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# Some common fixed point theorems for non-Archimedean L-fuzzy metric spaces

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Abstract. The aim of paper is to obtain some results on fixed point theorems for coincidence commuting mappings, implicit relations, contractive mappings and fuzzy $\psi$-contractive mappings in non-Archimedean $L$-fuzzy metric space.

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## 1. Introduction

The notion of fuzzy sets was introduced by Zadeh [15]. Many authors have studied fixed point theorems in fuzzy metric space [1, 2, 5, 6, 7, 9]. Saadati et al [14] introduced the concept of $L$-fuzzy metric space as an extension of fuzzy metric and intuitionistic fuzzy metric spaces. In 2008, Mihet proved a Banach Contraction Theorem in M-complete non-Archimedean fuzzy metric space. As the generalization of non-Archimedean fuzzy metric space, R. Saadati and S. Mansour [13] introduced the concept of non-Archimedean $L$-fuzzy metric space and showed that every contractive mapping on non-Archimedean $L$-fuzzy metric space has a unique fixed point. The aim of this paper is to obtain some results on fixed point theorems for coincidence commuting mappings, implicit relations, contractive mappings and fuzzy$\psi$-contractive mappings in non-Archimedean $L$-fuzzy metric space.

**Definition 1.1.** Let $L = (L, \leq_L)$ be a complete lattice and $U$ a non empty set called universe. An $L$-fuzzy set $A$ on $U$ is defined as a mapping. $A : U \to L$. For each $u$ in $U, A(u)$ represents the degree (in $L$) to which satisfies $A$.

Classically, a triangular norm $T$ on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $(1,x) = x$ for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $L = (L, \leq_L)$.

**Definition 1.2.** A triangular norm($t$-norm) on $L$ is a mapping $\tau : L^2 \to L$ satisfying the following conditions:

(a) $\tau(x, 1_L) = x \ \forall \ x \in L$ (boundary condition)
(b) $\tau(x,y) = \tau(y,x) \ \forall \ (x,y) \in L^2$ (commutativity)
(c) $\tau(x\tau(y,z)) = \tau(\tau(x,y),z))$ (associativity)
(d) $x \leq_L x'$ and $y \leq_L y' \Rightarrow \tau(x,y) \leq_L \tau(x',y')$ (monotonicity)

The $t$-norm $\tau$ is Hadzic type if $\tau(x,y) \geq_L \wedge(x,y)$ for every $x, y \in L$ where

$$\wedge(x,y) = \begin{cases} x & \text{if } \ x \leq_L y \\ y & \text{if } \ y \leq_L x \end{cases}$$

Triangle norms are recursively defined by $\tau^2 = \tau$ and

$$\tau^n(x_{(1)}, \dots, x_{(n+1)}) = \tau(\tau^{n-1}(x_{(1)}, \dots, x_{(n)}), x_{(n+1)})$$

for $n \geq 2, x_{(i)} \in L$ and $i \in \{1, 2, \dots, n+1\}$.

**Definition 1.3.** A negator on $L$ is any decreasing mapping $N : L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$. If $N(N(x)) = x$ for all $x \in L$, then $N$ is called an involutive negator.

In this research the negator $N : L \to L$ is fixed. The negator $N_S$ on $([0,1], \leq)$ defined as $N_S(x) = 1-x$, for all $x \in [0,1]$, is called the standard negator on $([0,1], \leq)$.

**Definition 1.4.** The triple $(X, M, \tau)$ is said to be an $L$-fuzzy metric space if $X$ is an non empty arbitrary set, $\tau$ is a continuous $t$-norm on $L$ and $M$ is an $L$-fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for every $x, y, z$ in $X$ and $t, s$ in $(0, \infty)$:

(a) $M(x,y,t) >_L 0_L$
(b) $M(x,y,t) = M(y,x,t) = 1_L$ for all $t > 0$ if and only if $x = y$
(c) $\tau(M(x,y,t), M(y,z,s)) \leq_L M(x,z,t+s)$
(d) $M(x,y,\cdot) : (0,\infty) \to L$ is continuous
(e) $\lim_{t\to\infty} M(x,y,t) = 1_L$.

In this case, $M$ is called an $L$-fuzzy metric.

