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Rough intuitionistic fuzzy sets in semigroups

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Abstract. In this paper, we discuss some basic properties of rough intuitionistic fuzzy set in semigroup. Intuitionistic fuzzy analogues of some results concerning rough semiprime (semiprimary) ideals in semigroup are obtained.

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1. Introduction

Most of problems in our every day life are imprecise, have various uncertainties. To deal with these uncertainties, some kind of theories were given like theory of fuzzy set [11], intuitionistic fuzzy sets [1], rough sets [10], soft sets [4] etc., which we can use as a mathematical tools to deal with uncertainties. In 1965, Zadeh [11] initiated the novel concept of fuzzy set theory, thereafter in 1982, the concept of rough set theory was first given by Pawlak [10] and then in 1999, Molodtsov [4] initiated the concept of soft theory, all these are used for modeling incomplete knowledge, vagueness and uncertainties. In fact, all these concepts having a good application in other disciplines and real life problems are now catching momentum. But, it is seen that all these theories have their own difficulties, that is why in this paper we are going to study rough intuitionistic fuzzy set in semigroup, which is also a new mathematical tool to deal with uncertainties.

In fact, there have been also attempts to fuzzify various mathematical structures like topological spaces, groups, rings, etc. and also concepts like relations, measure, probability, and automata etc. As a result, so many authors contributed different articles on these concepts and applied it on different branches of pure and applied mathematics. Like, Biswas and Nanda [2], in 1994, introduced the notion of rough

subgroups and in 1997, Kuroki [7] introduced the notion of a rough ideal in a semigroup. Based on an equivalence relation, in 1990, Dubois and Prade [3] introduced the lower and upper approximations of fuzzy sets in a Pawlak approximation space to obtain an extended notion called rough fuzzy sets. In 2008, Kazanci and Davvaz [9] introduced rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Recently in [5], the notions of rough intuitionistic fuzzy two-sided (bi-, (1,2)-) ideals in semigroups are introduced.
In this paper, in section 3, we substitute a semigroup with identity instead of the universe in pawlak approximation space and obtain generalization of some important properties that are presented in [7]. In section 4, we define rough intuitionistic fuzzy semiprime (semiprimary) ideals in semigroup and verify some basic properties.
Proofs of certain results in the sequel are routine. However, we include them for the sake of completeness.

2. Preliminaries

This section contains some basic definitions and results which will be needed in the sequel. In this paper, unless otherwise stated explicitly, S always denotes a commutative semigroup with identity. The symbol □ marks the end of a proof.

Definition 2.1 ([1]). An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \},$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and degree of nonmembership of the element $x \in X$ to A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$.
For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$.

Definition 2.2 ([1]). If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS in a non-empty set X, then

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ $\forall x \in X$,
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$,
(3) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$, where for all $x \in X$,
$(\mu_A \cap \mu_B)(x) = \mu_A(x) \wedge \mu_B(x)$ and $(\nu_A \cup \nu_B)(x) = \nu_A(x) \vee \nu_B(x)$.
(4) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$, where for all $x \in X$,
$(\mu_A \cup \mu_B)(x) = \mu_A(x) \vee \mu_B(x)$ and $(\nu_A \cap \nu_B)(x) = \nu_A(x) \wedge \nu_B(x)$.

Definition 2.3. Let $A = (\mu_A, \nu_A)$ be an IFS in S and let $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \le 1$. Then the set

$$A_{(\alpha,\beta)} = \{ x \in S \mid \mu_A(x) \ge \alpha, \nu_A(x) \le \beta \}$$

is called a (α, β)-level subset of A.
The set of all $(\alpha, \beta) \in Im(\mu_A) \times Im(\nu_A)$ such that $\alpha + \beta \le 1$ is called the image of $A = (\mu_A, \nu_A)$, denoted by $Im(A)$.

