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T-locality groups

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Abstract. The aim of this paper is to introduce the concepts of T-locality groups, where T stands for any continuous triangular norm. Our construct mainly will deal and relate with both fuzzy T-locality spaces and fuzzy TL-uniform spaces. We establish some basic results and characterization theorems of T-locality groups. We give the necessary and sufficient conditions for a group structure and a fuzzy T-locality system to be compatible. Moreover, we show that all initial and final lifts exist uniquely in the concrete category of T-locality groups and hence all initial and final T-locality groups exist and can be characterized. As consequences the T-locality subgroups, T-locality product groups and T-locality quotient groups are exist.

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1. Introduction

In 1995, N. N. Morsi introduced the fuzzy T-locality spaces, for each continuous triangular norm T. In 2006, we deduced the fuzzy TL-uniform spaces which compatible with fuzzy T-locality spaces. In this manuscript, we introduce a new structure of T-locality groups. We show that these structure is conforms well with fuzzy T-locality spaces. We give some other important results of T-locality groups and we give the notions of the left and right translations for a T-locality group. Also, we show that every T-locality group is TL-uniformizable and we characterize the uniformly continuous functions. Moreover, we study the initial and final T-locality groups.

We proceed as follows : In Section 2, we present some basic definitions and ideas on the classes of fuzzy sets, I-topological spaces, fuzzy T-locality spaces and fuzzy TL-uniform spaces. In Section 3, we introduce our definition of T-locality groups and we prove some of their properties, we show that for the T-locality group, the left and right translations are homeomorphisms. Some other results for these T-locality groups are studied. We also generalize the two important characterization theorems, which give necessary and sufficient conditions for a T-locality system and a group structure to be compatible. Moreover, we study the relations between T-locality groups and fuzzy TL-uniform spaces. Some examples of T-locality groups are given. In Section 4, we show that all initial lifts exist uniquely in the concrete category T-LocGrp of T-locality groups and hence all initial T-locality groups exist and can be characterized, thus the category T-LocGrp is a topological category. Therefore all final lifts and all final T-locality groups also exist. The subgroups and the product groups of T-locality groups in the categorical sense are special initial T-locality groups and hence exist and can be characterized. The quotient groups of T-locality groups are characterized as special final T-locality groups.

2. Preliminaries

A triangular norm T (cf. [18]) is a binary operation on the unit interval $I = [0, 1]$ that is associative, symmetric, isotone in each argument and has neutral element 1. The basic three (continuous) triangular norms are their simplest, namely Min (also denoted by $\wedge$), $\prod$ (product) and T_m (the Lukasiewicz conjunction), where for all $\alpha, \beta \in I$, $\alpha \prod \beta = \alpha\beta$ and $\alpha T_m \beta = (\alpha + \beta) \underline{\triangle} 1$. The binary operation $\underline{\triangle}$ above is the truncated subtraction, defined on non-negative real numbers by

$$r \underline{\triangle} s = \max\{r - s, 0\}, r, s \geq 0.$$

A continuous triangular norm T is uniformly continuous, that is for all $\epsilon > 0$ there is $\theta = \theta_{T,\epsilon} > 0$ such that for every $(\alpha, \beta) \in I \times I$, we have

$$(\alpha T \beta) - \epsilon \leq (\alpha - \theta) T (\beta - \theta) \leq \alpha T \beta \leq (\alpha + \theta) T (\beta + \theta) \leq (\alpha T \beta) + \epsilon. \tag{2.1}$$

Obviously, for every real numbers $r, s \geq 0, \epsilon > 0$ and the above $\theta = \theta_{T,\epsilon} > 0$, we have

$$(r T s) \underline{\triangle} \epsilon \leq (r \underline{\triangle} \theta) T (s \underline{\triangle} \theta). \tag{2.2}$$

For a continuous triangular norm T the following binary operation on I,

$$\jmath(\alpha, \gamma) = \sup\{\theta \in I : \alpha T \theta \leq \gamma\}, \alpha, \gamma \in I, \tag{2.3}$$

is called the residual implication of T [15]. For this implication, we shall use the following property [17], $\forall \alpha, \beta, \theta, \gamma \in I$:

$$\jmath(\alpha, \alpha) = 1. \tag{2.4}$$

$$\jmath(\alpha T \beta, \theta T \gamma) \geq \jmath(\alpha, \theta) T \jmath(\beta, \gamma). \tag{2.5}$$

A fuzzy set λ in a universe set X, introduced by Zadeh in [19], is a function $\lambda : X \to I = [0, 1]$. The collection of all fuzzy sets of X is denoted by I^X. The height of a fuzzy set λ is the following real number :

$$\text{hgt}\lambda = \sup\{\lambda(x) : x \in X\}.$$

If $H \subseteq X$, then its characteristic function is denoted by $\mathbf{1}_H$ and the set of all (crisp) subsets of X is denoted by 2^X. We also denote the constant fuzzy set of X with value $\alpha \in I$ by $\underline{\alpha}$.

Given two fuzzy sets $\mu, \lambda \in I^X$, we denote by $\mu T \lambda$ the following fuzzy set of X : $(\mu T \lambda)(x) = \mu(x) T \lambda(x), x \in X$. The degree of containment of μ in λ according to $\jmath$ is the real number in I [7], defined by :

$$\jmath < \mu, \lambda >= \inf_{x \in X} \jmath(\mu(x), \lambda(x)).$$

We follow Lowen's definition of a fuzzy interior operator on a set X [12]. This is an operator $^o : I^X \to I^X$ that satisfies $\mu^o \leq \mu, (\mu \wedge \lambda)^o = \mu^o \wedge \lambda^o$ for all $\mu, \lambda \in I^X$ and $\underline{\alpha}^o = \underline{\alpha}$ for all $\alpha \in I$. We may define an an I-topology in the usual way, namely assuming a fuzzy set μ to be open if and only if $\mu^o = \mu$. We denote this I-topology by τ. The pair $(X, ^o)$ is called an I-topological space.

A function $f : (X, ^o) = (X, \tau) \to (Y, ^{o\backslash}) = (Y, \tau^\backslash)$, between two I-topological spaces, is said to be continuous [12] if $f^\leftarrow(\mu) \in \tau$ for all $\mu \in \tau^\backslash$, where $(f^\leftarrow(\mu))(x) = \mu(f(x)), \forall x \in X$. It is said to be open if $f(\lambda) \in \tau^\backslash$ for all $\lambda \in \tau$.

In [13], Lowen introduced the concepts of *Initial and final I-topological spaces*. Consider a family of I-topological spaces $(Y_r, \tau_r)_{r \in \Lambda}$ and for each $r \in \Lambda$, a mapping $f_r : X \to Y_r$. The *initial I-topology* of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$ is meant the I-topology τ on X for which the conditions of an initial lift in the category of I-topological spaces are fulfilled, that is,

(i) All mappings $f_r : (X, \tau) \to (Y_r, \tau_r)$ are continuous,

(ii) For any I-topological space (Z, σ) and any mapping $f : (Z, \sigma) \to (X, \tau)$ is continuous if and only if for all $r \in \Lambda$ the mappings $f_r \circ f : (Z, \sigma) \to (Y_r, \tau_r)$ are continuous. The union $\bigcup_{r \in \Lambda} f_r^\leftarrow(\tau_r)$ of the family $(f_r^\leftarrow(\tau_r))_{r \in \Lambda}$ where

$$f_r^\leftarrow(\tau_r) = \{f_r^\leftarrow(\mu) : \mu \in \tau_r\},$$

is a subbase for an I-topology on X, for which the conditions (i) and (ii) of the initial lift in the category of I-topological spaces are fulfilled [12, 13], called the initial I-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$, and $f_r^\leftarrow(\tau_r)$ is the initial I-topology of τ_r with respect to f_r. Therefore, all initial lifts and all initial I-topological spaces exist uniquely in the category of I-topological spaces and hence the category of I-topological spaces is a topological category. Consequently, all final lifts also exist.

Assume that $f_r : X_r \to Y$ is a mapping of X_r to Y. By the *final I-topology* of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$ we mean the I-topology τ on Y which fulfills the conditions of a final lift in the category of I-topological spaces, that is,

(i) All mappings $f_r : (X_r, \tau_r) \to (Y, \tau)$ are continuous,

(ii) For any I-topological space (Z, σ) and any mapping $f : (Y, \tau) \to (Z, \sigma)$ is continuous if and only if for all $r \in \Lambda$ the mappings $f \circ f_r : (X_r, \tau_r) \to (Z, \sigma)$ are continuous. It is shown in [13] that the infimum $\bigcap_{r \in \Lambda} f_r(\tau_r)$ of the family $(f_r(\tau_r))_{r \in \Lambda}$ with respect to the finer relation on I-topologies, where

$$f_r(\tau_r) = \{\lambda \in I^Y : f_r^\leftarrow(\lambda) \in \tau_r\}$$

is the final I-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$.

I-filters and I-filterbases were introduced by R. Lowen in [14]. An I-filter in a universe X is a nonempty collection $\mathfrak{I} \subset I^X$ which satisfies ; $\underline{0} \notin \mathfrak{I}$, $\mathfrak{I}$ is closed under finite meets and contains all the fuzzy supersets of its individual members.

An I-filterbase in X is a nonempty collection $\mathfrak{B} \subset I^X$ which satisfies $\underline{0} \notin \mathfrak{B}$ and the meets of two members of $\mathfrak{B}$ contain a member of $\mathfrak{B}$.

Definition 2.1 ([16]). The T-saturation operator is the operator $^{\sim T}$ which sends an I-filterbase $\mathfrak{B}$ in X to the following subsets of I^X

$$\mathfrak{B}^{\sim T} = \{\mu \in I^X : \bigvee_{\gamma \in I_1} (\underline{\gamma} T \mu_\gamma) \leq \mu, \text{ where } \forall \gamma \in I_1, \mu_\gamma \in \mathfrak{B}\},$$

said to be the T-saturation of $\mathfrak{B}$.

The fuzzy T-locality spaces (T-locality spaces, for short) were introduced by N.N.Morsi, for more definitions and properties, we can refer to [16].

