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Fuzzy h-ideals with operators in Γ-hemirings

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Abstract. In this paper, we introduce the concept of fuzzy left h-ideals with operators in Γ-hemirings and establish a new fuzzy left h-ideal with operators. In particular, we consider the characterizations of M-Noetherian M-Γ-hemirings. Finally, we investigate cartesian products of M-fuzzy left h-ideals in M-Γ-hemirings.

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1. Introduction

Semirings play an important role in studying matrices and determinants. Many researchers studied the theory of matrices and determinants over semirings [11, 14]. A special semiring with a zero and endowed with the commutative addition is said to be a hemiring. Although ideals in semirings are useful for ways, they do not in general coincide with the usual ring ideals if S is a ring. Indeed, many results in rings apparently have no analogues in semirings using only ideals. We note that the ideals of semirings play a crucial role in the structure theory. Henriksen [9] defined a more restricted class of ideals in semirings, which is called k-ideals, a still more restricted of ideals in hemirings has been given by Iizuka [11]. In 2004, Jun [15] considered the fuzzy h-ideals of hemirings. By using the fuzzy h-ideals, Zhan et al. described the h-hemiregular hemirings [22]. Furthermore, many researchers gave some basic definitions and results related with fuzzy h-ideals of hemirings [3, 5, 20].

The concept of Γ-ring was first introduced in 1966 by Barnes [1], a concept more general than a ring. After the paper of Barnes, many researchers are engaged in studying some particular Γ-ring. In 1992, applying the concept of fuzzy sets to the theory of Γ-ring, Y. B. Jun and C. Y. Lee [13] gave the notion of fuzzy ideals in Γ-ring and some properties of fuzzy ideals of Γ-ring. After that, Hong and Jun

[10] defined the normalized fuzzy ideals and fuzzy maximal ideals in a Γ-ring and Jun [12] further characterized the fuzzy prime ideals of a Γ-ring. In particular, Dutta-Chanda studied the fuzzy ideals of a Γ-ring and characterized the Γ-fields and Notherian Γ-rings by considering the fuzzy ideals via operator rings of Γ-rings. The concept of Γ-semiring was introduced by M. K. Rao [18], these concepts are extended by Dutta and Sardar [7]. And some properties of such Γ-semiring have been studied, for example, see [8, 17, 19, 23].

Dudek [4, 6] discussed quasigroups and BCC-algebras with operators, respectively. In 2007, Zhan et al. [21] investigated fuzzy h-ideals with operators in hemirings. Now, in this paper, we consider this theory to Γ-hemirings, we introduce the concept of fuzzy left h-ideals with operators in M-Γ-hemirings S and establish a new fuzzy left h-ideal with operators. Using the left M-h-ideals, we establish M-fuzzy left h-ideals of S. Moreover, we introduce the concept of M-Noetherian M-Γ-hemirings and cartesian product of M-fuzzy left h-ideals, we prove that if μ and ν are M-fuzzy left h-ideals of S, then $\mu \times \nu$ is an M-fuzzy left h-ideal of $S \times S$.

2. Preliminaries

Definition 2.1 ([23])**.** Let S and Γ be two additive semigroups. Then S is said to be a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$) satisfies the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;
(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;
(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$;
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a zero of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with zero is said to be a Γ-hemiring.

A left ideal of a Γ-hemirings S is a subset A of S which is closed under the addition such that $S\Gamma A \subseteq A$, where $S\Gamma A = \{x\alpha y | x, y \in S, \alpha \in \Gamma\}$.

A left ideal A of Γ-hemirings S is called a left h-ideal of S, respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z$ implies $x \in A$.

Right h-ideals are defined similarly.

Definition 2.2 ([23])**.** A fuzzy set μ of Γ-hemirings S is said to be a fuzzy left h-ideal of S if it satisfies the following conditions:

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ *for all* $x, y \in S$,
(ii) $\mu(x\alpha y) \geq \mu(y)$ *for all* $x, y \in S$ *and* $\alpha \in \Gamma$,
(iii)$x + a + z = b + z$ *implies* $\mu(x) \geq \min\{\mu(a), \mu(b)\}$ *for all* $a, b, x, z \in S$.