Definition 2.4. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS of S. Then the composition $A \circ B$ is defined by

$$A \circ B = (\mu_A \circ \mu_B, \nu_A \circ \nu_B),$$

where for all $x \in S$,

$$(\mu_A \circ \mu_B)(x) = \bigvee_{x = yz} [\mu_A(y) \wedge \mu_B(z)]$$

and

$$(\nu_A \circ \nu_B)(x) = \bigwedge_{x = yz} [\nu_A(y) \vee \nu_B(z)].$$

Definition 2.5. An ideal P of S is called semiprime, if for $a \in S$, $a^2 \in P \Rightarrow a \in P$.

Definition 2.6. An ideal P of S is called semiprimary, if for $a, b \in S$, $ab \in P \Rightarrow a^n \in P \text{ or } b^m \in P$ for some $m, n \in Z^+$.

Definition 2.7 ([8])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy subsemigroup of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \quad and \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

Definition 2.8 ([8])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy two-sided ideal (or, simply intuitionistic fuzzy ideal) of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y) \quad and \quad \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y).$$

Theorem 2.9. *An IFS A of S is an intuitionistic fuzzy ideal of S iff the level subsets $A_{(\alpha, \beta)}$, $(\alpha, \beta) \in Im(A)$, are ideals of S.*

Following two definitions are intuitionistic fuzzy form of the fuzzy semiprime (semiprimary) ideals, defined in [6].

Definition 2.10. An intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of S is called intuitionistic fuzzy semiprime ideal if $\forall a \in S$, $A(a^2) = A(a)$ i.e. $\mu_A(a^2) = \mu_A(a)$ and $\nu_A(a^2) = \nu_A(a)$.

Definition 2.11. An intuitionistic fuzzy ideal $A = (\mu_A, \nu_A)$ of S is called intuitionistic fuzzy semiprimary ideal if $\forall a, b \in S$, either $A(ab) \leq A(a^n)$ i.e. $\mu_A(ab) \leq \mu_A(a^n)$ and $\nu_A(ab) \geq \nu_A(a^n)$ for some $n \in Z^+$ or else $A(ab) \leq A(b^m)$ i.e. $\mu_A(ab) \leq \mu_A(b^m)$ and $\nu_A(ab) \geq \nu_A(b^m)$ for some $m \in Z^+$.

3. Approximations of intuitionistic fuzzy sets

Particularly, in this section, S denotes a semigroup with identity.
Let θ be a congruence relation on S, that is, θ is an equivalence relation on S such that

$$(a, b) \in \theta \Rightarrow (ax, bx) \in \theta \text{ and } (xa, xb) \in \theta \;\; \forall x \in S.$$

For a congruence relation θ on S, we have $[a]_\theta [b]_\theta \subseteq [ab]_\theta \;\; \forall a, b \in S$, where $[a]_\theta$ denotes the θ-congruence class containing the element $a \in S$. A congruence relation θ on S is called *complete* if $[a]_\theta [b]_\theta = [ab]_\theta$ for all $a, b \in S$. Let us consider θ be a congruence relation of S. If X is a non-empty subset of S, then the sets $\theta_*(X) = \{ x \in S \mid [x]_\theta \subseteq X \}$ and $\theta^*(X) = \{ x \in S \mid [x]_\theta \cap X \neq \phi \}$ are respectively called the θ-lower and θ-upper approximations of the set X and $\theta(X) = (\theta_*(X), \theta^*(X))$ is called rough set with respect to θ if $\theta_*(X) \neq \theta^*(X)$.

If $A = (\mu_A, \nu_A)$ be an *IFS* of S. Then the *IFS* $\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$ and $\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$ are respectively called *θ-lower* and *θ-upper approximation* of the *IFS* $A = (\mu_A, \nu_A)$ where $\forall x \in S$,

$$\theta_*(\mu_A)(x) = \bigwedge_{a \in [x]_\theta} \mu_A(a), \qquad \theta_*(\nu_A)(x) = \bigvee_{a \in [x]_\theta} \nu_A(a),$$

$$\theta^*(\mu_A)(x) = \bigvee_{a \in [x]_\theta} \mu_A(a), \qquad \theta^*(\nu_A)(x) = \bigwedge_{a \in [x]_\theta} \nu_A(a).$$

For an IFS $A = (\mu_A, \nu_A)$ of S, $\theta(A) = (\theta_*(A), \theta^*(A))$ is called rough intuitionistic fuzzy set with respect to θ if $\theta_*(A) \neq \theta^*(A)$.