If, in the above definition, the triangle inequality $(c)$ is replaced by

$$\text{(NA)} \qquad \tau(M(x,y,t), M(y,z,s)) \leq_L M(x,z,\max\{t,s\}) \ \ \forall \ x,y,z \in X, \ t,s > 0$$

or equivalently,

$$\tau(M(x,y,t), M(y,z,t)) \leq_L M(x,z,t) \ \ \forall \ x,y,z \in X, \ t > 0.$$

Then the triple $(X, M, \tau)$ is called a Non Archimedean $L$-fuzzy metric space.

For t$\in (0, \infty)$, we define the closed ball $B[x,r,t]$ with centre $x \in X$ and radius $r \in L \setminus \{0_L, 1_L\}$, as

$$B[x,r,t] = \{y \in X : M(x,y,t) \geq_L N(r)\}.$$

**Definition 1.5** ([13]). A sequence $\{x_n\}_{n\in N}$ in an $L$-fuzzy metric space $(X, M, \tau)$ is called a right(left) Cauchy sequence if, for each $\varepsilon \in L\ \{0_L\}ad$, $t > 0$, there exists $n_0 \in N$ such that $M(x_m, x_n, t) >_L N(\varepsilon)$, for all $m \geq N \geq n_0$ $(n \geq m \geq n_0)$.

The sequence $\{x_n\}_{n\in N}$ is called convergent to $x \in X$ in an $L$-fuzzy metric space $(X, M, \tau)$ (denoted by $x_n \xrightarrow{M} X$) if $M(x_n, x, t) = M(x, x_n, t) = 1_L$, whenever $n \to +\infty$ for every $t > 0$. An $L$-fuzzy metric space is said to be right (left) complete if and only if every right (left) Cauchy sequence is convergent.

**Definition 1.6** ([13]). Let $(X, M, \tau)$ be an $L$-fuzzy metric space and let $N$, be a negator on $L$. Let $A$ be a subset of $X$, then the LF-diameter $f$ the set $A$ is the function defined as:

$$\delta_A(s) = \sup_{t<s} \inf_{x,y\in A} M(x, y, t).$$

A sequence $\{A_n\}_{n\in N}$ of subsets of an $L$-fuzzy metric space is called decreasing sequence if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$.

The following lemma gives conditions under which the intersection of such sequences is nonempty.

**Lemma 1.7** ([13]). *Let $(X, M, \tau)$ be a left complete $L$-fuzzy metric space and let $\{A_n\}_{n\in N}$ be a decreasing sequence of nonempty closed subsets of $X$ such that $\delta_{A_n}(t) \to 1_L$ as $n \to \infty$. Then $A = \bigcap_{n=1}^{\infty} A_n$ contains exactly one point.*

**Corollary 1.8** ([13]). *Let $(X, M, \tau)$ be a left complete $L$-fuzzy metric space and let $\{A_i\}_{i\in I}$ be a family of closed subsets of $X$, which has the finite intersection property and for each $\varepsilon > 0$, contains a set of LF-diameter less than $\varepsilon$, then $\bigcap_{i\in I} A_i \neq \phi$.*

**Definition 1.9.** Let $(X, M, \tau)$ be a left complete $L$-fuzzy metric space. $A$ mapping $\Delta : X \to X$ is said to be contractive if whenever $x$ and $y$ are distinct point in $X$, we have

$$M(\wedge x, \wedge y, t) >_L M(x, y, t).$$

## 2. Main Results

In this section, we prove some results of [3, 4, 8, 10, 11, 12] in non-Archimedean $L$-fuzzy metric space.

**Theorem 2.1.** *Let $(X, M, \tau)$ be complete non-Archimedean $L$-fuzzy metric space and $f, g, S$ and $T$ be self maps on $X$ satisfying*

(I) $M(Sx, Ty, t) \geq_L \min\{M(fx, gy, t), M(fx, Sx, t), M(gy, Ty, t)\}$

$$\textit{for all } x, y \in X, \textit{ with } fx \neq gy,$$

(II) $fS = Sf, fg = gf, fT = Tf, gS = Sg, gT = Tg, ST = TS,$

(III) $S(X) \subseteq f(X), T(X) \subseteq g(X)$

(IV) *$fg(X)$ is complete and $fg$ is one-one mapping.*

*Then $f$ and $S$ have a coincident point or $g$ and $T$ have a coincident point in $X$.*

*Proof.* For $a \in X$, let $B_a = [fga, \eta, t)]$ where

$$\eta(a,t)] = N\{\min\{M(fga, Sga, t), M(fga, Tfa, t)\}\}$$

denote the closed sphere centered at $fga$ with the radius

$$\min\{M(fga, Sga, t), M(fga, Tfa, t)\}\}. \qquad \square$$

Let $A$ be the collection of all the spheres for all $a \in fg(X)$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on $A$.