Fuzzy right h-ideals of S are defined similarly.

Example 2.3 ([23])**.** Let S be a hemiring with the multiplicative identity 1. Then S is a Γ-hemirings, where $\Gamma = S$ and $a\alpha b$ denotes the product of elements a, α, b in S. Now any fuzzy h-ideal of the hemiring S is a fuzzy h-ideal of the Γ-hemiring S.

3. M-fuzzy left h-ideals

Definition 3.1. A Γ-hemiring S with operators is an algebraic system consisting of a Γ-hemiring S, a set M and a function defined in the product set $M \times \Gamma \times S$ and having values in S such that if the product $m\alpha x$ denotes the elements in S determined by the element m of M, x of S and the element α, β of Γ, then

$$m\alpha(x + y) = m\alpha x + m\alpha y$$

and

$$m\alpha(x\beta y) = (m\alpha x)\beta(m\alpha y)$$

hold for any $x, y \in S$, $m \in M$ and $\alpha, \beta \in \Gamma$. We usually use the phrase "S is an M-Γ-hemiring" instead of a "Γ-hemiring with operators".

Example 3.2. Let $(S, +)$ be a semigroup, where S is the sets of all non-negative integers and the operation ais the usual additive operation. Let $(\Gamma, +) = \{1\}$. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = a \cdot \alpha \cdot b$ for all $a, b \in S$ and $\alpha \in \Gamma$, where "$\cdot$" is usual multiplication. Then, it can be easily verified that S, under the above multiplication and the structure Γ-mapping, is a Γ-hemiring. We consider $M = \{0, 1\}$, then S is an M-Γ-hemiring.

Definition 3.3. A left h-ideal I of an M-Γ-hemiring S is called a left M-h-ideal of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 3.4. Let S be an M-Γ-hemiring and μ a fuzzy h-ideal of S. If the inequality $\mu(m\alpha x) \geqslant \mu(x)$ holds for any $x \in S$, $m \in M$ and $\alpha \in \Gamma$, then μ is said to be a fuzzy left h-ideal with operators of S. We use the phrases "an M-fuzzy left h-ideal of S" instead of "an fuzzy h-ideal with operators of S".

Proposition 3.5. *Let A be a non-empty subset of an M-Γ-hemiring S, and μ a fuzzy set in S defined by*

$$\mu(x) = \begin{cases} s, & if \quad x \in A; \\ t, & otherwise, \end{cases}$$

for all $x \in S$, where $s > t$ in $[0, 1]$. Then μ is an M-fuzzy left h-ideal of S if and only if A is a left M-h-ideal of S.

Proof. Suppose that A is a left M-h-ideal of S. we know that μ is a fuzzy left h-ideal of S. Let $x \in S$, $m \in M$ and $\alpha \in \Gamma$. If $x \in A$, then $m\alpha x \in A$ as A is a left M-h-ideal of S, and so $\mu(m\alpha x) = s = \mu(x)$. If $x \notin A$, then $\mu(x) = t \leqslant \mu(m\alpha x)$. Thus μ is an M-fuzzy left h-ideal of S.

Conversely, if μ is an M-fuzzy left h-ideal of S, then it is easy to show that A is a left h-ideal of S. Then, for any $x \in A$, $m \in M$ and $\alpha \in \Gamma$, $\mu(m\alpha x) \geqslant \mu(x) = s$ and so, $\mu(m\alpha x) = s$. This shows that $m\alpha x \in A$. Consequently, A is a left M-h-ideal of S. □

For any subset A of a Γ-hemiring S, χ_A will denote the characteristic function of A.