Theorem 3.1 ([5]). *Let θ be a congruence relation on S. If A and B are an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta^*(A \circ B) \subseteq \theta^*(A) \cap \theta^*(B).$$

Theorem 3.2 ([5]). *Let θ be a congruence relation on S. If A and B are an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta_*(A \circ B) \subseteq \theta_*(A) \cap \theta_*(B).$$

Let θ, ϕ be two binary relations on a semigroup S. Then the product $\theta.\phi$ of θ and ϕ is defined as follows

$$\theta.\phi = \{(a,b) \in S \times S \mid (a,c) \in \theta \; and \; (c,b) \in \phi \; for \; some \; c \in S\}.$$

Assuming θ, ϕ are congruence relations on a semigroup S, as is well-known, $\theta.\phi$ is a congruence if and only if $\theta.\phi = \phi.\theta$.

Theorem 3.3. *Let θ, ϕ be congruence relations on S such that $\theta.\phi = \phi.\theta$. If A is an intuitionistic fuzzy subsemigroup of S, then*

$$\theta^*(A) \circ \phi^*(A) \subseteq (\theta.\phi)^*(A).$$

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subsemigroup of S. Now

$$\theta^*(A) \circ \phi^*(A) = (\theta^*(\mu_A) \circ \phi^*(\mu_A), \theta^*(\nu_A) \circ \phi^*(\nu_A)).$$

To show $\theta^*(A) \circ \phi^*(A) \subseteq (\theta.\phi)^*(A)$, we have to prove that for all $x \in S$,

$$(\theta^*(\mu_A) \circ \phi^*(\mu_A))(x) \leq (\theta.\phi)^*(\mu_A)(x)$$

and

$$(\theta^*(\nu_A) \circ \phi^*(\nu_A))(x) \geq (\theta.\phi)^*(\nu_A)(x).$$

Now let $x \in S$, then

$$(\theta^*(\mu_A) \circ \phi^*(\mu_A))(x) = \bigvee_{x = yz} [\theta^*(\mu_A)(y) \wedge \phi^*(\mu_A)(z)]$$

$$= \bigvee_{x = yz} [(\bigvee_{a \in [y]_\theta} \mu_A(a)) \wedge (\bigvee_{b \in [z]_\phi} \mu_A(b))]$$

$= \bigvee_{x=yz} [\bigvee_{a \in [y]_\theta, b \in [z]_\phi} (\mu_A(a) \wedge \mu_A(b))]$
$\leq \bigvee_{x=yz} \bigvee_{a \in [y]_\theta, b \in [z]_\phi} \mu_A(ab)$ since A is intuitionistic fuzzy subsemigroup
$\leq \bigvee_{x=yz} \bigvee_{ab \in [yz]_{\theta.\phi}} \mu_A(ab)$ [since $a \in [y]_\theta$, $b \in [z]_\phi \Rightarrow ab \in [yb]_\theta$,
$yb \in [yz]_\phi \Rightarrow ab \in [yz]_{\theta.\phi}$ by definition of $\theta.\phi$]
$= \bigvee_{c \in [x]_{\theta.\phi}} \mu_A(c) = (\theta.\phi)^*(\mu_A)(x)$.

Similarly we can prove $(\theta^*(\nu_A) \circ \phi^*(\nu_A))(x) \geq (\theta.\phi)^*(\nu_A)(x)$. Thus we have $\theta^*(A) \circ \phi^*(A) \subseteq (\theta.\phi)^*(A)$. □

A congruence relation θ on S is called an idempotent congruence if the quotient semigroup S/θ is an idempotent semigroup. An IFS A of S is called semiprime if $A(a^2) = A(a)$ for all $a \in S$.