Definition 2.2 ([16]). A T-locality space is an I-topological space $(X,^o)$ whose fuzzy interior operator is induced by some indexed family $\mathfrak{B} = (\mathfrak{B}(x))_{x\in X}$, of I-filerbases in X, in the following manner :

$$\mu^o(x) = \sup_{\nu \in \mathfrak{B}(x)} \mathfrak{J} < \nu, \mu >, \mu \in I^X, x \in X. \tag{2.6}$$

The family $\mathfrak{B}$ is said to be a T-locality basis for $(X,^o)$, and $\mathfrak{B}^{\sim T}$ is called a T-locality system of $(X,^o)$. The I-topology of $(X,^o)$ will be denoted by $\tau(\mathfrak{B})$. Also, a T-locality base $\mathfrak{B}$ and a T-locality system $\mathfrak{B}^{\sim T}$ induce the same T-locality space, that is $\tau(\mathfrak{B}) = \tau(\mathfrak{B}^{\sim T})$.

Theorem 2.3 ([16]). *A family of I-filterbases in X, $\mathfrak{B} = (\mathfrak{B}(x))_{x\in X}$, will be a T-locality base in X if and only if it satisfies the following two conditions, for all $x \in X$*

(TLB 1) $\nu(x) = 1$ *for all* $\nu \in \mathfrak{B}(x)$.

(TLB 2) *Every* $\nu \in \mathfrak{B}(x)$ *has a T-kernel. This consists of two families* $(\nu_\gamma \in \mathfrak{B}(x))_{\gamma \in I_1}$ *and* $(\nu_{y\gamma\theta} \in \mathfrak{B}(y))_{(y,\gamma,\theta)\in X\times I_1\times I_0}$ *such that for all* $(y,\gamma,\theta) \in X \times I_1 \times I_0$, $[(\gamma T \nu_\gamma(y)) \underline{\triangle} \theta] T \nu_{y\gamma\theta} \leq \nu$.

Definition 2.4 ([9]). A T-locality space $(X, \tau(\mathfrak{B}))$ is said to be L-Regular, if for every $(H, x, \epsilon) \in 2^X \times X \times I_0$ are such that there is $\nu \in \mathfrak{B}(x)$ with $\mathrm{hgt}(\nu \wedge 1_H) < \epsilon$, then there are an open set μ and $\rho \in \mathfrak{B}(x)$ such that, $1_H \leq \mu$ and $\mathrm{hgt}(\rho T \mu) < \epsilon$.

Theorem 2.5 ([16]). *Let $(X,^o)$ and $(Y,^{o\backslash})$ be T-locality spaces with T-locality basis $\mathfrak{B}$ and $\mathfrak{C}$, respectively, and $x \in X$. Then a function $f : (X,^o) \to (Y,^{o\backslash})$ will be continuous at the point $x \in X$, if and only if for all $\rho \in \mathfrak{C}(f(x))$, we have $f^{\leftarrow}(\rho) \in (\mathfrak{B}(x))^{\sim T}$ if and only if for all $\rho \in \mathfrak{C}(f(x))$ and all $\gamma \in I_1$ there is $\rho_\gamma \in \mathfrak{B}(x)$ such that $\underline{\gamma} T \rho_\gamma \leq f^{\leftarrow}(\rho)$ if and only if for all $\rho \in \mathfrak{C}(f(x))$ and all $\gamma \in I_1$ there is $\rho_\gamma \in \mathfrak{B}(x)$ such that $\underline{\gamma} T f(\rho_\gamma) \leq \rho$. If follows that f will be continuous if it is continuous at all points of its domain.*

Now, we deduce the following result on the T-locality spaces.

Proposition 2.6. *Let $(X, \tau(\mathfrak{B}_1))$ and $(Y, \tau(\mathfrak{B}_2))$ be two T-locality spaces with basis $\mathfrak{B}_1 = (\mathfrak{B}_1(x))_{x\in X}$ and $\mathfrak{B}_2 = (\mathfrak{B}_2(y))_{y\in Y}$ in X and Y, respectively. Then their T-product $(X \times Y, \tau(\mathfrak{B}_1) \otimes_T \tau(\mathfrak{B}_2))$ is a T-locality space with a base $\mathfrak{B} = \mathfrak{B}_1 \otimes_T \mathfrak{B}_2$, defined by*

$$\mathfrak{B}(x, y) = \{\nu_1 \otimes_T \nu_2 : \nu_1 \in \mathfrak{B}_1(x), \nu_2 \in \mathfrak{B}_2(y)\}.$$

where $(\nu_1 \otimes_T \nu_2)(x, y) = \nu_1(x) T \nu_2(y)$, for every $(x, y) \in X \times Y$.

Proof. First, we show that for each $(x,y) \in X \times Y$, $\mathfrak{B}(x,y)$ is an I-filterbase. Obviously, $\mathfrak{B} \neq \varnothing$ and $\underline{0} \notin \mathfrak{B}$. Let $\lambda_1, \lambda_2 \in \mathfrak{B}(x,y)$. Then there are $\nu_1, \nu_2 \in \mathfrak{B}_1(x)$ and $\mu_1, \mu_2 \in \mathfrak{B}_2(y)$ such that $\lambda_1 = \nu_1 \otimes_T \mu_1$ and $\lambda_2 = \nu_2 \otimes_T \mu_2$. So, for every $(x,y) \in X \times Y$, we have

$$\begin{aligned}
(\lambda_1 \wedge \lambda_2)(x,y) &= \lambda_1(x,y) \wedge \lambda_2(x,y) = (\nu_1 \otimes_T \mu_1)(x,y) \wedge (\nu_2 \otimes_T \mu_2)(x,y) \\
&= [\nu_1(x) T \mu_1(y)] \wedge [\nu_2(x) T \mu_2(y)] \\
&\geq [\nu_1(x) \wedge \nu_2(x)] T [\mu_1(y) \wedge \mu_2(y)], \text{clear} \\
&= (\nu_1 \wedge \nu_2)(x) T (\mu_1 \wedge \mu_2)(y) \\
&\geq \nu(x) T \mu(y), \text{by hypothesis, where } \nu \in \mathfrak{B}_1(x) \text{ and } \mu \in \mathfrak{B}_2(y) \\
&= (\nu \otimes_T \mu)(x,y) \\
&= \lambda(x,y), \text{where } \lambda = (\nu \otimes_T \mu) \in \mathfrak{B}(x,y).
\end{aligned}$$

Hence, the intersection of any two members of $\mathfrak{B}(x,y)$ contain a member of $\mathfrak{B}(x,y)$, which proving that $\mathfrak{B}(x,y)$ is an I-filterbase in $X \times Y$.
Now, we fulfill the conditions of Theorem 2.3 :
(TLB 1) For every $\lambda \in \mathfrak{B}(x,x)$ and $(x,x) \in X \times Y$, we have

$$\begin{aligned}
\lambda(x,x) &= (\nu \otimes_T \mu)(x,x), \text{for some } \nu \in \mathfrak{B}_1(x) \text{ and } \mu \in \mathfrak{B}_2(x) \\
&= \nu(x) T \mu(x) \\
&= 1T1, \text{by hypothesis} \\
&= 1.
\end{aligned}$$

(TLB 2) Let $\lambda \in \mathfrak{B}(x,y)$ and $(x,y) \in X \times Y$. Then there are $\nu \in \mathfrak{B}_1(x)$ and $\mu \in \mathfrak{B}_2(x)$ such that $\lambda = \nu \otimes_T \mu$.
Now, since ν has a T-kernel, that is a two families $(\nu_\gamma \in \mathfrak{B}_1(x))_{\gamma \in I_1}$ and $(\nu_{z\gamma\theta} \in \mathfrak{B}_1(z))_{(z,\gamma,\theta) \in X \times I_1 \times I_0}$ such that for all $(z,\gamma,\theta) \in X \times I_1 \times I_0$,

$$[(\gamma T \nu_\gamma(z)) \underline{\triangle} \theta] T \nu_{z\gamma\theta} \leq \nu.$$

Also, since μ has a T-kernel, that is a two families $(\mu_\gamma \in \mathfrak{B}_2(y))_{\gamma \in I_1}$ and $(\mu_{s\gamma\theta} \in \mathfrak{B}_2(s))_{(s,\gamma,\theta) \in Y \times I_1 \times I_0}$ such that for all $(s,\gamma,\theta) \in Y \times I_1 \times I_0$,

$$[(\gamma T \mu_\gamma(s)) \underline{\triangle} \theta] T \nu_{s\gamma\theta} \leq \mu.$$

Hence, for every $\alpha \in I_1$ and $\epsilon \in I_0$, we can get, by continuity of T, $\gamma \in I_1$ in such a way that $\alpha = \gamma T \gamma$ and then $\theta = \theta_{T,\epsilon}$ be as in (2.1). For all which by taking $\lambda_\alpha = \nu_\gamma \otimes_T \mu_\gamma \in \mathfrak{B}(x,y)$ and $\lambda_{zs\alpha\epsilon} = \nu_{z\gamma\theta} \otimes_T \mu_{s\gamma\theta} \in \mathfrak{B}(x,y)$, we have

$$\begin{aligned}
&[(\alpha T \lambda_\alpha(z,s)) \underline{\triangle} \epsilon] T \lambda_{zs\alpha\epsilon} = \{[\gamma T \gamma T (\nu_\gamma \otimes_T \mu_\gamma)(z,s)] \underline{\triangle} \epsilon\} T \lambda_{zs\alpha\epsilon} \\
&\quad = \{[\gamma T \gamma T \nu_\gamma(z) T \mu_\gamma(s)] \underline{\triangle} \epsilon\} T \{\nu_{z\gamma\theta} \otimes_T \mu_{s\gamma\theta}\} \\
&\quad \leq [(\gamma T \nu_\gamma(z)) \underline{\triangle} \theta] T [(\gamma T \mu_\gamma(s)) \underline{\triangle} \theta] T [\nu_{z\gamma\theta} \otimes_T \mu_{s\gamma\theta}], \text{by(2.2)} \\
&\quad = \{[(\gamma T \nu_\gamma(z)) \underline{\triangle} \theta] T \nu_{z\gamma\theta}\} \otimes_T \{[(\gamma T \mu_\gamma(s)) \underline{\triangle} \theta] T \mu_{s\gamma\theta}\}, \text{clear} \\
&\quad \leq \nu \otimes_T \mu = \lambda.
\end{aligned}$$

This proves that λ has a T-kernel and thus $\mathfrak{B}$ satisfies (TLB 2). Which winds up the proof. □

In [10], Höhle defines for every $\psi, \varphi \in I^{X\times X}$ and $\lambda \in I^X$:
The T-section of ψ over λ by

$$(\psi < \lambda >_T)(x) = \sup_{z\in X}[\lambda(z)T\psi(z,x)],\ x \in X.$$

The T-composition of ψ, φ by

$$(\psi o_T \varphi)(x,y) = \sup_{z\in X}[\varphi(x,z)T\psi(z,y)],\ x,y \in X.$$

The symmetric of ψ by ${}_s\psi(x,y) = \psi(y,x)$, $x,y \in X$.