Corollary 3.6. *Let A be a non-empty subset of an M-Γ-hemirings S. Then A is a left M-h-ideal of S if and only if χ_A is an M-fuzzy left h-ideal of S.*

Proposition 3.7. *Let μ be an M-fuzzy left h-ideal of an M-Γ-hemiring S. For any $m \in M$, $\alpha \in \Gamma$, define a fuzzy set $\mu[m\alpha]$ in S by $\mu[m\alpha](x) = \mu(m\alpha x)$, $\forall x \in S$. Then $\mu[m\alpha]$ is a fuzzy left h-ideal of S.*

Proof. It is obvious. □

For any $t \in [0, 1]$, the set

$$U(\mu; t) = \{x \in S | \mu(x) \geq t\}$$

is called a level subset of μ.

The following is a simple consequence of the transfer principle for fuzzy sets in [16].

Lemma 3.8 ([23]). *A fuzzy set μ in a Γ-hemiring S is a fuzzy left h-ideal of S if and only if the each non-empty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is a left h-ideal of S.*

Theorem 3.9. *A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy left h-ideal of S if and only if the each non-empty level subset $U(\mu; t)$ of μ is a left M-h-ideal of S.*

Proof. Let μ be an M-fuzzy left h-ideal of S, and assume that $U(\mu; t) \neq \varnothing$ for $t \in [0, 1]$. Then by Lemma 3.8, $U(\mu; t)$ is a left h-ideal of S. For every $x \in U(\mu; t)$, $\alpha \in \Gamma$, and $m \in M$, we have

$$\mu(m\alpha x) \geqslant \mu(x) \geqslant t,$$

and so $m\alpha x \in U(\mu; t)$. Hence $U(\mu; t)$ is a left M-h-ideal of S.

Conversely, suppose that $U(\mu; t) \neq \varnothing$ is a left M-h-ideal of S. Then μ is a fuzzy left h-ideal of S by Lemma 3.8. Now assume that there exist $y \in S$, $\gamma \in \Gamma$ and $k \in M$ such that

$$\mu(k\gamma y) < \mu(y).$$

Taking

$$t_0 = \frac{1}{2}(\mu(k\gamma y) + \mu(y)),$$

we get $t_0 \in [0, 1]$ and

$$\mu(k\gamma y) < t_0 < \mu(y)$$

This implies that $k\gamma y \notin U(\mu; t_0)$ and $y \in U(\mu; t_0)$, this leads a contradiction. And therefor

$$\mu(k\gamma y) \geq \mu(y),$$

for all $y \in S$, $\gamma \in \Gamma$ and $k \in M$. This completes the proof. □

Proposition 3.10. *Let μ and ν be two fuzzy sets in an M-Γ-hemiring S. If they are M fuzzy left h-ideals of S, then so is $\mu \cap \nu$, where $\mu \cap \nu$ is defined by*

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad x \in S.$$

Proof. For $a, b \in S$,

$$\begin{aligned} (\mu \cap \nu)(a + b) &= \min\{\mu(a + b), \nu(a + b)\} \\ &\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\} \\ &= \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\} \\ &= \min\{(\mu \cap \nu)(a), (\mu \cap \nu)(b)\}. \end{aligned}$$

For all $\alpha \in \Gamma$, since $\mu(a\alpha b) \geq \mu(b)$, and $\nu(a\alpha b) \geq \mu(b)$, it follows that

$$\begin{aligned}(\mu\cap\nu)(a\alpha b) &= \min\{\mu(a\alpha b), \nu(a\alpha b)\}\\ &\geq \min\{\mu(b), \nu(b)\}\\ &= (\mu\cap\nu)(b).\end{aligned}$$

Now, $\mu\cap\nu$ is a fuzzy left ideal of S. Let $a, b, x, z \in S$ be such that $x+a+z = b+z$. Then

$$\begin{aligned}(\mu\cap\nu)(x) &= \min\{\mu(x), \nu(x)\}\\ &\geq \min\{\min\{\mu(a), \mu(b)\}, \min\{\nu(a), \nu(b)\}\}\\ &= \min\{\min\{\mu(a), \nu(a)\}, \min\{\mu(b), \nu(b)\}\}\\ &= \min\{(\mu\cap\nu)(a), (\mu\cap\nu)(b)\}.\end{aligned}$$