Theorem 3.4. *Let θ be an idempotent congruence relation on S. If A is an IFS of S, then $\theta_*(A)$ is semiprime.*

Proof. Let $A = (\mu_A, \nu_A)$ be an IFS of S. To prove $\theta_*(A)$ is semiprime, we have to prove $\theta_*(\mu_A)(a^2) = \theta_*(\mu_A)(a)$ and $\theta_*(\nu_A)(a^2) = \theta_*(\nu_A)(a)$ for all $a \in S$. Now let $a \in S$, then

$\theta_*(\mu_A)(a) = \bigwedge_{x \in [a]_\theta} \mu_A(x) = \bigwedge_{x \in [a]_\theta.[a]_\theta} \mu_A(x)$
{ since S/θ is idempotent semigroup, $[a]_\theta = [a]_\theta.[a]_\theta = [a^2]_\theta$ }
$= \bigwedge_{x \in [a^2]_\theta} \mu_A(x) = \theta_*(\mu_A)(a^2)$.

Similarly we have $\theta_*(\nu_A)(a) = \theta_*(\nu_A)(a^2)$. Therefore $\theta_*(A)$ is semiprime. □

Theorem 3.5. *Let θ be an idempotent congruence relation on S. If A is an IFS of S, then $\theta^*(A)$ is semiprime.*

Proof. Proof is similar to Theorem 3.4. □

Theorem 3.6. *Let θ be an idempotent congruence relation on S. If A and B are IFS of S, then*

$$\theta^*(A) \cap \theta^*(B) \subseteq \theta^*(A \circ B).$$

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be IFS of S. Then

$$\theta^*(A \circ B) = (\theta^*(\mu_A \circ \mu_B), \theta^*(\nu_A \circ \nu_B))$$

and

$$\theta^*(A) \cap \theta^*(B) = (\theta^*(\mu_A) \cap \theta^*(\mu_B), \theta^*(\nu_A) \cup \theta^*(\nu_B)).$$

To show $\theta^*(A) \cap \theta^*(B) \subseteq \theta^*(A \circ B)$, we have to prove that for all $x \in S$,

$$(\theta^*(\mu_A) \cap \theta^*(\mu_B))(x) \leq \theta^*(\mu_A \circ \mu_B)(x)$$

and

$$(\theta^*(\nu_A) \cup \theta^*(\nu_B))(x) \geq \theta^*(\nu_A \circ \nu_B)(x).$$

Now let $x \in S$, then

$(\theta^*(\mu_A) \cap \theta^*(\mu_B))(x) = \theta^*(\mu_A)(x) \wedge \theta^*(\mu_B)(x)$

$= \bigvee_{a \in [x]_\theta} \mu_A(a) \wedge \bigvee_{b \in [x]_\theta} \mu_B(b) = \bigvee_{a \in [x]_\theta, b \in [x]_\theta} [\mu_A(a) \wedge \mu_B(b)]$

$\leq \bigvee_{ab \in [x]_\theta . [x]_\theta} [\mu_A(a) \wedge \mu_B(b)] = \bigvee_{ab \in [x]_\theta} [\mu_A(a) \wedge \mu_B(b)]$

[since θ is an idempotent congruence i.e. $[x]_\theta . [x]_\theta = [x]_\theta \; \forall x \in S$,]

$= \bigvee_{\alpha \in [x]_\theta} \bigvee_{\alpha = ab} [\mu_A(a) \wedge \mu_B(b)]$

$= \bigvee_{\alpha \in [x]_\theta} (\mu_A \circ \mu_B)(\alpha) = \theta^*(\mu_A \circ \mu_B)(x)$.

Again,

$(\theta^*(\nu_A) \cup \theta^*(\nu_B))(x) = \theta^*(\nu_A)(x) \vee \theta^*(\nu_B)(x)$

$= \bigwedge_{a \in [x]_\theta} \nu_A(a) \vee \bigwedge_{b \in [x]_\theta} \nu_B(b) = \bigwedge_{a \in [x]_\theta, b \in [x]_\theta} [\nu_A(a) \vee \nu_B(b)]$

$\geq \bigwedge_{ab \in [x]_\theta . [x]_\theta} [\nu_A(a) \vee \nu_B(b)] = \bigwedge_{ab \in [x]_\theta} [\nu_A(a) \wedge \nu_B(b)]$

[since θ is an idempotent congruence i.e. $[x]_\theta . [x]_\theta = [x]_\theta \; \forall x \in S$]

$= \bigwedge_{\alpha \in [x]_\theta} \bigwedge_{\alpha = ab} [\nu_A(a) \vee \nu_B(b)]$

$= \bigwedge_{\alpha \in [x]_\theta} (\nu_A \circ \nu_B)(\alpha) = \theta^*(\nu_A \circ \nu_B)(x)$.