The fuzzy TL-uniform spaces (TL-uniform spaces, for short) were introduced by K. A. Hashem and N. N. Morsi, for more definitions and properties, we can refer to [9].

Definition 2.7 ([9])**.** (i) A TL-uniform base on a set X is a subset $\partial \subset I^{X\times X}$ which fulfills the following properties

(TLUB 1) ∂ is an I-filterbase,

(TLUB 2) For all $\varphi \in \partial$ and $x \in X$, we have $\varphi(x,x) = 1$,

(TLUB 3) For all $\varphi \in \partial$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \partial$ with $\underline{\gamma}T(\varphi_\gamma o_T \varphi_\gamma) \leq \varphi$,

(TLUB 4) For all $\varphi \in \partial$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \partial$ with $\underline{\gamma}T\varphi_\gamma \leq_s \varphi$.

(ii) A TL-uniformity on X is a T-saturated TL-uniform base on X.

(iii) If Σ is a TL-uniformity on X, then we shall say that ∂ is a basis for Σ if ∂ is an I-filterbase and $\partial^{\sim T} = \Sigma$.

It follows that for a TL-uniformity Σ on a set X and all $\varphi \in \Sigma$, we find that ${}_s\varphi \in \Sigma$. The pair (X, Σ) consisting of a set X and a TL-uniformity Σ on X is called TL-uniform space.

Definition 2.8 ([9])**.** Let (X, Σ) and (Y, ω) be TL-uniform spaces, with bases ∂ and $\partial^\setminus$, respectively, and $f : X \to Y$ be a function. We say that f is uniformly continuous if for every $\varphi \in \partial^\setminus$ and $\gamma \in I_1$, there is $\psi \in \nu$ such that $\underline{\gamma}T\psi \leq (f \times f)^{\leftarrow}(\varphi)$.

Proposition 2.9 ([9])**.** *If Σ is a TL-uniformity on a set X, then the indexed family $(\Sigma(x))_{x\in X}$ given by $\Sigma(x) = \{\psi < 1_x >_T : \psi \in \Sigma\}$ is a T-locality system on X.*

3. T-locality groups

The concept of T-locality group is introduce in this section and some of their properties and results are deduced, we show that for a given T-locality group the left and right translations are homeomorphisms. Also, we study the relations between T-locality groups and TL-uniform spaces. Precisely, we show that every T-locality group is TL-uniformizable and induces TL-uniformities.

In what follows, we consider $(G, *)$ as a group with e as the identity element, and for every $\lambda : G \to I$, we define ${}_s\lambda : G \to I$, as ${}_s\lambda(x) = \lambda(x^{-1})$, for each $x \in G$, where x^{-1} is the inverse element of x.

Now, we define the structure of T-locality groups as follows:

Let $(G, *)$ be a group and $(G, \tau(\mathfrak{B}))$ a T-locality space with base $\mathfrak{B}$ on G. Then the triple $(G, *, \tau(\mathfrak{B}))$ is called a T-locality group if the following mappings Γ :

$(G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \to (G, \tau(\mathfrak{B}))$ defined by

$$\Gamma(x, y) = x * y = xy \text{ for all } x, y \in G \ ;$$

$\Upsilon : (G, \tau(\mathfrak{B})) \to (G, \tau(\mathfrak{B}))$ defined by $\Upsilon(x) = x^{-1}$, for all $x \in G$, are continuous.

Theorem 3.1. *For any group $(G, *)$, we have $(G, *, \tau(\mathfrak{B}))$ is a T-locality group if and only if the mapping $\Omega : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \to (G, \tau(\mathfrak{B}))$ defined by*

$$\Omega(x, y) = xy^{-1} \text{ for all } x, y \in G \ ;$$

is continuous.

Proof. Let $(G, *, \tau(\mathfrak{B}))$ be a T-locality group and $h : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \to (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B}))$ the mapping defined by $h(x, y) = (x, y^{-1})$. Then h is the product of the identity mapping $\mathcal{J}_G$ and the continuous mapping Υ, therefore obviously, h is continuous. Hence, $\Omega = \Gamma \circ h$ is the composition of continuous mappings Γ and h, that is, Ω is continuous.

On the other hand, let Ω be a continuous mapping and $i : (G, \tau(\mathfrak{B})) \to (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B}))$ the canonical injection map defined by $i(x) = (e, x)$, where e is the identity element of G. Then $\Upsilon = \Omega \circ i$ is the composition of continuous mappings and therefore is continuous. Since $\Gamma = \Omega \circ h$ and since $h = \mathcal{J}_G \times \Upsilon$ is the product of continuous mappings $\mathcal{J}_G$ and Υ, then h is continuous and therefore Γ also is continuous. Hence $(G, *, \tau(\mathfrak{B}))$ is a T-locality group. □

If $*$ is a binary operation on G, then we define a binary operation $\odot_T$ on I^G by, for all $\lambda, \nu \in I^G$ and $x \in G$

$$(\lambda \odot_T \nu)(x) = \sup_{yz = x \in G} [\lambda(y) T \nu(z)].$$

Lemma 3.2. *If $(G, *)$ is a group and $\lambda : G \to I$, then for all $x, y \in G$, we have*

$$(1_x \odot_T \lambda)(y) = \lambda(x^{-1}y) \text{ and } (\lambda \odot_T 1_x)(y) = \lambda(yx^{-1}).$$

Proof. Let $\lambda : G \to I$ and $x, y \in G$. Then

$$\begin{aligned}(1_x \odot_T \lambda)(y) &= \sup_{zs=y} [1_x(z) T \lambda(s)] = \sup_{xs=y} \lambda(s) \\ &= \sup_{s=x^{-1}y} \lambda(s) = \lambda(x^{-1}y).\end{aligned}$$

Analogously, we can show that $(\lambda \odot_T 1_x)(y) = \lambda(yx^{-1})$. Which winds up the proof. □

For each group $(G, *)$ and $\alpha \in G$, the left and right translations are the homomorphisms $\mathfrak{L}_\alpha : (G, *) \to (G, *)$ defined by $\mathfrak{L}_\alpha(x) = \alpha x$ and $\mathfrak{R}_\alpha : (G, *) \to (G, *)$, defined by $\mathfrak{R}_\alpha(x) = x\alpha$, for each $x \in G$, respectively.

The left and right translation in T-locality groups fulfill the following results.

Proposition 3.3. *Let $(G, *, \tau(\mathfrak{B}))$ be a T-locality group. Then for each $\alpha \in G$, we have*

(i) *$\mathfrak{L}_\alpha$ and $\mathfrak{R}_\alpha$ are homeomorphisms,*

(ii) *$(1_\alpha \odot_T \lambda) = \mathfrak{L}_\alpha(\lambda)$ and $(\lambda \odot_T 1_\alpha) = \mathfrak{R}_\alpha(\lambda)$, for every $\lambda \in I^G$,*

(iii) *$\nu \in (\mathfrak{B}(e))^{\sim T}$ if and only if $\mathfrak{L}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$ if and only if $\mathfrak{R}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$,*

(iv) $\nu \in (\mathfrak{B}(\alpha))^{\sim T}$ *if and only if* $\mathfrak{L}_{\alpha^{-1}}(\nu) \in (\mathfrak{B}(e))^{\sim T}$ *if and only if* $\mathfrak{R}_{\alpha^{-1}}(\nu) \in (\mathfrak{B}(e))^{\sim T}$,

(v) *If* $\mathfrak{B}$ *is* T*-saturated, then* $\lambda \in \mathfrak{B}(e)$ *if and only if* $(1_\alpha \odot_T \lambda) \in \mathfrak{B}(\alpha)$.

Proof. (i) The left translation $\mathfrak{L}_\alpha$ is the composition of the mapping Γ defined above and the injection map $i : (G, \tau(\mathfrak{B})) \to (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B}))$ defined by $i(x) = (\alpha, x)$, that is $\mathfrak{L}_\alpha = \Gamma \circ i$. Hence, $\mathfrak{L}_\alpha$ is continuous and bijective. Since $(\mathfrak{L}_\alpha)^{\leftarrow} = \mathfrak{L}_{\alpha^{-1}}$, then $(\mathfrak{L}_\alpha)^{\leftarrow}$ is also continuous. Therefore, $\mathfrak{L}_\alpha$ is a homeomorphism. Similarly, one can prove that $\mathfrak{R}_\alpha$ is a homeomorphism.

(ii) Let $\lambda \in I^G$ and $\alpha \in G$. Then for every $y \in G$, we have

$$\begin{aligned}(\mathfrak{L}_\alpha(\lambda))(y) &= \sup\{\lambda(z) : z \in (\mathfrak{L}_\alpha)^{\leftarrow}(y)\}\\ &= \sup\{\lambda(z) : z \in \mathfrak{L}_{\alpha^{-1}(y)}\}\\ &= \sup\{\lambda(z) : z = \alpha^{-1}y\}\\ &= \lambda(\alpha^{-1}y)\\ &= (1_\alpha \odot_T \lambda)(y), \text{ by Lemma 3.2.}\end{aligned}$$

That is, $(1_\alpha \odot_T \lambda) = \mathfrak{L}_\alpha(\lambda)$. Similarly, we can prove $(\lambda \odot_T 1_\alpha) = \mathfrak{R}_\alpha(\lambda)$.