Consider a totally ordered sub family $A_1$ of $A$. Since $fg(X)$ is complete and by above corollary 1.8, we have $\bigcap_{B_a \in A_1} B_a = B \neq \phi$. Let $fgb \in B$ where $b \in fg(X)$ and $B_a \in A_1$. Then $fgb \in B_a$. Hence

$$\begin{aligned} M(fgb, fga, t) &\geq_L N\{N\{\min\{M(fga, Sga, t), M(fga, Tfa, t)\}\}\} \\ &\geq_L \min\{M(fga, Sga, t), M(fga, Tfa, t)\}. \end{aligned} \tag{2.1}$$

If $a = b$ then $B_a = Bb$. Assume that $a \neq b$. Since $fg$ is one-one, we have $fga \neq fgb$. Let $x \in B_b$. Then

$$\begin{aligned} M(x, fgb, t) &\geq_L N\{N\{\min\{M(fgb, Sgb, t), M(fgb, Tfb, t)\}\}\} \\ &\geq_L \min\{M(fgb, Sgb, t), M(fgb, Tfb, t)\} \\ &\geq_L \min\{M(fgb, fga, t), M(fga, Tfa, t), M(Tfa, Sgb, t), \\ &\qquad M(fgb, fga, t), M(fga, Sga, t), M(Sga, Tfb, t)\} \\ &\geq_L \min\{M(fgb, Sga, t), M(fga, Tfa, t), \\ &\quad \min\{M(fgb, gfa, t), M(fgb, Sgb, t), M(gfa, Tfa, t)\}, \\ &\quad \min\{M(fga, gfb, t), M(fga, Sga, t), M(gfb, Tfb, t)\}\} \\ &= \min\{M(fga, Sga, t), M(fga, Tfa, t)\} \quad \text{from (2.1), (I) (II).} \end{aligned}$$

Now

$$\begin{aligned} M(x, fga, t) &\geq_t N\{N\{\min\{M(x, fgb, t), M(fgb, fga, t)\}\}\} \\ &\geq_L \min\{M(x, fgb, t), M(fgb, fga, t)\} \\ &\geq_L \min\{M(fga, Sga, t), M(fga, Tfa, t)\}. \end{aligned}$$

Thus $x \in B_b$. Hence $B_b \subseteq B_a$ for every $B_b \in A_1$. Thus $B_b$ is an upper bound in $A$ for the family $A_1$ and hence by Zorn's lemma, there is a maximal element in $A$, say $B_z, z \in fg(X)$. There exists $w \in X$ such that $z = fgw$. Suppose $S(gfgw) \neq f(gfgw)$ and $T(ffgw) \neq g(ffgw)$. From (I) we have

$$\begin{aligned} M(Sgfgw, TSfgw, t) &\geq_L N\{N\{\min\{M(fgfgw, gSfgw, t), M(fgfgw, Sgfgw, t), \\ &\qquad M(gSfgw, TSfgw, t)\}\}\} \\ &\geq_L \min\{M(fgfgw, gSfgw, t), M(fgfgw, Sgfgw, t), \\ &\qquad M(gSfgw, TSfgw, t)\} \\ &= M(fgfgw, gSfgw, t) \end{aligned} \tag{2.2}$$

$$\begin{aligned} M(STfgw, Tffgw, t) &\geq_L N\{N\{\min\{M(fTfgw, gffgw, t), M(fTfgw, STfgw, t), \\ &\qquad M(gffgw, Tffgw, t)\}\}\} \end{aligned}$$

$$\geq_L \min\{M(fTfgw, gffgw, t), M(fTfgw, STfgw, t), M(gffgw, Tffgw, t)$$

$$= M(fTfgw, gffgw, t) \tag{2.3}$$

$$M(fggSw, SggSw, t) \geq_L N\{N\{\min\{M(fggSw, TSfgw, t), M(TSfgw, Tffgw, t), M(Tffgw, SggSw, t)\}\}\}$$

$$\geq_L \min\{M(fggSw, TSfgw, t), M(TSfgw, Tffgw, t), M(Tffgw, SggSw, t)\}$$

$$\geq_L \min\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t), \min\{M(fggSw, gffgw, t), M(fggSw, SggSw, t), M(gffgw, Tffgw, t)\}\} \quad \text{from (2.2), (2.3), (I), (II)}$$