Thus we have $\theta^*(A) \cap \theta^*(B) \subseteq \theta^*(A \circ B)$. □

Theorem 3.7. *Let θ be an idempotent congruence relation on S. If A and B are IFS of S, then*

$$\theta_*(A) \cap \theta_*(B) \subseteq \theta_*(A \circ B).$$

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be IFS of S. Then

$$\theta_*(A \circ B) = (\theta_*(\mu_A \circ \mu_B), \theta_*(\nu_A \circ \nu_B))$$

and

$$\theta_*(A) \cap \theta_*(B) = (\theta_*(\mu_A) \cap \theta_*(\mu_B), \theta_*(\nu_A) \cup \theta_*(\nu_B)).$$

To show $\theta_*(A) \cap \theta_*(B) \subseteq \theta_*(A \circ B)$, we have to prove that for all $x \in S$,

$$(\theta_*(\mu_A) \cap \theta_*(\mu_B))(x) \leq \theta_*(\mu_A \circ \mu_B)(x)$$

and

$$(\theta_*(\nu_A) \cup \theta_*(\nu_B))(x) \geq \theta_*(\nu_A \circ \nu_B)(x).$$

Now let $x \in S$, then

$(\theta_*(\mu_A) \cap \theta_*(\mu_B))(x) = \theta_*(\mu_A)(x) \wedge \theta_*(\mu_B)(x)$

$= \bigwedge_{a \in [x]_\theta} \mu_A(a) \wedge \bigwedge_{b \in [x]_\theta} \mu_B(b) = \bigwedge_{a \in [x]_\theta, b \in [x]_\theta} [\mu_A(a) \wedge \mu_B(b)]$

$\leq \bigwedge_{a \in [x]_\theta, b \in [x]_\theta} \bigvee_{ab = \alpha\beta} [\mu_A(\alpha) \wedge \mu_B(\beta)]$

$$= \bigwedge_{a\in[x]_\theta,\, b\in[x]_\theta} (\mu_A \circ \mu_B)(ab) = \bigwedge_{ab\in[x]_\theta} (\mu_A \circ \mu_B)(ab)$$

[since θ is an idempotent congruence i.e. $[x]_\theta . [x]_\theta = [x]_\theta \ \forall x \in S$]

$$= \bigwedge_{c\in[x]_\theta} (\mu_A \circ \mu_B)(c) = \theta_*(\mu_A \circ \mu_B)(x).$$

Again,

$$(\theta_*(\nu_A) \cup \theta_*(\nu_B))(x) = \theta_*(\nu_A)(x) \vee \theta_*(\nu_B)(x)$$

$$= \bigvee_{a\in[x]_\theta} \nu_A(a) \vee \bigvee_{b\in[x]_\theta} \nu_B(b) = \bigvee_{a\in[x]_\theta,\, b\in[x]_\theta} [\nu_A(a) \vee \nu_B(b)]$$

$$\geq \bigvee_{a\in[x]_\theta,\, b\in[x]_\theta} \bigwedge_{ab=\alpha\beta} [\nu_A(\alpha) \vee \nu_B(\beta)]$$

$$= \bigvee_{a\in[x]_\theta,\, b\in[x]_\theta} (\nu_A \circ \nu_B)(ab) = \bigvee_{ab\in[x]_\theta} (\nu_A \circ \nu_B)(ab)$$

[since θ is an idempotent congruence i.e. $[x]_\theta . [x]_\theta = [x]_\theta \ \forall x \in S$]

$$= \bigvee_{c\in[x]_\theta} (\nu_A \circ \nu_B)(c) = \theta_*(\nu_A \circ \nu_B)(x).$$