(iii) Let $\nu \in (\mathfrak{B}(e))^{\sim T}$. Then for every $\gamma \in I_1$, there is

$$\nu_\gamma \in \mathfrak{B}(e) = \mathfrak{B}(\alpha^{-1}\alpha) = \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha)) \text{ such that } \underline{\gamma} T \nu_\gamma \leq \nu.$$

Since $\mathfrak{L}_{\alpha^{-1}}$ is continuous, then in view of Theorem 2.5, we get for every $\theta \in I_1$, there is $\nu_{\gamma\theta} \in \mathfrak{B}(\alpha)$ such that $\underline{\theta} T \nu_{\gamma\theta} \leq (\mathfrak{L}_{\alpha^{-1}})^{\leftarrow}(\nu_\gamma) = \mathfrak{L}_\alpha(\nu_\gamma)$, Thus $\underline{\gamma} T \underline{\theta} T \nu_{\gamma\theta} \leq \underline{\gamma} T \mathfrak{L}_\alpha(\nu_\gamma) = \mathfrak{L}_\alpha(\underline{\gamma} T \nu_\gamma) \leq \mathfrak{L}_\alpha(\nu)$. By putting $\beta = (\gamma T \theta) \in I_1$ and $\nu_\beta = \nu_{\gamma\theta} \in \mathfrak{B}(\alpha)$, we have $\underline{\beta} T \nu_\beta \leq \mathfrak{L}_\alpha(\nu)$, which implies $\mathfrak{L}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$.

Conversely, let $\mathfrak{L}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$. Then for every $\gamma \in I_1$, there is $\nu_\gamma \in \mathfrak{B}(\alpha) = \mathfrak{B}(\alpha e) = \mathfrak{B}(\mathfrak{L}_\alpha(e))$ such that $\underline{\gamma} T \nu_\gamma \leq \mathfrak{L}_\alpha(\nu)$. Since $\mathfrak{L}_\alpha$ is continuous, then again by Theorem 2.5, we get for every $\theta \in I_1$, there is $\nu_{\gamma\theta} \in \mathfrak{B}(e)$ such that $\underline{\theta} T \nu_{\gamma\theta} \leq (\mathfrak{L}_\alpha)^{\leftarrow}(\nu_\gamma)$. By putting $\beta = (\gamma T \theta) \in I_1$ and $\nu_\beta = \nu_{\gamma\theta} \in \mathfrak{B}(e)$, we get

$$\begin{aligned}\underline{\beta} T \nu_\beta = \underline{\gamma} T \underline{\theta} T \nu_{\gamma\theta} &\leq \underline{\gamma} T (\mathfrak{L}_\alpha)^{\leftarrow}(\nu_\gamma) = (\mathfrak{L}_\alpha)^{\leftarrow}(\underline{\gamma} T \nu_\gamma)\\ &\leq (\mathfrak{L}_\alpha)^{\leftarrow}(\mathfrak{L}_\alpha(\nu)) = \nu, \text{ for } \mathfrak{L}_\alpha \text{ is ingective.}\end{aligned}$$

This implies that $\nu \in (\mathfrak{B}(e))^{\sim T}$. Analogously, we can show that $\nu \in (\mathfrak{B}(e))^{\sim T}$ iff $\mathfrak{R}_\alpha(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$.

(iv) Follows immediately from (iii).

(v) Let $\mathfrak{B}$ be a T-saturated I-filterbase. Then

$\lambda \in \mathfrak{B}(e)$

iff $\lambda \in \mathfrak{B}(\alpha^{-1}\alpha) = \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha))$, $\alpha \in G$

iff $\exists \nu \in \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha))$ such that $\nu \leq \lambda$

iff $\exists \nu \in \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha))$, for which $(\mathfrak{L}_{\alpha^{-1}})^{\leftarrow}(\nu) \in (\mathfrak{B}(\alpha))^{\sim T}$ and $\nu \leq \lambda$, by Theorem 2.5

iff $\exists \nu \in \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha))$, for which $\mathfrak{L}_\alpha(\nu) \in \mathfrak{B}(\alpha)$ and $\nu \leq \lambda$, by hypothesis

iff $\exists \nu \in \mathfrak{B}(\mathfrak{L}_{\alpha^{-1}}(\alpha))$, for which $\mathfrak{L}_\alpha(\nu) \in \mathfrak{B}(\alpha)$ and $\mathfrak{L}_\alpha(\nu) \leq \mathfrak{L}_\alpha(\lambda)$

iff $\mathfrak{L}_\alpha(\lambda) \in \mathfrak{B}(\alpha)$

iff $(1_\alpha \odot_T \lambda) \in \mathfrak{B}(\alpha)$, by (ii).

This completes the proof. □

We shall now give some characterization theorems of T-locality groups. The first gives necessary and sufficient conditions for a group structure and a T-locality system to be compatible and the second gives necessary and sufficient conditions for an I-filterbase to be a T-locality group.

Theorem 3.4. *If $(G,*)$ is a group, then the triple $(G,*,\tau(\mathfrak{B}))$ is a T-locality group if and only if the following are hold :*
(i) *for every $x \in G$, we have $(\mathfrak{B}(x))^{\sim T} = \{(1_x \odot_T \nu) : \nu \in (\mathfrak{B}(e))^{\sim T}\}$,*
(ii) *for all $\nu \in \mathfrak{B}(e)$ and $\gamma \in I_1$, there is $\nu_\gamma \in \mathfrak{B}(e)$, with $\underline{\gamma} T(\nu_\gamma \odot_T {}_s\nu_\gamma) \le \nu$.*

Proof. Let $(G,*,\tau(\mathfrak{B}))$ be a T-locality group. Then (i) follows from Proposition 3.3. Now, since the map $\Omega : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \to (G, \tau(\mathfrak{B}))$ is continuous at all $(x,y) \in G \times G$. Then, by view of Proposition 2.6 and Theorem 2.5, we have for every $\nu \in \mathfrak{B}(e) = \mathfrak{B}(ee^{-1}) = \mathfrak{B}(\Omega(e,e))$ and $\gamma \in I_1$, there is $(\nu_\gamma \otimes_T \nu_\gamma) \in \mathfrak{B}(e,e)$ such that $\underline{\gamma} T\Omega(\nu_\gamma \otimes_T \nu_\gamma) \le \nu$.
Hence, for every $x \in G$, we get

$$\begin{aligned}\nu(x) &\ge [\underline{\gamma} T\Omega(\nu_\gamma \otimes_T \nu_\gamma)](x)\\ &= \gamma T(\Omega(\nu_\gamma \otimes_T \nu_\gamma))(x)\\ &= \gamma T \sup\{\nu_\gamma(y) T \nu_\gamma(z) : (y,z) \in \Omega^{\leftarrow}(x)\}\\ &= \gamma T \sup\{\nu_\gamma(y) T \nu_\gamma(z) : \Omega(y,z) = x\}\\ &= \gamma T \sup\{\nu_\gamma(y) T \nu_\gamma(z) : yz^{-1} = x\}\\ &= \gamma T \sup\{\nu_\gamma(y) T_s \nu_\gamma(z^{-1}) : yz^{-1} = x\}\\ &= \gamma T(\nu_\gamma \odot_T {}_s\nu_\gamma)(x),\end{aligned}$$

that is, $\underline{\gamma} T(\nu_\gamma \odot_T {}_s\nu_\gamma) \le \nu$.
Which holds (ii).

Conversely, let the stated conditions be hold. If $\nu \in \mathfrak{B}(\Omega(e,e)) = \mathfrak{B}(ee^{-1}) = \mathfrak{B}(e)$ and $\gamma \in I_1$, then from (ii) we can get $\nu_\gamma \in \mathfrak{B}(e)$ such that $\underline{\gamma} T(\nu_\gamma \odot_T {}_s\nu_\gamma) \le \nu$.
So, as above, we can reach to $\underline{\gamma} T\Omega(\nu_\gamma \otimes_T \nu_\gamma) \le \nu$, which meaning, by view of Theorem 2.5, that the mapping $\Omega : (G \times G, \tau(\mathfrak{B}) \otimes_T \tau(\mathfrak{B})) \to (G, \tau(\mathfrak{B}))$ is continuous at $(e,e) \in G \times G$.
Also, as it follows from (i) and Proposition 3.3, that the left translation $\mathfrak{L}_\alpha$ is continuous at the elements $\alpha^{-1}, e \in G$. Therefore, the continuity of the mapping Ω follows from the following composition :
$\Omega = \mathfrak{L}_{\alpha\theta^{-1}} \circ \Omega \circ (\mathfrak{L}_{\alpha^{-1}} \times \mathfrak{L}_{\theta^{-1}}) : G \times G \to G$, where we have $(\alpha,\theta) \to (e,e) \to (ee^{-1}) \to e \to \alpha\theta^{-1}$, for all $(\alpha,\theta) \in G \times G$.
Completing, by Theorem 3.1, that $(G,*,\tau(\mathfrak{B}))$ is a T-locality group. This proves our assertion. □

Theorem 3.5. *Let $(G,*)$ be a group and consider a collection $\mathcal{F} \subset I^G$, which satisfies that :*
(i) *$\mathcal{F}$ is an I-filterbase;*
(ii) *For all $\lambda \in \mathcal{F}$, we have $\lambda(e) = 1$;*
(iii) *For all $\lambda \in \mathcal{F}$, we get ${}_s\lambda \in \mathcal{F}$;*
(iv) *For all $\lambda \in \mathcal{F}$ and $\gamma \in I_1$, there is $\lambda_\gamma \in \mathcal{F}$, with $\underline{\gamma} T(\lambda_\gamma \odot_T \lambda_\gamma) \le \lambda$.*
Then there exists a unique T-locality system compatible with the group structure of

G such that $\mathcal{F}$ is a T-locality basis at $e \in G$. This T-locality system is given by, for every $x \in G$

$$(\mathfrak{B}(x))^{\sim T} = \{(1_x \odot_T \lambda) \in I^G : \lambda \in \mathcal{F}\}^{\sim T}. \tag{3.1}$$

Moreover $(G, \tau(\mathfrak{B}))$ is an L-Regularity T-locality space.

Proof. It follows already from the preceding theorem that if a T-locality system exists, compatible with the group structure of G, it must be given by (3.1) and so it is unique. Now, we show that $\mathfrak{B} = (\mathfrak{B}(x))_{x\in G}$ is a T-locality base in G as : Obviously, $\mathfrak{B}(x)$ is an I-filterbase.