Therefore $\mu\cap\nu$ is a fuzzy left h-ideal of S. Let $m \in M$, $\alpha \in \Gamma$ we have

$$\begin{aligned}(\mu\cap\nu)(m\alpha x) &= \min\{\mu(m\alpha x), \nu(m\alpha x)\}\\ &\geq \min\{\mu(x), \nu(x)\}\\ &= (\mu\cap\nu)(x).\end{aligned}$$

Consequently, $\mu\cap\nu$ is an M-fuzzy left h-ideal of S. □

Theorem 3.11 ([23]). *Let $\{A_t | t \in \Lambda \subseteq [0,1]\}$ be a collection of M-h-ideals of an M-Γ-hemiring S such that*
(i) $S = \bigcup_{t\in\Lambda} A_t$,
(ii) $t < s$ if and only if $A_s \subset A_t$ for all $t, s \in \Lambda$.
Define a fuzzy set μ in S by

$$\mu(x) = \sup\{t \in \Lambda | x \in A_t\}, \quad \forall x \in S.$$

Then μ is an M-fuzzy left h-ideal of S.

Definition 3.12. An M-Γ-hemiring S is said to satisfy the ascending (descending) chain condition (briefly, $ACC(DCC)$) if for every ascending (descending) sequence $A_1 \subseteq A_2 \subseteq \cdots (A_1 \supseteq A_2 \supseteq \cdots)$ of left M-h-ideals of S there exists a nature number n such that

$$A_i = A_n, \quad \forall i \geq n.$$

Definition 3.13. An M-Γ-hemiring S is said to M-Noetherian if every left M-h-ideals of S satisfies ACC for left M-h-ideals.

Theorem 3.14. *Let $\{A_n | n \in N\}$ be a family of left M-h-ideals of an M-Γ-hemiring S which is nested, that is, $S = A_1 \supset A_2 \supset \cdots$ Let μ be a fuzzy set in S defined by*

$$\mu(x) = \begin{cases} \frac{n}{n+1}, & for \quad x \in A_n/A_{n+1}, \quad n = 1, 2, 3, \cdots; \\ 1, & for \quad x \in \bigcap_{n=1}^{\infty} A_n, \end{cases}$$

for all $x \in S$. Then μ is an M-fuzzy left h-ideal of S.

Proof. Suppose that $x \in A_k/A_{k+1}$ and $y \in A_r/A_{r+1}$ for $k = 1, 2, \cdots; r = 1, 2, \cdots$. Without loss of generality, we may assume that $k \leq r$. Then clearly $y \in A_k$, so $x + y \in A_k$. Hence

$$\mu(x+y) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}.$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x + y \in \bigcap_{n=1}^{\infty} A_n$, and clearly that

$$\mu(x+y) = 1 = \min\{\mu(x), \mu(y)\}.$$

If $x \in \bigcap_{n=1}^{\infty} A_n$, and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exits $l \in N$ such that $y \in A_l/A_{l+1}$, it follows that $x + y \in A_l$, so that

$$\mu(x+y) \geq \frac{l}{l+1} = \min\{\mu(x), \mu(y)\}.$$

Similarly, we know that

$$\mu(x+y) \geq \frac{l}{l+1} = \min\{\mu(x), \mu(y)\}.$$

whenever $x \notin \bigcap_{n=1}^{\infty} A_n$, and $y \in \bigcap_{n=1}^{\infty} A_n$.