Thus we have $\theta_*(A) \cap \theta_*(B) \subseteq \theta_*(A \circ B)$. □

Theorem 3.8. *Let θ be an idempotent congruence relation on S. If A and B are intuitionistic fuzzy right ideal and intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta^*(A) \cap \theta^*(B) = \theta^*(A \circ B).$$

Proof. This follows from Theorems 3.1 and 3.6. □

Theorem 3.9. *Let θ be an idempotent congruence relation on S. If A and B are intuitionistic fuzzy right ideal and intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta_*(A) \cap \theta_*(B) = \theta_*(A \circ B).$$

Proof. This follows from Theorems 3.2 Theorem 3.7. □

4. Rough intuitionistic fuzzy semiprime(semiprimary) ideals

Definition 4.1. Let θ be a congruence relation on S. An IFS A of S is called upper rough intuitionistic fuzzy semiprime (semiprimary) ideal of S if $\theta^*(A)$ is an intuitionistic fuzzy semiprime (semiprimary) ideal of S.

Example 4.2. As example 4.8 in [5], Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Let θ be a congruence relation on S such that the θ-congruence classes are the subsets $\{a\}, \{d\}, \{b, c\}$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in S\}$ be an intuitionistic fuzzy subset of S, defined by

$$A = \{\langle a, 0.3, 0.4\rangle, \langle b, 0.2, 0.3\rangle, \langle c, 0.4, 0.5\rangle, \langle d, 0.3, 0.4\rangle\}$$

Since for every $x \in S$, $\theta^*(\mu_A)(x) = \bigvee_{\alpha \in [x]_\theta} \mu_A(\alpha)$ and $\theta^*(\nu_A)(x) = \bigwedge_{\alpha \in [x]_\theta} \nu_A(\alpha)$, so the upper approximation $\theta^*(A) = \{\langle x, \theta^*(\mu_A)(x), \theta^*(\nu_A)(x)\rangle \mid x \in S\}$ is given by

$$\theta^*(A) = \{\langle a, 0.3, 0.4\rangle, \langle b, 0.4, 0.3\rangle, \langle c, 0.4, 0.3\rangle, \langle d, 0.3, 0.4\rangle\}$$

Then it can be easily verified that

$$\theta^*(\mu_A)(xy) \geq \theta^*(\mu_A)(x) \vee \theta^*(\mu_A)(y),$$

$$\theta^*(\nu_A)(xy) \leq \theta^*(\nu_A)(x) \wedge \theta^*(\nu_A)(y)$$

for all $x, y \in S$. Therefore $\theta^*(A)$ is an intutionistic fuzzy two-sided ideal of S. Also it can be easily verified that

$$\forall a \in S, \ \theta^*(A)(a^2) = \theta^*(A)(a)$$

and $\forall a, b \in S$,

$$\textit{either} \quad \theta^*(A)(ab) \leq \theta^*(A)(a^n) \quad \textit{for some } n \in Z^+$$

$$\textit{or else} \quad \theta^*(A)(ab) \leq \theta^*(A)(b^m) \quad \textit{for some } m \in Z^+.$$

Therefore $\theta^*(A)$ is an intutionistic fuzzy semiprime (semiprimary) ideal of S. Hence A is upper rough intuitionistic fuzzy semiprime (semiprimary) ideal of S.

Definition 4.3. Let θ be a congruence relation on S. An IFS A of S is called lower rough intuitionistic fuzzy semiprime (semiprimary) ideal of S if $\theta_*(A)$ is an intuitionistic fuzzy semiprime (semiprimary) ideal of S.

Definition 4.4. If $\theta^*(A)$ and $\theta_*(A)$ are both intuitionistic fuzzy semiprime (semiprimary) ideals of S then A is called a rough intuitionistic fuzzy semiprime (semiprimary) ideal of S.

Theorem 4.5. *Let θ be a congruence relation on S. If $A = (\mu_A, \nu_A)$ is an IFS of S and $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$, then*
(1) $(\theta^*(A))_{(\alpha,\beta)} = \theta^*(A_{(\alpha,\beta)})$,
(2) $(\theta_*(A))_{(\alpha,\beta)} = \theta_*(A_{(\alpha,\beta)})$.