(TLB1) For every $\nu \in \mathfrak{B}(x)$ and $x \in G$, we have

$$\begin{aligned}\nu(x) &= (1_x \odot_T \lambda)(x), \text{ for some } \lambda \in \mathcal{F}\\ &= \lambda(x^{-1}x), \text{ by Lemma 3.2}\\ &= \lambda(e)\\ &= 1, \text{ by (ii).}\end{aligned}$$

(TLB2) Let $\nu \in \mathfrak{B}(x)$. Then for every $z \in G$, we have

$$\begin{aligned}\nu(z) &= (1_x \odot_T \lambda)(z), \text{ for some } \lambda \in \mathcal{F}\\ &= \lambda(x^{-1}z), \text{ by Lemma 3.2}\\ &\geq [\underline{\gamma}T(\lambda_\gamma \odot_T \lambda_\gamma)](x^{-1}z), \text{ by (iv)}\\ &= \gamma T \sup_{z\in G}[\lambda_\gamma(x^{-1}y)T\lambda_\gamma(y^{-1}z)]\\ &\geq \gamma T \sup_{z\in G}\{\lambda_\gamma(x^{-1}y)T[\underline{\alpha}T(\lambda_{\gamma\alpha} \odot_T \lambda_{\gamma\alpha})](y^{-1}z)\}, \text{ by (iv) again}\\ &= \gamma T\alpha T \sup_{z\in G}\{\lambda_\gamma(x^{-1}y)T \sup_{rs=y^{-1}z}[\lambda_{\gamma\alpha}(r)T\lambda_{\gamma\alpha}(s)]\}\\ &\geq \gamma T\alpha T\lambda_\gamma(x^{-1}y)T\lambda_{\gamma\alpha}(y^{-1}z)T\lambda_{\gamma\alpha}(e)\\ &= \gamma T\alpha T(1_x \odot_T \lambda_\gamma)(y)T(1_y \odot_T \lambda_{\gamma\alpha})(z)T1, \text{ by Lemma 3.2 and (ii)}\end{aligned}$$

Since, for all $\theta \in I_0$, we can get (by continuity of T) $\alpha = \alpha_\theta \in I_1$, for which $[\gamma T\alpha T(1_x \odot_T \lambda_\gamma)(y)] \geq [(\gamma T(1_x \odot_T \lambda_\gamma)(y))\underline{\triangle}\theta]$, hence by taking $\nu_{x\gamma} = (1_x \odot_T \lambda_\gamma) \in \mathfrak{B}(x)$ and $\nu_{y\gamma\theta} = (1_y \odot_T \lambda_{\gamma\alpha}) \in \mathfrak{B}(y)$, we get $[(\gamma T\nu_{x\gamma}(y))\underline{\triangle}\theta]T\nu_{y\gamma\theta} \leq \nu$. That is ν has a T-kernel, and therefore $\mathfrak{B} = (\mathfrak{B}(x))_{x\in G}$ is a T-locality base in G. We show that the T-locality space $(G, \tau(\mathfrak{B}))$ is an L-Regularity as follows : Let $H \in 2^G, x \in G$ and $\varepsilon \in I_0$ be are given such that there is $\nu \in \mathfrak{B}(x)$ with $\mathrm{hgt}(\nu \wedge 1_H) < \varepsilon$. Consequently, we can find ε_0 very small such that $\mathrm{hgt}(\nu \wedge 1_H) + \varepsilon_0 < \varepsilon$. Since $\nu \in \mathfrak{B}(x)$, then for all $\gamma \in I_1$ there are $\lambda, \lambda_\gamma \in \mathcal{F}$ such that $\nu = (1_X \odot_T \lambda)$ and $\lambda \geq \underline{\gamma}T(\lambda_\gamma \odot_T \lambda_\gamma)$.

Hence

$$
\begin{aligned}
\varepsilon &> hgt[(1_x \odot_T \lambda) \wedge 1_H] + \varepsilon_0 \\
&= \sup_{z\in H} (1_x \odot_T \lambda)(z) + \varepsilon_0 \\
&= \sup_{z\in H} \lambda(x^{-1}z) + \varepsilon_0 \\
&\geq sup_{z\in H}[\underline{\gamma}T(\lambda_\gamma \odot_T \lambda_\gamma)](x^{-1}z) + \varepsilon_0, \text{ by (iv)} \\
&= \sup_{z\in H}\sup_{y\in G}[\gamma T \lambda_\gamma(x^{-1}y) T \lambda_\gamma(y^{-1}z)] + \varepsilon_0 \\
&\geq (\gamma+\theta)T \sup_{z\in H}\sup_{y\in G}[\lambda_\gamma(x^{-1}y) T \lambda_\gamma(y^{-1}z)], \theta = \theta_{T,\varepsilon_0} \text{ as in (2.1)} \\
&= (\gamma+\theta)T \sup_{y\in G}[\lambda_\gamma(x^{-1}y) T \sup_{z\in H} {}_s\lambda_\gamma(z^{-1}y)] \\
&= (\gamma+\theta)T \sup_{y\in G}[(1_x \odot_T \lambda_\gamma)(y) T \sup_{z\in H}(1_z \odot_T {}_s\lambda_\gamma)(y)] \\
&= (\gamma+\theta)T \sup_{y\in G}[(1_x \odot_T \lambda_\gamma) T \sup_{z\in H}(1_z \odot_T {}_s\lambda_\gamma)](y).
\end{aligned}
$$

Choose $\gamma_0 \in I_1$ for which $(\gamma_0 + \theta) = 1$, and taking

$$\rho = (1_x \odot_T \lambda_{\gamma_0}), \mu = [\vee_{z\in H}(1_z \odot_T {}_s\lambda_{\gamma_0})]^o,$$

we get $\rho \in \mathfrak{B}(x)$ and μ is an open set which satisfy, $1_H \leq \mu$ and $\text{hgt}(\rho T \mu) < \varepsilon$. Since, for every $x \in H$, we have

$$
\begin{aligned}
\mu(x) &= [\vee_{z\in H}(1_z \odot_T {}_s\lambda_{\gamma_0})]^o(x) \geq (1_x \odot_T {}_s\lambda_{\gamma_0})^o(x) \\
&= \sup_{\nu\in\mathfrak{B}(x)} \jmath < \nu, 1_x \odot_T {}_s\lambda_{\gamma_0} >, \text{ by (2.6)} \\
&\geq \jmath < 1_x \odot_T {}_s\lambda_{\gamma_0}, 1_x \odot_T {}_s\lambda_{\gamma_0} >, \text{ since } 1_x \odot_T {}_s\lambda_{\gamma_0} \in \mathfrak{B}(x), \text{ by (iii)} \\
&= 1, \text{ by (2.4)}.
\end{aligned}
$$

This proves the L-Regularity of $(G, \tau(\mathfrak{B}))$ and completing the proof. □

Proposition 3.6. *Let $(G, *)$ be a group and for all $\lambda \in I^G$, we define $\lambda_L, \lambda_R : G \times G \to I$, by $\lambda_L(x,y) = \lambda(x^{-1}y), \lambda_R(x,y) = \lambda(yx^{-1}), x, y \in G$. Then for every $\nu \in I^G$, the foolowing hold :*

(i) $\lambda_L < \nu >_T = \nu \odot_T \lambda$ *and* $\lambda_R < \nu >_T = \lambda \odot_T \nu$ *;*
(ii) $(\lambda T \nu)_L = \lambda_L T \nu_L$ *and* $(\lambda T \nu)_R = \lambda_R T \nu_R$ *;*
(iii) $({}_s\lambda)_L = {}_s(\lambda_L)$ *and* $({}_s\lambda)_R = {}_s(\lambda_R)$ *;*
(iv) $(\lambda \odot_T \nu)_L = \nu_L \circ_T \lambda_L$ *and* $(\lambda \odot_T \nu)_R = \nu_R \circ_T \lambda_R$.

Proof. (i) For every $\nu \in I^G$ and $y \in G$, we have

$$
\begin{aligned}
(\lambda_L < \nu >_T)(y) &= \sup_{z\in G}[\nu(z) T \lambda_L(z,y)] = \sup_{z\in G}[\nu(z) T \lambda(z^{-1}y)] \\
&= \sup_{z,s=z^{-1}y}[\nu(z) T \lambda(s)] = \sup_{zs=y}[\nu(z) T \lambda(s)] \\
&= (\nu \odot_T \lambda)(y),
\end{aligned}
$$

and

$$\begin{aligned}(\lambda_R < \nu >_T)(y) &= \sup_{z\in G}[\nu(z)T\lambda_R(z,y)] = \sup_{z\in G}[\nu(z)T\lambda(yz^{-1})]\\ &= \sup_{z,s=yz^{-1}}[\nu(z)T\lambda(s)] = \sup_{z,s=yz^{-1}}[\lambda(s)T\nu(z)]\\ &= \sup_{sz=y}[\lambda(s)T\nu(z)] = (\lambda \odot_T \nu)(y).\end{aligned}$$

This proves the required equalities.

(ii) Obviously hold.

(iii) Let $\lambda \in I^G$ and $x, y \in G$. Then

$$\begin{aligned}({}_s\lambda)_L(x,y) &= {}_x\lambda(x^{-1}y) = \lambda((x^{-1}y)^{-1}) = \lambda(y^{-1}x) = \lambda_L(y,x)\\ &= {}_s(\lambda_L)(x,y).\end{aligned}$$

Thus $({}_s\lambda)_L = {}_s(\lambda_L)$, and similarly we can prove $({}_s\lambda)_R = {}_s(\lambda_R)$.

(iv) For $\lambda, \nu \in I^G$ and $x, y \in G$, we have

$$\begin{aligned}(\lambda \odot_T \nu)_L(x,y) &= (\lambda \odot_T \nu)(x^{-1}y) = \sup_{rs=x^{-1}y}[\lambda(r)T\nu(s)]\\ &= \sup_{(x^{-1}z)(z^{-1}y)=x^{-1}y}[\lambda(x^{-1}z)T\nu(z^{-1}y)]\\ &= \sup_{z\in G}[\lambda_L(x,z)T\nu_L(z,y)] = (\nu_L \circ_T \lambda_L)(x,y), \text{ and}\\ (\lambda \odot_T \nu)_R(x,y) &= (\lambda \odot_T \nu)(yx^{-1}) = \sup_{rs=yx^{-1}}[\lambda(r)T\nu(s)]\\ &= \sup_{(zx^{-1})(yz^{-1})=yx^{-1}}[\lambda(zx^{-1})T\nu(yz^{-1})]\\ &= \sup_{z\in G}[\lambda_R(x,z)T\nu_R(z,y)] = (\nu_R \circ_T \lambda_R)(x,y).\end{aligned}$$

Rendering (iv) and winds up the proof. □

In the following, we devote to proving that every T-locality group is TL-uniformizable. In doing so, we introduce the concepts of left and right TL-uniformities. Generally, these two TL-uniformities are not equal unless the group under consideration is commutative.