$$= \min\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t)\} \tag{2.4}$$

$$M(Tffgw, TffTw, t) N\{N\{\min\{M(Tffgw, STfgw, t), M(STfgw, Sgfgw, t), M(Sgfgw, TffTw, t)\}\}\}$$

$$\min\{M(Tffgw, STfgw, t), M(STfgw, Sgfgw, t), M(Sgfgw, TffTw, t)\}$$

$$\min\{M(fTfgw, gffgw, t), M(fgfgw, gSfgw, t), \min\{M(fgfgw, gffTw, t), M(fgfgw, Sgfgw, t), M(gffTw, TffTw, t)\}\} \quad \text{from (2.2), (2.3), (I), (II)}$$

$$= \min\{M(fTfgw, gffgw, t), M(fgfgw, gSfgw, t)\} \tag{2.5}$$

From (2.2), (2.4) we have

$$\min\{M(Sgfgw, TSfgw, t), M(fggSw, SggSw, t)\} \geq_L \min\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t)\} \tag{2.6}$$

From (2.3), (2.5) we have

$$\min\{M(STfgw, Tffgw, t), M(Tffgw, TffTw, t)\} \geq_L \min\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t)\} \tag{2.7}$$

If $\min\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t)\} = M(fgfgw, gSfgw, t)$, then from (2.6), $fgfgw \notin BgSw \Rightarrow fgz \notin BgSw$. Hence $B_z \not\subset B_{gSw}$. It is a contradiction to the maximality of $B_z$ in $A$, since $gSw \subseteq gf(X) = fg(X)$. If min $\{M(fgfgw, gSfgw, t), M(fTfgw, gffgw, t)\} = M(fTfgw, gffgw, t)$, then from (2.7), $fgfgw \notin B_{fTw} \Rightarrow fgz \notin B_{fTw}$. Hence $B_z \not\subset B_{fTw}$. It is a contradiction to the maximality of $B_z$ in $A$, since $fTw \subseteq fg(X)$. Hence either $S(gfgw) = f(gfgw)$ or $T(ffgw) = g(ffgw)$. Thus either $f$ and $S$ or $g$ and $T$ have a coincident point in $X$. Taking $f = g$ we get the following result:

**Corollary 2.2.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space, $f, S, T : X \to X$ satisfying*

(a) *$f(X)$ is complete,*

(b) *$M(Sx, Ty, t) \geq_L \min\{M(fx, fy, t), M(fx, Sx, t), M(fy, Ty, t)\}$ for $x, y \in X$, $x \neq y$;*
(c) *$fS = Sf$, $fT = Tf$, $ST = TS$,*
(d) *$S(X) \subseteq f(X)$, $T(X) \subseteq f(X)$.*

*Then either $fw = Sw$ or $fw = Tw$ for some $w \in Xs$.*

Taking $S = T$, we get the following corollary:

**Corollary 2.3.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space. If $f$ and $T$ are two self maps satisfying*

$$T(X) \subseteq f(X), \tag{2.8}$$

*and*

$$M(Tx, Ty, t) \geq_L \min\{M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t)\} \quad \forall\ x, y \in X,\ x \neq y \tag{2.9}$$

*then there exists $z \in X$ such that $fz = Tz$.*

*Further if $f$ and $T$ are coincidentally commuting at $z$ then $z$ is the unique common fixed point of $f$ and $T$.*

If we take $f = I$ (identity map) in above theorem, it becomes the following result:

**Corollary 2.4.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space. If $T : X \to X$ is a mapping such that for every $x, y \in X, x \neq y$,*

$$M(Tx, Ty, t) \geq_L \min\{M(x, Tx, t), M(x, y, t), M(y, Ty, t)\} \tag{2.10}$$

*Then $T$ has a unique fixed point.*

Now we extend Corollary 2.3 when $T$ is a multivalued map. Let $C(X)$ denote the class of all non empty compact subsets of $X$. For $A, B \in C(X)$, the Hausdorff metric is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} \tag{2.11}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$.