Now if $y \in A_r/A_{r+1}$ for some $r = 1, 2, \cdots$, then $x\alpha y \in A_r$ for all $x \in S$ and $\alpha \in \Gamma$. Hence

$$\mu(x\alpha y) \geq \frac{r}{r+1} = \mu(y).$$

If $y \in \bigcap_{n=1}^{\infty} A_n$, then $x\alpha y \in \bigcap_{n=1}^{\infty} A_n$ for all $x \in S$ and $\alpha \in \Gamma$. So

$$\mu(x\alpha y) = 1 = \mu(y).$$

Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. If $a, b \in A_r/A_{r+1}$ for some $r = 1, 2, \cdots$, then $x \in A_r$ as A_r is a left M-h-ideal of S. Thus

$$\mu(x) \geq \frac{r}{r+1} = \min\{\mu(a), \mu(b)\}.$$

If $a, b \in \bigcap_{n=1}^{\infty} A_n$, then $x \in \bigcap_{n=1}^{\infty} A_n$, and so

$$\mu(x) = 1 = \min\{\mu(a), \mu(b)\}.$$

Assume that $a \in A_r/A_{r+1}$ for some $r = 1, 2, \cdots$, and $b \in \bigcap_{n=1}^{\infty} A_n$, (or, $a \in \bigcap_{n=1}^{\infty} A_n$ for some and $b \in A_r/A_{r+1}$ for some $r = 1, 2, \cdots$). Then $x \in A_r$, and so

$$\mu(x) = \frac{r}{r+1} = \min\{\mu(a), \mu(b)\}.$$

Consequently, μ is a fuzzy left h-ideal of S.

The last, let $x \in \bigcap_{n=1}^{\infty} A_n$, $m \in M$ and $\alpha \in \Gamma$. Then $\mu(x) = 1$ and $m\alpha x \in \bigcap_{n=1}^{\infty} A_n$, so

$$\mu(m\alpha x) = 1 = \mu(x).$$

If $x \in A_r/A_{r+1}$, $m \in M$ and $\alpha \in \Gamma$, then $m\alpha x \in A_r$, we have

$$\mu(m\alpha x) \geq \frac{r}{r+1} = \mu(x).$$

So, μ is an M-fuzzy left h-ideal of S. □

4. Cartesian product of M-fuzzy left h-ideals

A fuzzy relation on any set S is a fuzzy set $\mu : S \times S \longrightarrow [0, 1]$.

If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is a fuzzy relation on ν if $\mu(x, y) \leqslant \min\{\nu(x), \nu(y)\}, \quad \forall x, y \in S$.

Definition 4.1 ([2])**.** Let μ and ν be fuzzy sets in a set S. Then the Cartesian product of μ is defined by $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} \ \ \forall x, y \in S$.

Lemma 4.2 ([2])**.** *Let μ and ν be fuzzy sets in a set S. Then*

(i) $\mu \times \nu$ is a fuzzy relation on S,

(ii) $U(\mu \times \nu; t) = U(\mu; t) \times U(\nu; t)$ for all $t \in [0, 1]$.

Definition 4.3 ([2]). If ν is a fuzzy set in a set S, then the strongest fuzzy relation on S that is a fuzzy relation on ν is μ_ν, which is given by $\mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}\ \ \forall x,y \in S$.

Lemma 4.4 ([2]). *For a given fuzzy set ν on a set S, let μ_ν be strongest fuzzy relation on S. Then for $t \in [0,1]$, we have that $U(\mu_\nu;t) = U(\nu;t) \times U(\nu;t)$.*

The following proposition is an immediate consequence of Lemma 4.4, and we omit the proof.

Proposition 4.5. *If ν is a fuzzy left h-ideal of an M- Γ-hemiring S. Then the level left h-ideals of μ_ν are given by $U(\mu_\nu;t) = U(\nu;t) \times U(\nu;t)\ \ \forall t \in [0,1]$.*

Let S_1 and S_2 be two M-Γ-hemirings. Now we can easy to check that $S_1 \times S_2$ is an M-Γ-hemiring by the operations which we define as follows:
(*i*) $(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2)$;
(*ii*) $(x_1,x_2)\alpha(y_1,y_2) = (x_1\alpha y_1, x_2\alpha y_2)$;
(*iii*) $m\alpha(x_1,x_2) = (m\alpha x_1, m\alpha x_2)$,
for all $x_1,x_2,y_1,y_2 \in S$, $\alpha \in \Gamma$ and $m \in M$.

