \psi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) \qquad (4.2) \\
&\leq \varphi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right).
\end{aligned}$$

It implies that $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$ for all $n \geq 1$ and so $\{M(x_{n-1}, x_n, t)\}$ is a non-decreasing sequence of positive real numbers in $(0, 1]$.
Let $S(t) = \lim_{n\to\infty} M(x_{n-1}, x_n, t)$. Now, we show that $S(t) = 1$ for all $t > 0$. If not, there exists some $t > 0$ such that $S(t) < 1$. Then, on making $n \to \infty$ in (4.2), we obtain

$$\varphi\left(\frac{1}{S(t)} - 1\right) \leq \varphi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right)$$

which is a contradiction. Therefore $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty$.
Now, for each positive integer p, we have

$$M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, t/p) * M(x_{n+1}, x_{n+2}, t/p) * \cdots * M(x_{n+p-1}, x_{n+p}, t/p).$$

It follows that

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

and hence $\{x_n\}$ is a G-Cauchy sequence.
Since Y is G-complete, then there exists $z \in Y$ such that $\lim_{n\to\infty} x_n = z$. On the other hand, by the condition (i) of Definition 4.3, it follows that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, 2, \ldots, m$. As Y is G-complete, from each A_i, $i = 1, 2, \ldots, m$, one can extract a subsequence of $\{x_n\}$ that converges to z. In virtue of the fact that each A_i, $i = 1, 2, \ldots, m$, is closed, we conclude that $z \in \bigcap_{i=1}^{m} A_i$ and so $\bigcap_{i=1}^{m} A_i \neq \varnothing$. Obviously, $\bigcap_{i=1}^{m} A_i$ is closed and G-complete. Now, consider the restriction of g on $\bigcap_{i=1}^{m} A_i$, i.e., $g|\bigcap_{i=1}^{m} A_i : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i$ which satisfies the assumptions of Theorem 3.2 and thus, $g|\bigcap_{i=1}^{m} A_i$ has a unique fixed point in $\bigcap_{i=1}^{m} A_i$, say u, which is obtained by iteration from the starting point $x_0 \in Y$. To this aim, we have to show that $x_n \to u$ as $n \to \infty$, where $x_n = gx_{n-1}$ $(n \geq 1)$. We have proved that, for every $x_0 \in X$, the sequence $\{x_n\}$ converges to some $z \in X$. Then,

by (4.1), we have

$$\varphi\left(\frac{1}{M(x_n,u,t)}-1\right) \leq \varphi\left(\frac{1}{M(x_{n-1},u,t)}-1\right)-\psi\left(\frac{1}{M(x_{n-1},u,t)}-1\right).$$

Now, letting $n \to \infty$, we get

$$\varphi\left(\frac{1}{M(z,u,t)}-1\right) \leq \varphi\left(\frac{1}{M(z,u,t)}-1\right)-\psi\left(\frac{1}{M(z,u,t)}-1\right)$$

which is a contradiction if $M(z,u,t) < 1$, and so, we conclude that $u = z$. Obviously, u is a unique fixed point of g. □

Remark 4.5. Theorem 4.4 extends and generalizes the related results of [20, 26, 29] in fuzzy metric spaces via cyclic weak (φ,ψ)-contraction.

Example 4.6. Let $X = \mathbb{R}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X, t > 0$. Assume $A_1 = A_2 = \cdots = A_m = [0,1]$, so that $Y = \bigcup_{i=1}^{m} A_i = [0,1]$ and define $g : Y \to Y$ by $gx = \frac{x^2}{4}$ for all $x \in Y$. Furthermore, if $\varphi, \psi : [0,\infty) \to [0,\infty)$ are defined by $\psi(s) = \frac{s}{4}$ and $\varphi(s) = \frac{s}{2}$ for all $s \geq 0$, we have

$$\begin{aligned}\varphi\left(\frac{1}{M(gx,gy,t)}-1\right) &= \frac{\left|x^2-y^2\right|}{8t}\\ &\leq \frac{|x-y|}{4t} = \varphi\left(\frac{1}{M(x,y,t)}-1\right)-\psi\left(\frac{1}{M(x,y,t)}-1\right).\end{aligned}$$

Clearly, g is a cyclic weak (φ,ψ)-contraction and all the conditions of Theorem 4.4 are satisfied. Therefore g has a unique fixed point $0 \in \bigcap_{i=1}^{m} A_i$.

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Penumarthy Parvateesam Murthy (ppmurthy@gmail.com)
Department of Pu-re and Applied Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University), Koni, Bilaspur (Chhattisgarh), 495 009, India

Urmila Mishra (mishra.urmila@gmail.com)
Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University), Koni, Bilaspur (Chhattisgarh), 495 009, India

Rashmi (rashmikwt@gmail.com)
Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University), Koni, Bilaspur (Chhattisgarh), 495 009, India

Calogero Vetro (cvetro@math.unipa.it)
Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123 Palermo, Italy