**Theorem 2.5.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space. Let $f : X \to X$ and $T : X \to C(X)$ be satisfying*

$$T(X) \subseteq f(X) \quad \forall\ x \in X, \tag{2.12}$$

$$H(Tx, Ty, t) \geq_L \min\{M(fx, fy, t), M(fx, Tx, t), M(fy, Ty, t)\} \quad \forall\ x, y \in X,\ x \neq y \tag{2.13}$$

*then there exists $z \in X$ such that $fz \in Tz$.*
*Further assume that*

$$M(fx, fu, t) \geq_L H(Tfy, Tu, t)\ \forall\ x, y, u \in X, fx \in Ty \tag{2.14}$$

*and*

$$f \text{ and } T \text{ are coincidentally commuting at } z. \tag{2.15}$$

*Then $fz$ is the unique common fixed point of $f$ and $T$.*

*Proof.* Let $B_a = [fa, \eta, t]$ where $\eta(a,t) = N\{N\{1 - M(fa, Ta, t)\}\}$ denote the closed sphere centered at fa with the radius 1-$M(fa, Ta, t)$ and let $A$ be the collection of these spheres for all $a \in X$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on $A$. Let $A_1$ be a totally ordered sub family of $A$. Since $(X, M, \tau)$ is complete and by above corollary 1.8, we have

$$\bigcap_{B_a \in A_1} B_a = B \neq \phi.$$

Let $fb \in B$ and $B_a \in A_1$. Then $fb \in B_a$. Hence

$$M(fb, fa, t) \geq_L M(fa, Ta, t) \tag{2.16}$$

If $a = b$ then $B_a = B_b$. Assume that $a \neq b$. Let $x \in B_b$. Then

$$M(x, fb, t) \geq_L N\{N\{M(fb, Tb, t)\}\}. \geq_L M(fb, Tb, t).$$

Since $Ta$ is compact, there exists $u \in Ta$ such that

$$M(fa, u, t) = M(fa, Ta, t). \tag{2.17}$$

Consider

$$\begin{aligned} M(fb, Tb, t) &= \inf_{c \in Tb} M(fb, c, t) \\ &\geq_L \min\{M(fb, fa, t), M(fa, u, t), \inf_{c \in Tb} M(u, c, t)\} \\ &\geq_L \min\{M(fb, Ta, t), M(Ta, Tb, t)\} \qquad \text{(from (2.16) and (2.17))} \\ &\geq_L \min\{M(fb, Ta, t), M(Ta, fb, t)\} \qquad \text{(from (2.16) and (2.13))} \end{aligned}$$

Thus

$$M(x, fa, t) \geq_L M(fb, Tb, t) M(fa, Ta, t) \tag{2.18}$$

Now,

$$\begin{aligned} M(x, fa, t) &\geq_L \min\{M(x, fa, t), M(fb, fa, t)\} \\ &\geq_L M(fa, Ta, t) \qquad \text{(from (2.16) and (2.18)).} \end{aligned}$$

Thus $x \in B_a$. Hence $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus $B_b$ is an upper bound in $A$ for the family $A_1$ and hence by Zorn's Lemma, $A$ has a maximal element, say $B_z$, $z \in X$.

Suppose $fz \notin Tz$. Since $Tz$ is compact, there exists $k \in Tz$ such that $M(fz, Tz, t) = M(fz, k, t)$. From (2.12), there exists $w \in X$ such that $k = fw$. Thus

$$M(fz, Tz, t) = M(fz, fw, t). \tag{2.19}$$

Clearly $z \neq w$. Now,

$$\begin{aligned} M(fw, Tw, t) &\geq_L H(Tz, Tw, t) \\ &\geq_L \min\{M(fz, fw, t), M(fz, Tz, t), M(fw, Tw, t)\} \\ &= M(fz, fw, t) \qquad \text{(from (2.19))} \end{aligned}$$

Hence, $fz \notin B_w$. Thus $B_z \not\subset B_w$. It is a contradiction to the maximality of $B_z$. Hence $fz \in Tz$.

Further assume (2.14) and (2.15). Write $fz = p$. Then $p \in Tz$. From (2.14),

$M(p, fp, t) = M(fz, fp, t) \geq_L H(Tfz, Tp, t) = H(Tp, Tp, t) = 0$. This implies that $fp = p$. From (2.15), $p = fp \in fTz \subseteq Tfz = Tp$. Thus $fz = p$ is a common fixed point of $f$ and $T$.