Theorem 4.6. *Let μ and ν be M-fuzzy left h-ideals of an M-Γ-hemiring S. Then $\mu \times \nu$ is an M-fuzzy left h-ideal of $S \times S$.*

Proof. Let $(x_1,x_2),(y_1,y_2) \in S \times S$ and $\alpha \in \Gamma$. Then we have

$$\begin{aligned}(\mu\times\nu)((x_1,x_2)+(y_1,y_2)) &= (\mu\times\nu)(x_1+y_1,x_2+y_2)\\ &= \min\{\mu(x_1+y_1),\nu(x_2+y_2)\}\\ &\geq \min\{\min\{\mu(x_1),\mu(y_1)\},\min\{\nu(x_2),\nu(y_2)\}\}\\ &= \min\{\min\{\mu(x_1),\nu(x_2)\},\min\{\mu(y_1),\nu(y_2)\}\}\\ &= \min\{(\mu\times\nu)(x_1,x_2),(\mu\times\nu)(y_1,y_2)\},\end{aligned}$$

and then

$$\begin{aligned}(\mu\times\nu)((x_1,x_2)\alpha(y_1,y_2)) &= (\mu\times\nu)(x_1\alpha y_1,x_2\alpha y_2)\\ &= \min\{\mu(x_1\alpha y_1),\nu(x_2\alpha y_2)\}\\ &\geq \min\{\mu(y_1),\nu(y_2)\}\\ &= (\mu\times\nu)(y_1,y_2).\end{aligned}$$

Now let $(a_1,a_2),(b_1,b_2),(x_1,x_2),(z_1,z_2) \in S \times S$ be such that

$$(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2),$$

i.e. $(x_1+a_1+z_1, x_2+a_2+z_2) = (b_1+z_1, b_2+z_2)$, it follows that

$$x_1+a_1+z_1 = b_1+z_1, x_2+a_2+z_2 = b_2+z_2,$$

so that

$$\begin{aligned}(\mu\times\nu)(x_1,x_2) &= \min\{\mu(x_1),\nu(x_2)\}\\ &\geq \min\{\min\{\mu(a_1),\mu(b_1)\},\min\{\nu(a_2),\nu(b_2)\}\}\\ &= \min\{\min\{\mu(a_1),\nu(a_2)\},\min\{\mu(b_1),\nu(b_2)\}\}\\ &= \min\{(\mu\times\nu)(a_1,a_2),(\mu\times\nu)(b_1,b_2)\}.\end{aligned}$$

Therefor $\mu \times \nu$ is a fuzzy left h-ideal of $S \times S$. Now let $x = (x_1, x_2) \in S \times S, m \in M$, then

$$\begin{aligned} (\mu \times \nu)(m\alpha x) &= (\mu \times \nu)(m\alpha(x_1, x_2)) \\ &= (\mu \times \nu)(m\alpha x_1, m\alpha x_2) \\ &= \min\{\mu(m\alpha x_1), \nu(m\alpha x_2)\} \\ &\geq \min\{\mu(x_1), \nu(x_2)\} \\ &= (\mu \times \nu)(x_1, x_2) \\ &= (\mu \times \nu)(x). \end{aligned}$$

Hence, $\mu \times \nu$ is an M-fuzzy left h-ideal of $S \times S$. □

Theorem 4.7. *Let ν be a fuzzy set in an M-Γ-hemiring S and let μ_ν be the strongest fuzzy relation on S. Then ν is an M-fuzzy left h-ideal of S if and only if μ_ν is an M-fuzzy left h-ideal of $S \times S$.*