Theorem 3.7. *Let $(G, *, \tau(\mathfrak{B}))$ be a T-locality group, if we define $\partial_L = \{\lambda_L \in I^{G\times G} : \lambda \in \mathfrak{B}(e)\}$ and $\partial_R = \{\lambda_R \in I^{G\times G} : \lambda \in \mathfrak{B}(e)\}$, then ∂_L and ∂_R are TL-uniform bases.*

Proof. If $(G, *, \tau(\mathfrak{B}))$ be a T-locality group, then $(G, \tau(\mathfrak{B}))$ is a T-locality space with T-locality basis $\mathfrak{B} = (\mathfrak{B}(x))_{x\in G}$. We claim that ∂_L is TL-uniform base.

(TLUB1) Obviously ∂_L is an I-filterbase.

(TLUB2) If $\varphi \in \partial_L$, then there is $\lambda \in \mathfrak{B}(e)$ such that $\varphi = \lambda_L$, and for all $x \in G$, we get $\varphi(x,x) = \lambda_L(x,x) = \lambda(x^{-1}x) = \lambda(e) = 1$.

(TLUB3) If $\varphi \in \partial_L$, then there exists a $\lambda \in \mathfrak{B}(e)$ such that $\varphi = \lambda_L$. Thus by virtue of Theorem 3.4 (i), for all $\gamma \in I_1$, we can find $\lambda_\gamma \in \mathfrak{B}(e)$ for which

$\underline{\gamma}T(\lambda_{\gamma\odot_T\lambda_\gamma}) \leq \lambda$. By taking $\varphi_\gamma = (\lambda_\gamma)_L \in \partial_L$, we can obtain

$$\begin{aligned}\underline{\gamma}T(\varphi_\gamma \circ_T \varphi_\gamma) &= \underline{\gamma}T[(\lambda_\gamma)_L \circ_T (\lambda_\gamma)_L)]\\ &= \underline{\gamma}T[\lambda_\gamma \odot_T \lambda_\gamma]_L, \text{ by Proposition 3.6 (iv)}\\ &= [\underline{\gamma}T(\lambda_\gamma \odot_T \lambda_\gamma)]_L, \text{ clear}\\ &\leq \lambda_L = \varphi.\end{aligned}$$

(TLUB4) If $\varphi \in \partial_L$, then there is $\lambda \in \mathfrak{B}(e)$ such that $\varphi = \lambda_L$. Consequently, by Theorem 3.4 (ii), for all $\gamma \in I_1$, there exists a $\lambda_\gamma \in \mathfrak{B}(e)$ for which $\underline{\gamma}T\lambda_\gamma \leq {}_s\lambda$. Therefore, by Proposition 3.6 (iii), we get

$$\underline{\gamma}T(\lambda_\gamma)_L = (\underline{\gamma}T\lambda_\gamma)_L \leq ({}_s\lambda)_L = {}_s(\lambda_L) \text{ implies } \underline{\gamma}T\varphi_\gamma \leq {}_s\varphi.$$

This shows in accordance with Definition 2.7, that the collection ∂_L is a TL-uniform base, which in turn gives rise to a TL-uniformity $\Sigma_L = \partial_L^{\sim T}$, and similarly for ∂_R. This compleres the proof. □

We will call Σ_L and Σ_R, respectively, the left and right TL-uniformity associated with $\mathfrak{B}$.

Definition 3.8. An I-topological space (X, τ) is called TL-uniformizable if there is a TL-uniformity Σ on X such that $\tau = \tau(\Sigma)$.

Theorem 3.9. *Every T-locality group is TL-uniformazable.*

Proof. Let $(G, *, \tau(\mathfrak{B}))$ be a T-locality group. Then $(G, \tau(\mathfrak{B}))$ is a T-locality space with the T-locality system $\mathfrak{B}^{\sim T} = ((\mathfrak{B}(x))^{\sim T})_{x\in G}$. Now, suppose that $(\Sigma(x))_{x\in G}$ is a T-locality system associated with the left TL-uniformity Σ. Then we get

$$\begin{aligned}\Sigma(x) &= \{\psi < 1_x >_T : \psi \in \Sigma\}, \text{ by Proposition 2.9}\\ &= \{\lambda_L < 1_x >_T : \lambda \in (\mathfrak{B}(e))^{\sim T}\}, \text{ by Theorem 3.5}\\ &= \{1_x \odot_T \lambda : \lambda \in (\mathfrak{B}(e))^{\sim T}\}, \text{ clear}\\ &= (\mathfrak{B}(x))^{\sim T}, \text{ by Proposition 3.3 (iv)}\end{aligned}$$

Therefore, in view of Definition 2.2, we have $\tau(\mathfrak{B}) = \tau(\mathfrak{B}^{\sim T}) = \tau(\Sigma)$. Which proves that $(G, *, \tau(\mathfrak{B}))$ is TL-uniformizable. □

Proposition 3.10. *Let $(G, *, \tau(\mathfrak{B}))$ and $(E, \sharp, \tau(\xi))$ be T-locality groups. If $\partial^{\sim T}$ and $\mathcal{W}^{\sim T}$ are the associated left TL-uniformities on G and E, respectively, then $f : (G, \partial^{\sim T}) \to (E, \mathcal{W}^{\sim T})$ is uniformly continuous if and only if for all $\rho \in \xi(e^\setminus)$ and $\gamma \in I_1$, there is $\lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma}T(1_x \odot_T \lambda) \leq f^{\leftarrow}(1_{f(x)} \ominus_T \rho)$, for each $x \in G$, where e and $e^\setminus$ are identity elements of G and E, respectively.*

Proof. In view of Definition 3.8, we have $f : (G, \partial^{\sim T}) \to (E, \mathcal{W}^{\sim T})$ is uniformly continuous

iff $\forall \varphi \in \mathcal{W}, \gamma \in I_1 \exists \psi \in \partial$ such that $\underline{\gamma}T\psi \leq (f \times f)^{\leftarrow}(\varphi)$

iff $\forall \rho \in \xi(e^\setminus), \gamma \in I_1 \exists \lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma}T\lambda_L \leq (f \times f)^{\leftarrow}(\rho_L)$

iff $\forall \rho \in \xi(e^\setminus), \gamma \in I_1 \exists \lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma}T\lambda_L(x, y) \leq \rho_L(f(x), f(y)), \forall x, y \in G$

iff $\forall \rho \in \xi(e^\setminus), \gamma \in I_1 \exists \lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma}T\lambda(x^{-1}y) \leq \rho((f(x))^{-1}, f(y)), \forall x, y \in G$

iff $\forall \rho \in \xi(e^{\backslash}), \gamma \in I_1 \exists \lambda \in \mathfrak{B}(e)$ such that $\gamma T(1_x \odot_T \lambda)(y) \leq (1_{f(x)} \ominus_T \rho)(f(y)), \forall x, y \in G$, by Lemma 3.2

iff $\forall \rho \in \xi(e^{\backslash}), \gamma \in I_1 \exists \lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma} T(1_x \odot_T \lambda) \leq f^{\leftarrow}(1_{f(x)} \ominus_T \rho), \forall x \in G$. This winds up the proof. □

Proposition 3.11. *Let $(G, *, \tau(\mathfrak{B}))$ and $(E, \sharp, \tau(\xi))$ be T-locality groups, with e and $e^{\backslash}$ as the identity elements of G and E, respectively. Then*

(i) *If $\partial^{\sim T}$ and $\mathcal{W}^{\sim T}$ are the associated right TL-uniformities on G and E, respectively, then $f : (G, \partial^{\sim T} \to (E, \mathcal{W}^{\sim T})$ is uniformly continuous if and only if for all $\rho \in \xi(e^{\backslash})$ and $\gamma \in I_1$, there is $\lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma} T(\lambda \odot_T 1_x) \leq f^{\leftarrow}(\rho \ominus_T 1_{f(x)})$, for each $x \in G$.*

(ii) *If $\partial_L^{\sim T}$ and $\mathcal{W}_R^{\sim T}$ (resp. $\partial_R^{\sim T}$ and $\mathcal{W}_L^{\sim T}$)are the associated left and right (resp. right and left) TL-uniformities on G and E, respectively, then $f : G \to E$ is uniformly continuous if and only if for all $\rho \in \xi(e^{\backslash})$ and $\gamma \in I_1$, there is $\lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma} T(1_x \odot_T \lambda) \leq f^{\leftarrow}(\rho \ominus_T 1_{f(x)})$ (resp. $\underline{\gamma} T(\lambda \odot_T 1_x) \leq f^{\leftarrow}(1_{f(x)} \ominus_T \rho)$), for each $x \in G$.*

Proof. Analogous to that of Proposition 3.10. □

Proposition 3.12. *Let $(G, *, \tau(\mathfrak{B}))$ and $(E, \sharp, \tau(\xi))$ be T-locality groups. If $\partial^{\sim T}$ and $\mathcal{W}^{\sim T}$ are the associated left TL-uniformities on G and E, respectively, then a continuous homomorphism $f : G \to E$ is uniformly continuous.*

Proof. Let $f : G \to E$ be a continuous homomorphism, $\gamma \in I_1$ and $\rho \in \xi(e^{\backslash}) = \xi(f(e))$. Then by Theorem 2.5, there is $\lambda \in \mathfrak{B}(e)$ such that $\underline{\gamma} T\lambda \leq f^{\leftarrow}(\rho)$ and hence, we obtain for every $x, y \in G$ that

$$\begin{aligned}
[\underline{\gamma} T(1_x \odot_T \lambda)](y) &= \gamma T(1_x \odot_T \lambda)(y) = \gamma T\lambda(x^{-1}y), \text{ by Lemma 3.2} \\
&\leq (f^{\leftarrow}(\rho))(x^{-1}y) = \rho(f(x^{-1}y)), \\
&= \rho(f(x^{-1})f(y)), \text{ for } f \text{ homomorphism} \\
&= \rho((f(x))^{-1}f(y)), \text{ clear} \\
&= (1_{f(x)} \ominus_T \rho)(f(y)), \text{ by Lemma 3.2 again} \\
&= [f^{\leftarrow}(1_{f(x)} \ominus_T \rho)](y).
\end{aligned}$$

That is $\underline{\gamma} T(1_x \odot_T \lambda) \leq f^{\leftarrow}(1_{f(x)} \ominus_T \rho)$, which proves that f is uniformly continuous. □

Definition 3.13. A T-locality space $(G, ^o)$ is called homogeneous space if for all $x, y \in G$, there is a homeomorphism $h : (G, ^o) \to (G, ^o)$ such that $h(x) = y$.