Proof. (1) We have $x \in \theta^*(A_{(\alpha,\beta)}) \Leftrightarrow [x]_\theta \cap A_{(\alpha,\beta)} \neq \phi \Leftrightarrow \exists a \in [x]_\theta, \ \mu_A(a) \geq \alpha, \nu_A(a) \leq \beta \Leftrightarrow \bigvee_{a \in [x]_\theta} \mu_A(a) \geq \alpha, \ \bigwedge_{a \in [x]_\theta} \nu_A(a) \leq \beta \Leftrightarrow \theta^*(\mu_A)(x) \geq \alpha, \theta^*(\nu_A)(x) \leq \beta \Leftrightarrow x \in (\theta^*(A))_{(\alpha,\beta)}$.

(2) we have $x \in \theta_*(A_{(\alpha,\beta)}) \Leftrightarrow [x]_\theta \subseteq A_{(\alpha,\beta)} \Leftrightarrow \forall a \in [x]_\theta, \ \mu_A(a) \geq \alpha, \nu_A(a) \leq \beta \Leftrightarrow \bigwedge_{a \in [x]_\theta} \mu_A(a) \geq \alpha, \ \bigvee_{a \in [x]_\theta} \nu_A(a) \leq \beta \Leftrightarrow \theta_*(\mu_A)(x) \geq \alpha, \theta_*(\nu_A)(x) \leq \beta \Leftrightarrow x \in (\theta_*(A))_{(\alpha,\beta)}$. □

Theorem 4.6 ([9]). *Let θ congruence relation on S and I be an ideal of S, then $\theta^*(I)$ is an ideal of S. Moreover if $\theta_*(I) \neq \phi$, then $\theta_*(I) = I$.*

Theorem 4.7. *Let θ be a congruence relation on S and P be a semiprime (semiprimary) ideal of S such that $\theta_*(P) \neq \phi$, then $\theta_*(P)$ is a semiprime (semiprimary) ideal of S.*

Proof. If $\theta_*(P) \neq \phi$, then by Theorem 4.6, $\theta_*(P) = P$. Hence the result follows. □

Theorem 4.8. *Let θ be a complete congruence relation on S and P be a semiprimary ideal of S then $\theta^*(P)$ is a semiprimary ideal of S.*

Proof. Since P is an ideal of S, by Theorem 4.6, $\theta^*(P)$ is an ideal of S. Now let $ab \in \theta^*(P)$, where $a, b \in S$. If a^n does not belong to $\theta^*(P)$, $\forall n \in Z^+$, then $[ab]_\theta \cap P \neq \phi$ and $[a^n]_\theta \cap P = \phi$, $\forall n \in Z^+$. Since θ is complete, $\exists x \in [a]_\theta$, $y \in [b]_\theta$ such that $xy \in P$ and x^n does not belong to P, $\forall n \in Z^+$. Since P is semiprimary ideal, we get $y^m \in P$ for some $m \in Z^+$. Again since $y \in [b]_\theta$, we get $y^m \in [b^m]_\theta$. So $[b^m]_\theta \cap P \neq \phi$, which implies $b^m \in \theta^*(P)$. Hence $\theta^*(P)$ is a semiprimary ideal of S. □

Theorem 4.9 ([5]). *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy ideal of S then A is rough intuitionistic fuzzy ideal of S.*

Theorem 4.10. *An intuitionistic fuzzy ideal A of S is intuitionistic fuzzy semiprime iff the level ideals $A_{(\alpha,\beta)}$, $(\alpha,\beta) \in Im(A)$, are semiprime ideals of S.*

Proof. Let $A = (\mu_A, \nu_A)$ be intuitionistic fuzzy semiprime and $a \in S$. If $a^2 \in A_{(\alpha,\beta)}$, then $\mu_A(a) = \mu_A(a^2) \geq \alpha$ and $\nu_A(a) = \nu_A(a^2) \leq \beta$. So $a \in A_{(\alpha,\beta)}$ and hence $A_{(\alpha,\beta)}$ is semiprime.