Suppose $q \in X, q \neq p$ is such that $q = fq \in Tq$. From (2.13) and (2.14) we have

$$\begin{aligned} M(p, q, t) = M(fp, fq, t) &\geq_L H(Tfp, Tq, t) \\ &= H(Tp, Tq, t) \\ &\geq_L \min\{M(fp, fq, t), M(fp, Tp, t), M(fq, Tq, t)\} \\ &= M(p, q, t). \end{aligned}$$

This implies that $p = q$. Thus $p = fz$ is the unique common fixed point of $f$ and $T$. We denote by $F_X$ the set of maps $f : X \to [0, +\infty)$. □

**Definition 2.6.** A function $f \in F_X$ is said to be metric locally constant (shortly, m.l.c.) provided that for any $x \in X$ and any $y$ in the open $B(x, f(x))$ one has $f(x) = f(y.)$

**Remark 2.7.** Let $(X, M, \tau)$ be non-Archimedean $L$-fuzzy metric space. If $a, b \in X, \lambda > 0$, and $b \in B(a, \lambda, t)$ then $B(a, \lambda, t) = B(b, \lambda, t)$.

(a) If $a, b \in X, 0 < \delta \leq \lambda$, then either $B(a, \lambda, t) \cap B(b, \lambda, t) = \phi$ or $B(b, \lambda, t) \subseteq B(a, \lambda, t)$. Hence, if a ball $B(a, \lambda, t)$ contains a ball $B(b, \lambda, t)$, then either the balls are the same or $\delta < \lambda$.

(b) Every ball is clopen (closed and open) in the topology defined by $M$.

**Theorem 2.8.** *Let $(X, M, \tau)$ be complete non-Archimedean $L$-fuzzy metric space and $T : X \to X$ contractive mapping. Then there exist subset $B \subseteq X$ such that $T : B \to B$ and that the function $f(x) = M(x, Tx, t), x \in B$, is m.l.c.*

*Proof.* Let $B_a = B(a, 1 - M(a, Ta, t), t)$ denote the closed spheres centered at a with the radii 1-$M(a, Ta, t)$ and let $A$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \leq B_b \quad \text{iff} \quad B_b \subseteq B_a \text{ is a partial order on } A.$$

Let $A_1$ be a totally ordered subfamily of $A$. Since $(X, M, \tau)$ is complete and from above Corollary 1.8, we have

$$\bigcap_{B_a \in A_1} = B_a = B \neq \phi.$$

Let $b \in B$ and $B_a \in A_1$ then $b \in B_a$ so

$$M(b, a, t) \geq_L 1 - (1 - M(a, Ta, t)) = M(a, Ta, t).$$

If $a = b$ then $B_a = B_b$. Assume that $a \neq b$, for any $x \in B_b$

$$M(x, a, t) \geq_L \min\{M(x, b, t), M(b, a, t)\} \geq_L M(a, Ta, t)$$

and

$$\begin{aligned} M(x,b,t) &\geq_L 1-(1-M(b,Tb,t) \\ &= M(b,Tb,t) \\ &\geq_L \min\{M(b,a,t),M(a,Ta,t),M(Ta,Tb,t)\} \\ &= \min\{M(a,Ta,t),M(Ta,Tb,t)\} = M(a,Ta,t). \end{aligned}$$

So $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus $B_b$ is the upper bound for the family $A_1$. By Zorn's lemma there is a maximal element in $A_1$, say $B_z$. For any $b \in B_z$

$$\begin{aligned} M(b,Tb,t) &\geq_L \min\{M(b,z,t),M(z,Tz,t),M(Tz,Tb,t)\} \\ &\geq_L \min\{M(b,z,t),M(z,Tz,t),M(z,b,t)\} = M(z,Tz,t) \end{aligned}$$

$B_b \cap B_z$ is nonempty (contains $b$) so by above Remark 2.7,

$$B_b \subseteq B_z.$$

Since $Tb \in B_b$ we just prove that $T : B_z \to B_z$.

For $z = Tzf(x) = 0$ so theorem is proved.

For $z = Tz$ we are going to prove that $f(b) = f(z)$ for every $b \in B_z$.

We know that $M(b,Tb,t) \geq_L M(z,Tz,t)$ for any $b \in B_z$. Let us suppose that for some $b \in B_z$

$$M(b,Tb,t) \geq_L M(z,Tz,t).$$

As

$$M(b,z,t) \geq_L M(z,Tz,t)$$

then

$$\begin{aligned} M(z,Tz,t) &\geq_L \min\{M(z,b,t),M(b,Tz,t)\} \\ &\geq_L \min\{M(z,b,t),M(b,Tb,t),M(Tb,Tz,t)\} \\ &\geq_L \min\{M(z,b,t),M(b,Tb,t),M(b,z,t)\} \\ &= \min\{M(z,b,t),M(b,Tb,t)\} \\ &= M(z,b,t). \end{aligned}$$

We obtain that $M(z,Tz,t) = M(b,z,t)$.