Proof. Assume that ν is an M-fuzzy left h-ideal of S. Let $(x_1, x_2), (y_1, y_2) \in S \times S$, and $\alpha \in \Gamma$. Then

$$\begin{aligned} \mu_\nu((x_1, x_2) + (y_1, y_2)) &= \mu_\nu(x_1 + y_1, x_2 + y_2) \\ &= \min\{\nu(x_1 + y_1), \nu(x_2 + y_2)\} \\ &\geq \min\{\min\{\nu(x_1), \nu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} \\ &= \min\{\min\{\nu(x_1), \nu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\ &= \min\{\mu_\nu(x_1, x_2), \mu_\nu(y_1, y_2)\}, \end{aligned}$$

and

$$\begin{aligned} \mu_\nu((x_1, x_2)\alpha(y_1, y_2)) &= \mu_\nu(x_1\alpha y_1, x_2\alpha y_2) \\ &= \min\{\nu(x_1\alpha y_1), \nu(x_2\alpha y_2)\} \\ &\geq \min\{\nu(y_1), \nu(y_2)\} \\ &= \mu_\nu(y_1, y_2). \end{aligned}$$

Now let $(a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S \times S$ be such that

$$(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2).$$

So

$$x_1 + a_1 + z_1 = b_1 + z_1, x_2 + a_2 + z_2 = b_2 + z_2.$$

Thus

$$\begin{aligned} \mu_\nu(x_1, x_2) &= \min\{\nu(x_1), \nu(x_2)\} \\ &\geq \min\{\min\{\nu(a_1), \nu(b_1)\}, \min\{\nu(a_2), \nu(b_2)\}\} \\ &= \min\{\min\{\nu(a_1), \nu(a_2)\}, \min\{\nu(b_1), \nu(b_2)\}\} \\ &= \min\{\mu_\nu(a_1, a_2), \mu_\nu(b_1, b_2)\}. \end{aligned}$$

Therefor μ_ν is a fuzzy left h-ideal of $S \times S$. Now, for any $(x_1, x_2) \in S \times S, m \in M$, we have

$$\begin{aligned} \mu_\nu(m\alpha(x_1, x_2)) &= \mu_\nu(m\alpha x_1, m\alpha x_2) \\ &= \min\{\nu(m\alpha x_1), \nu(m\alpha x_2)\} \\ &\geq \min\{\nu(x_1), \nu(x_2)\} \\ &= \mu_\nu(x_1, x_2). \end{aligned}$$

Thus μ_ν is an M-fuzzy left h-ideal of $S \times S$.

Conversely, suppose that μ_ν is an M-fuzzy left h-ideal of $S \times S$. By Theorem 4.12 in [15], we know that ν is a fuzzy left h-ideal of S. Now, for any $x_1, x_2, y_1, y_2 \in S$, and $\alpha \in \Gamma$, by using Proposition 4.7 in [15], we have

$$\begin{aligned} \min\{\nu(x_1\alpha y_1), \nu(x_2\alpha y_2)\} &= \mu_\nu(x_1\alpha y_1, x_2\alpha y_2) \\ &= \mu_\nu((x_1, x_2)\alpha(y_1, y_2)) \\ &\geq \mu_\nu(y_1, y_2) \\ &= \min\{\nu(y_1), \nu(y_2)\}, \end{aligned}$$

and so $\nu(x_1\alpha y_1) \geq \min\{\nu(y_1), \nu(y_2)\}$. Taking $x_1 = x, y_1 = y$ and $y_2 = 0$, we get $\nu(x\alpha y) \geq \min\{\nu(y), \nu(0)\} = \nu(y)$.

Then let $m \in M$, we have

$$\begin{aligned} \min\{\nu(m\alpha x_1), \nu(m\alpha x_2)\} &= \mu_\nu(m\alpha x_1, m\alpha x_2) \\ &= \mu_\nu(m\alpha(x_1, x_2)) \\ &\geq \mu_\nu(x_1, x_2) \\ &= \min\{\nu(x_1), \nu(x_2)\}. \end{aligned}$$

Taking $x_1 = x_2 = x$, we have $\nu(m\alpha x) \geq \nu(x)$. Consequently, ν is an M-fuzzy left h-ideal of S. □

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