Proposition 3.14. *Every T-locality group is a homogeneous space.*

Proof. This follows easily from the fact that, if $(G, *, \tau(\mathfrak{B}))$ is a T-locality group, then for all $\alpha, \theta \in G$, the function $\mathcal{R}_{\alpha^{-1}\theta} : (G, *) \to (G, *)$ for which, $\mathcal{R}_{\alpha^{-1}\theta}(\alpha) = \alpha\alpha^{-1}\theta = \theta$, is a homomorphism. □

Example 3.15. (i) For every topological group $(G, *, \mathrm{T})$, we have the topologically generated group $(G, *, \mathfrak{W}(\mathrm{T}))$ is a T-locality group, since all topologically generated spaces are T-locality spaces (cf. [16]). Where, $\mathfrak{W}(\mathrm{T}) = \{\Sigma \in I^G : \Sigma$ is lower semicontinuous from (G, T) into $[0, 1]\}$.

(ii) Let $(G,*)$ be a group and for every $x \in G$, take $\mathfrak{B}(x) = \{(1_x \vee \underline{\alpha}) \in I^G : \alpha \geq 1/2\}$. Then obviously $\mathfrak{B} = (\mathfrak{B}(x))_{x\in G}$ is a T-locality base, because for every $x \in G$, $\mathfrak{B}(x)$ is an I-filterbase. Moreover, for every $\partial \in \mathfrak{B}(x)$, we have

$$\begin{aligned}\partial(x) &= (1_x \vee \underline{\alpha})(x), \text{ for some } \alpha \geq 1/2\\ &= 1,\end{aligned}$$

Which holds (TLB1). Also, for every $\partial \in \mathfrak{B}(x)$ and all $(y,\gamma,\theta) \in X \times I_1 \times I_0$, we can take $\partial_\gamma = (1_x \vee \underline{\frac{1}{2}}) \in \mathfrak{B}(x)$ and $\partial_{y\gamma\theta} = (1_y \vee \underline{\frac{1}{2}} \vee \underline{\gamma} \vee \underline{\theta}) \in \mathfrak{B}(y)$, which satisfy $[(\gamma T\partial_\gamma(y))\underline{\triangle}\theta]T\partial_{y\gamma\theta} \leq \partial$. Holds (TLB2). That is $(G,*,\tau(\mathfrak{B}))$ is a T-locality group. Furthermore the left TL-uniform base induced by $(G,*,\tau(\mathfrak{B}))$ is $\partial = \{\lambda_L \in I^{G\times G} : \lambda \in \mathfrak{B}(e)\} = \{(1_e \vee \underline{\alpha})_L \in I^{G\times G} : \alpha \geq 1/2\}$.

(iii) The set R^+ of all positive real numbers equipped with the usual multiplication is a group. Now, if we take

$$\mathcal{F} = \{(1_1 \vee 1_{(H\cup_s H)} \vee \underline{\frac{1}{2}}) \in I^{\mathrm{R}^+} : H \subseteq \mathrm{R}^+\} \subset I^{\mathrm{R}^+},$$

then, we have the collection $\mathcal{F}$ satisfies the conditions in Theorem 3.5, thus there is a T-locality base $\mathfrak{B} = (\mathfrak{B}(x))_{x\in\mathrm{R}^+}$ in R^+, given by for all $x \in \mathrm{R}^+$,

$$\mathfrak{B}(x) = \{[1_x \odot_T (1_1 \vee 1_{(H\cup_s H)} \vee \underline{\frac{1}{2}})] \in I^{\mathrm{R}^+} : H \subseteq \mathrm{R}^+\}.$$

Moreover, $(\mathrm{R}^+, \tau(\mathfrak{B}))$ is an L-Regularity T-locality space.

4. Initial and final T-locality groups

This section shows that the category T-LocGrp of T-locality groups is a topological category [1] and hence all initial and final T-locality groups exist and can be characterized.

For any class Λ, let $((H_r,\tau_r))_{r\in\Lambda}$ be a family of T-locality groups and $(f_r)_{r\in\Lambda}$ a family of homomorphisms of a group G into groups H_r. For any T-locality group (G,τ), the family $(f_r : (G,\tau) \to (H_r,\tau_r))_{r\in\Lambda}$ is called an initial lift of $(f_r : G \to (H_r,\tau_r))_{r\in\Lambda}$ in the category T-LocGrp provided that (G,τ) is the T-locality group for which the following conditions are fulfilled :

(i) All mappings $f_r : (G,\tau) \to (H_r,\tau_r)$ are continuous homomorphisms;

(ii) For any T-locality group (H,σ) and any mapping $f : (H,\sigma) \to (G,\tau)$ is continuous homomorphism if and only if for all $r \in \Lambda$ the mappings $f_r \circ f : (H,\sigma) \to (H_r,\tau_r)$ are continuous homomorphisms.

By an *initial T-locality group* we mean the T-locality group which provides an initial lift in the category T-LocGrp. To prove that all initial lifts and all initial T-locality groups exist in T-LocGrp we have to prove first that in the case $f_r : G \to H_r$ is an injective homomorphism for each $r \in \Lambda$, and τ is the initial T-locality of $(\tau_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$ we get that (G,τ) also is a T-locality group.

First, we shall consider the case of Λ being a singleton:

Proposition 4.1. *Let (H,σ) be a T-locality group and let $f : G \to H$ be an injective homomorphism of a group G into H. Then the initial T-locality space $(G, f^{\leftarrow}(\sigma))$ of (H,σ) with respect to f also is a T-locality group.*

Proof. Let $\Omega_G : (G \times G, f^{\leftarrow}(\sigma) \times f^{\leftarrow}(\sigma)) \to (G, f^{\leftarrow}(\sigma))$ and $\Omega_H : (H \times H, \sigma \times \sigma) \to (H, \sigma)$ be defined as in Theorem 3.1 and let $\lambda \in f^{\leftarrow}(\sigma)$. Then, there is $\Sigma \in \sigma$ such that $\lambda = f^{\leftarrow}(\Sigma)$. Since (H, σ) is a T-locality group, it follows that Ω_H is continuous and hence $\Omega_H^{\leftarrow}(\Sigma) \in \sigma \times \sigma$. Now, since f is a homomorphism, then for every $x, y \in G$, we have

$$\begin{aligned} \Omega_G^{\leftarrow}(\lambda)(x, y) &= \lambda(\Omega_G(x, y)) = (f^{\leftarrow}(\Sigma))(xy^{-1}) = \Sigma(f(xy^{-1})) \\ &= \Sigma(f(x)f(y^{-1})) = \Sigma(f(x)(f(y))^{-1}) \\ &= \Sigma(\Omega_H(f(x), f(y))) = (\Omega_H^{\leftarrow}(\Sigma))(f(x), f(y)) \\ &= [(f \times f)^{\leftarrow}(\Omega_H^{\leftarrow}(\Sigma))](x, y), \end{aligned}$$

that is, $\Omega_G^{\leftarrow}(\lambda) = (f \times f)^{\leftarrow}(\Omega_H^{\leftarrow}(\Sigma))$. Since $f^{\leftarrow}(\sigma)$ is the initial I-topology of σ with respect to f, then $f : (G, f^{\leftarrow}(\sigma)) \to (H, \sigma)$ is continuous and hence $f \times f : G \times G \to H \times H$ is obviously continuous. Therefore,

$$\Omega_G^{\leftarrow}(\lambda) = (f \times f)^{\leftarrow}(\Omega_H^{\leftarrow}(\Sigma)) \in (f \times f)^{\leftarrow}(\sigma \times \sigma) = f^{\leftarrow}(\sigma) \times f^{\leftarrow}(\sigma)),$$

which means that Ω_G is continuous and hence, by Theorem 3.1, $(G, f^{\leftarrow}(\sigma))$ is T-locality group. □

Now, consider the case of any class Λ :

Proposition 4.2. *Let $((H_r, \sigma_r))_{r\in\Lambda}$ be a family of T-locality groups and for all $r \in \Lambda$, let $f_r : G \to H_r$ be an injective homomorphism of a group G into H_r. Let τ be the initial I-topology of $(\sigma_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$. Then (G, τ) also is T-locality group.*

Proof. Let $\Omega_G : (G \times G, \tau \times \tau) \to (G, \tau)$ and $\Omega_{H_r} : (H_r \times H_r, \sigma_r \times \sigma_r) \to (H_r, \sigma_r)$ be defined as in Theorem 3.1. Since $f_r \circ \Omega_G = \Omega_{H_r} \circ (f_r \times f_r)$ and Ω_{H_r}, f_r are continuous, then $f_r \circ \Omega_G$ is continuous. From condition (ii) of the initial lift in the category of I-topological spaces, it follows that Ω_G is continuous and thus (G, τ) is T-locality group. □

The following theorem shows that the T-locality group mentioned in Propositions 4.1 and 4.2 fulfills the conditions of an initial lift in the concrete category T-LocGrp.