But

$$M(b,z,t) = M(z,Tz,t) \leq_L M(b,Tb,t)$$

implies that $z \in B_z$ but $z \notin B_b$ and hence

$$B_b \subsetneqq B_z$$

which contradicts the maximality of $B_z$ Thus we proved that $f$ is m.l.c. on $B = B_z$. □

## 3. Implicit relations

Let $\phi$ be the sets of all real continuous functions $\phi : (R^+)^4 \to R$, non decreasing in first argument and satisfying the following conditions:

(i) For $u, v \geq 0$, $\phi(u, v, v, u) \geq 0$ or $\phi(u, v, u, v) \geq 0$ imply $u \geq v$

(ii) The functions $\phi$ satisfies condition $(\phi_U)$ if

$$\phi(u, u, 1, 1) \geq 0 \quad \text{implies} \quad u \geq 1. \tag{3.1}$$

**Theorem 3.1.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space. If*

$$F(M(Tx, Ty, t), M(x, y, t), M(x, Tx, t), M(y, Ty, t)) \geq_L 0$$

*for every $x \neq y$ in $X$, where $F$ satisfies conditions $(\phi_U)$, then $T$ has at most one fixed point.*

*Proof.* Suppose that $T$ has two fixed points $z$ and $z'$ with $z \neq z'$. Then by (3.1) we have successively

$$F(M(Tz, Tz', t), M(z, z', t), M(z, Tz, t), M(z', Tz', t)) \geq_L 0$$

i.e.,

$$F(M(z, z', t), M(z, z', t), M(z, z, t), M(z', z', t)) \geq_L 0$$

i.e.,

$$F(M(z, z', t), M(z, z', t), 1, 1) \geq_L 0$$

which contradicts $(\phi_U)$. □

**Theorem 3.2.** *Let $(X, M, \tau)$ be complete non-Archimedean L-fuzzy metric space. If $T : X \to X$ satisfies the inequality (3.1) for every distinct points $x, y$ in $X$, where $F \in F_4$, then $T$ has a fixed point. Further, if $F$ satisfies in addition condition $(\phi_U)$, then the fixed point is unique.*

*Proof.* Let $B_a = B[a, \eta, t]$ where $\eta(a, t) = N\{N\{1 - M(a, Ta, t)\}\}$ denote the closed sphere centered at "$a$" with radius $1 - M(a, Ta, t)$ and let $A$ be the collection of these spheres for all $\in X$. The relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on $A$.

Now consider a totally ordered subfamily $A_1$ of $A$. Since $(X, M, \tau)$ be complete and from above Corollary 1.8, we have

$$\bigcap_{B_a \in A_1} B_a = B \neq 0.$$

Let $b \in B$, $b \neq a$, $B_a \in A_1$ and $x \in B_b$. Then

$$\begin{aligned} M(x, b, t) &\geq_L N\{N\{M(b, Tb, t)\}\} \\ &\geq_L M(b, Tb, t) \\ &\geq_L \min\{M(b, a, t), M(a, Ta, t), M(Ta, Tb, t)\} \\ &= \min\{M(a, Ta, t), M(Ta, Tb, t)\} \end{aligned} \tag{3.2}$$

On the other hand, by (3.1) we have successively

$$F(M(Ta,Tb,t),M(a,b,t),M(a,Ta,t),M(b,Tb,t)) \geq_L 0$$
$$F(M(Ta,Tb,t),M(a,Ta,t),M(a,Ta,t),M(Ta,Tb,t)) \geq_L 0$$

which implies

$$M(Ta,Tb,t) \geq_L M(a,Ta,t)$$

By (3.2) we have

$$M(x,b,t) \geq_L M(a,Ta,t).$$

Now, we have

$$M(x,a,t) \geq_L \min\{M(x,b,t),M(b,a,t),M(a,Ta,t)\}.$$

So $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus $B_b$ is the upper bound for the family $A$. By Zorn's lemma $A$ has a maximal element, say $B_z$, for some $z \in X$. We are going to prove that $z = Tz$.