Conversely, let $A_{(\alpha,\beta)}$, $(\alpha,\beta) \in Im(A)$, be semiprime and $a \in S$. Also let $\mu_A(a^2) = \alpha$, $\nu_A(a^2) = \beta$. Then $a^2 \in A_{(\alpha,\beta)}$, which implies $a \in A_{(\alpha,\beta)}$. So $\mu_A(a) \geq \alpha = \mu_A(a^2)$ and $\nu_A(a) \leq \beta = \nu_A(a^2)$. Again since A is intuitionistic fuzzy ideal, above implies $\mu_A(a) = \mu_A(a^2)$, $\nu_A(a) = \nu_A(a^2)$. Hence A is intuitionistic fuzzy semiprime. □

Theorem 4.11. *An intuitionistic fuzzy ideal A of S is intuitionistic fuzzy semiprimary iff the level ideals $A_{(\alpha,\beta)}$, $(\alpha,\beta) \in Im(A)$, are semiprimary ideals of S.*

Proof. Let $A = (\mu_A, \nu_A)$ be intuitionistic fuzzy semiprimary and $ab \in A_{(\alpha,\beta)}$, where $a, b \in S$. If a^n does not belong to $A_{(\alpha,\beta)}$, $\forall n \in Z^+$, then $\mu_A(a^n) < \alpha \leq \mu_A(ab)$ and $\nu_A(a^n) > \beta \geq \nu_A(ab)$. So $\mu_A(ab) \leq \mu_A(b^m)$ and $\nu_A(ab) \geq \nu_A(b^m)$ for some $m \in Z^+$. Hence $b^m \in A_{(\alpha,\beta)}$, which implies $A_{(\alpha,\beta)}$ is semiprimary.

Conversely, let $A_{(\alpha,\beta)}$, $(\alpha,\beta) \in Im(A)$, be semiprimary and $a, b \in S$. Also let $\mu_A(ab) = \alpha$, $\nu_A(ab) = \beta$. If $A(ab) > A(a^n)$, $\forall n \in Z^+$, then $\mu_A(a^n) < \alpha$ and $\nu_A(a^n) > \beta$. So a^n do not belong to $A_{(\alpha,\beta)}$, $\forall n \in Z^+$. Hence $b^m \in A_{(\alpha,\beta)}$ for some $m \in Z^+$, whence $\mu_A(b^m) \geq \alpha = \mu_A(ab)$ and $\nu_A(b^m) \leq \beta = \nu_A(ab)$. Therefore A is intuitionistic fuzzy semiprimary. □

Theorem 4.12. *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy semiprimary ideal of S, then A is rough intuitionistic fuzzy semiprimary ideal of S.*

Proof. Since A is an intuitionistic fuzzy semiprimary ideal, then by Theorem 4.11, $A_{(\alpha,\beta)}$, $(\alpha, \beta) \in Im(A)$, if non-empty, are semiprimary ideals of S. Again by theorem 4.7, $\theta_*(A_{(\alpha,\beta)})$, if non-empty, is semiprimary ideal of S. Hence by theorem 4.5, $(\theta_*(A))_{(\alpha,\beta)}$ is semiprimary ideal of S. Lastly, by Theorem 4.11, $\theta_*(A)$ is an intuitionistic fuzzy semiprimary ideal of S. Similarly using Theorem 4.11, Theorem 4.8 and Theorem 4.5, we have $\theta^*(A)$ is an intuitionistic fuzzy semiprimary ideal of S. Hence using definition 4.4, we get the required result. □

Theorem 4.13. *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy semiprime ideal of S, then it is lower rough intuitionistic fuzzy semiprime ideal of S.*

Proof. Using Theorem 4.10, Theorem 4.7 and Theorem 4.5, we get the required result. □

5. Conclusions

The present paper deals with the properties of rough intuitionistic fuzzy sets in semigroup. Then we introduce rough intuitionistic fuzzy semiprime (semiprimary) ideals and focus on some of its properties. It will be natural to continue this work by studying other algebraic structures.

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