Theorem 4.3. *Let $((H_r, \sigma_r))_{r\in\Lambda}$ be a family of T-locality groups and for all $r \in \Lambda$, let $f_r : G \to H_r$ be an injective homomorphism of a group G into H_r and let τ be the initial I-topology of $(\sigma_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$. Then $(f_r : (G, \tau) \to (H_r, \sigma_r))_{r\in\Lambda}$ is an initial lift of $(f_r : G \to (H_r, \sigma_r))_{r\in\Lambda}$ in the category T-LocGrp.*

Proof. First, Propositions 4.1 and 4.2 show that (G, τ) is a T-locality group. From condition (i) of an initial lift in the category of I-topological spaces, we get that condition (i) of initial lift in T-LocGrp holds, that is, all $f_r : (G, \tau) \to (H_r, \sigma_r)$ are continuous homomorphisms. Here, let (H, σ) be a T-locality group and f be a mapping from H into G. Then from condition (ii) of an initial lift in the category of I-topological spaces, we get $f : (H, \sigma) \to (G, \tau)$ is continuous if and only if all $f_r \circ f : (H, \sigma) \to (H_r, \sigma_r)$ are continuous. Now, if f is a homomorphism and we have all f_r are homomorphisms, then all $f_r \circ f$ are homomorphisms. Conversely, let all $f_r \circ f$ be homomorphisms. Since all f_r are homomorphisms we have for every $x, y \in H$, that

$$\begin{aligned} f_r(f(x,y)) &= (f_r \circ f)(xy) = [(f_r \circ f)(x)][(f_r \circ f)(y)] \\ &= [f_r(f(x))][f_r(f(y))] = f_r(f(x)f(y)). \end{aligned}$$

Moreover, since f_r is injective, we get $f(xy) = f(x)f(y)$, that is, f is a homomorphism. Hence, $f : (H, \sigma) \to (G, \tau)$ is continuous homomorphism if and only if all $f_r \circ f : (H, \sigma) \to (H_r, \sigma_r)$ are continuous homomorphisms, that is, condition (ii) of an initial lift in T-LocGrp is fulfilled. □

Theorem 4.3 states that all initial lifts exist uniquely in the concrete category T-LocGrp and this means that the category T-LocGrp is a topological category [1]. Hence, all initial T-locality groups exist.

By means of Theorem 4.3, the T-locality groups introduced in Propositions 4.1 and 4.2 coincide with the initial T-locality groups, that is, if $((H_r, \sigma_r))_{r\in\Lambda}$ is a family of T-locality groups, and for each $r \in \Lambda$, f_r is an injective homomorphism of a group G into H_r and τ is the initial I-topology of $(\sigma_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$, then (G, τ) is the initial T-locality group.

T-locality subgroups and *T-locality product groups* are special initial T-locality groups and hence the above implies the following result.

Corollary 4.4. (i) *If (G, τ) is a T-locality groups and S a subgroup of G, then the I-topological subspace (G, τ_S) also is T-locality group, called a T-locality subgroup.*

(ii) *If $((G_r, \tau_r))_{r\in\Lambda}$ is a family of T-locality groups and G is the product $\Pi_{r\in\Lambda} G_r$ of the family $(G_r)_{r\in\Lambda}$ of a groups and $\tau = \Pi_{r\in\Lambda}\tau_r$ is the product of the family $(\tau_r)_{r\in\Lambda}$ of I-topologies, then (G, τ) also is T-locality group, called a T-locality product group.*

Now, since the concrete category T-LocGrp is topological category, then all final lifts also uniquely exist [1]. This even means that also all final T-locality groups exist.

If $((G_r, \tau_r))_{r\in\Lambda}$ is a family of T-locality groups and $(f_r)_{r\in\Lambda}$ a family of homomorphisms of G_r into a group H, indexed by any class Λ. For any T-locality group (H, σ), the family $(f_r : (G_r, \tau_r) \to (H, \sigma))_{r\in\Lambda}$ is called a final lift of $(f_r : (G_r, \tau_r) \to H)_{r\in\Lambda}$ in the category T-LocGrp provided that (H, σ) is the T-locality group for which fulfills the following conditions :

(i) All mappings $f_r : (G_r, \tau_r) \to (H, \sigma)$ are continuous homomorphisms;

(ii) For any T-locality group (G, τ) and any mapping $f : (H, \sigma) \to (G, \tau)$ is continuous homomorphism if and only if for all $r \in \Lambda$ the mappings $f \circ f_r : (G_r, \tau_r) \to (G, \tau)$ are continuous homomorphisms.

By a *final T-locality group* we mean a T-locality group which provides a final lift in the category T-LocGrp.

The following propositions show that if for each $r \in \Lambda$, $f_r : (G_r, \tau_r) \to (H, \sigma)$ is a surjective homomorphism and σ is the final T-locality of $(\tau_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$, then (H, σ) also is a T-locality group.

To prove these results we need the following proposition which can be proved easily by means of the properties of the T-locality group.

Proposition 4.5. *If $f : (G, \tau) \to (H, f(\tau))$ is a surjective homomorphism from a T-locality group (G, τ) to a group H equipped with the final I-topology $f(\tau)$ of τ with respect to f, then f is an open function.*

Consider the case of Λ being a singleton :

Proposition 4.6. *Let (G, τ) be a T-locality group and let $f : (G, \tau) \to H$ be a homomorphism of a group G onto H. Then the final T-locality space $(H, f(\tau))$ of (G, τ) with respect to f also is a T-locality group.*

Proof. Let $\Omega_G : (G \times G, \tau \times \tau) \to (G, \tau)$ and $\Omega_H : (H \times H, f(\tau) \times f(\tau)) \to (H, f(\tau))$ be defined as in Theorem 3.1. Now, since f is a surjective homomorphism, then for every $\Sigma \in I^H$ and $x, y \in H$, we have

$$\begin{aligned}
(\Omega_H^{\leftarrow}(\Sigma))(x, y) &= \Sigma(\Omega_H(x, y)) \\
&= \sup\{\Sigma(\Omega_H(f(r), f(s))) : r, s \in G, (f \times f)(r, s) = (x, y)\} \\
&= \sup\{\Sigma(f(r)(f(s))^{-1}) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\}, \text{ clear} \\
&= \sup\{\Sigma(f(r)(f(s^{-1})) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\} \\
&= \sup\{\Sigma(f(rs^{-1})) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\} \\
&= \sup\{(f^{\leftarrow}(\Sigma))(rs^{-1}) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\} \\
&= \sup\{(f^{\leftarrow}(\Sigma))(\Omega_G(r, s)) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\} \\
&= \sup\{(\Omega_G^{\leftarrow}(f^{\leftarrow}(\Sigma)))(r, s) : (r, s) \in (f \times f)^{\leftarrow}(x, y)\} \\
&= [(f \times f)(\Omega_G^{\leftarrow}(f^{\leftarrow}(\Sigma)))](x, y),
\end{aligned}$$

that is, $\Omega_H^{\leftarrow}(\Sigma) = (f \times f)(\Omega_G^{\leftarrow}(f^{\leftarrow}(\Sigma)))$. If $\Sigma \in f(\tau)$, then $f^{\leftarrow}(\Sigma) \in \tau$, and from continuity of Ω_G we get $\Omega_G^{\leftarrow}(f^{\leftarrow}(\Sigma)) \in \tau \times \tau$. But from Proposition 4.5, we have f is an open, hence $f \times f : G \times G \to H \times H$ is obviously an open. Therefore,

$$\Omega_H^{\leftarrow}(\Sigma) = (f \times f)(\Omega_G^{\leftarrow}(f^{\leftarrow}(\Sigma))) \in f(\tau) \times f(\tau).$$

Which proves the continuity of Ω_H and this implies that $(H, f(\tau))$ is T-locality group. □

For any class Λ we have the following :

Proposition 4.7. *Let $((G_r, \tau_r))_{r \in \Lambda}$ be a family of T-locality groups and for all $r \in \Lambda$, let $f_r : G_r \to H$ be a homomorphism of a group G_r onto a group H. Let σ be the final I-topology of $(\tau_r)_{r \in \Lambda}$ with respect to $(f_r)_{r \in \Lambda}$. Then (H, σ) is a T-locality group.*

Proof. Let $\mu \in \sigma$. Since $f_r : (G_r, \tau_r) \to (H, \sigma)$ is cintinuous, then $f_r^{\leftarrow}(\mu) \in \tau_r$ for all $r \in \Lambda$. But from continuity of $\Omega_{G_r} : (G_r \times G_r, \tau_r \times \tau_r) \to (G_r, \tau_r)$, we get $\Omega_{G_r}^{\leftarrow}(f_r^{\leftarrow}(\mu)) \in \tau_r \times \tau_r$. Now, by a similar way to the proof of Proposition 4.6, we have $\Omega_H^{\leftarrow}(\mu) = (f_r \times f_r)(\Omega_{G_r}^{\leftarrow}(f_r^{\leftarrow}(\mu)))$, where $\Omega_H : (H \times H, \sigma \times \sigma) \to (H, \sigma)$, moreover all $f_r \times f_r$ are open, hence $\Omega_H^{\leftarrow}(\mu) \in \sigma \times \sigma$. This proves that Ω_H is continuous and thus (H, σ) is a T-locality group. □

Now we are going to show that the T-locality group given in Propositions 4.6 and 4.7 fulfills the conditions of a final lift in the concrete category T-LocGrp.

Theorem 4.8. *Let $((G_r, \tau_r))_{r\in\Lambda}$ be a family of T-locality groups and for all $r \in \Lambda$, let $f_r : G_r \to H$ be a surjective homomorphism of a group G_r into H and let σ be the final I-topology of $(\tau_r)_{r\in\Lambda}$ with respect to $(f_r)_{r\in\Lambda}$. Then $(f_r : (G_r, \tau_r) \to (H, \sigma))_{r\in\Lambda}$ is a final lift of $(f_r : G_r \to H)_{r\in\Lambda}$ in the category T-LocGrp. The proof goes similarly, using Propositions 4.6 and 4.7 and the properties of final lift in the category T-LocGrp, as in case of Theorem 4.3.*

From Theorem 4.8 we get that the T-locality groups introduced in Propositions 4.6 and 4.7 can be considered as the final T-locality groups.

T-locality quotient group is special final T-locality group and hence the above implies the following result.

Corollary 4.9. *If N is a normal subgroup of a T-locality group (G, τ) and G/N is the corresponding quotient group and if $h : G \to G/N$ is the canonical homomorphism defined by $h(x) = xN$ for all $x \in G$, then the I-topological quotient space $(G/N, h(\tau))$ also is T-locality group, called a T-locality quotient group.*

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