Let us suppose the contrary, i.e. that $z \neq Tz$. Inequality (3.1) we have that

$$F(M(Tz,T(Tz),t),M(z,Tz,t),M(z,Tz,t),M(Tz,T(T(z)),t) \geq_L 0$$

which implies

$$M(Tz,T(Tz),t) \geq_L M(z,Tz,t).$$

Now, if $y \in BTz$ then

$$M(y,Tz,t) \geq_L M(Tz,T(T(z),t) \geq_L M(z,Tz,t).$$

Therefore,

$$M(y,z,t) \geq_L \min\{M(y,Tz,t),M(T(z),TT(z),t)\} = M(z,Tz,t).$$

This mean that $y \in Bz$ and that $B_{Tz} \subseteq B_z$. On the other hand, $z \notin Bz$ since $M(z,Tz,t) > M(z,T(Tz),t)$. so $BTz \not\subset Bz$. This is a contradiction with the maximality of $Bz$. Hence, we have that $z = Tz$. If $F$ satisfies property $(\phi_U)$. By Theorem 3.1 it follows that $z$ is the unique fixed point of $T$. □

**Theorem 3.3.** *Let $(X,M,\tau)$ be non-Archimedean $L$-fuzzy metric space in which $\tau$ is Hadzic type and $\Delta : X \to X$ be a fuzzy $\psi$-contractive mapping. If there exists $x \in X$ such that $M(x,\Delta x,t) > 0\ \forall\ t > 0$, then $\Delta$ has a unique fixed point.*

*Proof.* Let $B_x = B[x,\eta,t]$ with $\eta(x,t) = N(M(x,\Delta x,t))$ and $t > 0$. Let $A$ be the collection of these balls for all $\in X$. The relation $B_x \leq B_y$ iff $B_y \subseteq B_x$ is a partial order in $A$. Consider a totally ordered subfamily $A_i$ of $A$. From Corollary 1.8, we have

$$\bigcap_{B_x \in A_1} Bx = B \neq \phi.$$

Let $y \in B$ and $B_x \in A_1$, then

$$M(x,y,t) \geq_L N(N(M(x,\wedge x,t))) = M(x,\wedge x,t) \tag{3.3}$$

Now, if $x_0 \in B_y$, then by using $\psi$-contractive mapping, we have

$$\begin{aligned} M(x_0, y, t) &\geq_L N(N(M(y, \wedge y, t))) \\ &\geq_L \tau^2(M(y, x, t), M(x, \wedge x, t), M(\wedge x, \wedge y, t)) \\ &\geq_L \tau^2(M(y, x, t), M(x, \wedge x, t), M(x, y, t)) \\ &\geq_L M(x, \wedge x, t). \end{aligned}$$

Thus

$$M(x_n, y, t) \geq_L M(x, \wedge x, t) \tag{3.4}$$

Now by using (3.3) and (3.4), we obtain

$$\begin{aligned} M(x_0, x, t) &\geq_L \tau(M(x_0, y, t), M(x, y, t)) \\ &\geq_L \tau(M(x, \wedge x, t), M(\wedge x, x, t)) \\ &\geq_L M(x, \wedge x, t). \end{aligned}$$

Therefore $x_0 \in B_x$ and $B_y \subseteq B_x$ implies that $B_x \leq B_y$ for all $B_x \in A_1$. Thus $B_y$ is an upper bound in $A$ for family $A_1$ and hence by Zorn's Lemma, $A$ has a maximal element, say $B_z, z \in X$. We claim that $z = \Delta z$.

Suppose that $z \neq \Delta z$. Since $\Delta$ is $\psi$-contractive and $\psi(t) > t$, therefore

$$\begin{aligned} M(\wedge z, \wedge^2 z, t) &\geq_L \psi M(z, \wedge z, t), \\ &\geq_L M(z, \wedge z, t), \end{aligned}$$

where $\Delta^2 = \Delta 0 \Delta$ and

$$\wedge z \in B[\wedge z, \eta(\wedge z, t), t] \cap B[z, \eta(z, t), t]$$

Therefore $B_{\Delta z} \subseteq B_z$ and $z$ is not in $B_{\Delta z}$. Thus $B_{\Delta z} \subset B_z$, which contradicts the maximality of $B_z$. Hence $\Delta$ has a fixed point.

Uniqueness easily follows from $\psi$-contractive condition. □

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