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On soft Hausdorff spaces

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Abstract. The aim of this paper is to study some properties of soft Hausdorff space introduced by Shabir and Naz. Firstly, we give a representation of soft sets and soft topological spaces. Secondly, we introduce some new concepts in soft topological space such as convergence of sequences, homeomorphism and investigate the relations between these concepts and Hausdorff axiom in soft topological space.

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1. Introduction

The concept of soft set theory has been initiated by Molodtsov [15] in 1999 as a general mathematical tool for modeling uncertainties. By a soft set we mean a pair (F, E), where E is a set interpreted as the set of parameters and the mapping $F : E \to P(X)$ is referred to as the soft structure on X. After the introduction of the notion of soft sets several researchers improved this concept. Maji et. al. [10, 11] pointed out several directions for the applications of soft sets. Aktaş and Çağman [1] introduced the soft group and also compared soft sets to fuzzy set and rough set. Babitha and Sunil [4] studied the soft set relation and discussed some related concepts. Jun [8] applied soft sets to the theory of BCK/BCI- algebras and reported the concept of soft BCK/BCI-algebra. Maji et al. [12] presented the concept of fuzzy soft set by combining the fuzzy set and soft set. Ahmad and Kharal [9] revised and improved some results in fuzzy soft set theory. Researches on soft set and fuzzy soft set theory have been progressing rapidly in several directions, some applications are presented in [2, 6, 13, 18]. Topological structure of soft sets also was studied by many authors: Shabir and Naz [19] defined the soft topological space and studied the concepts of soft open set, soft interior point, soft neighborhood of a point, soft separation axioms and subspace of a soft topological space. Aygünoğlu and Aygün [3] introduced the soft continuity of soft mapping, soft product topology and studied soft compactness and generalized Tychonoff theorem to the soft topological space. Min [14] gave some results on soft topological spaces. Zorlutuna et al. [21] also investigated soft interior point and soft neighborhood. Hussain and Ahmad [7] investigated and discussed the properties of soft interior, closure and boundary on soft topology. As a different approach to soft topology Varol et al. [16] interpreted a classical topology as a soft set over the powerset $P(X)$ and characterized also some other categories related to topology (crisp and fuzzy) as subcategories of the category of soft sets. The concept of fuzzy soft topology which is based on Chang' s fuzzy topology was introduced by Tanay and Kandemir [20]. After then, Varol and Aygün [17] investigated the topological structure of fuzzy soft sets.

In the present paper we begin, as a preliminaries, with the basic concepts of soft set theory. We recall some notions of soft topological spaces such as product soft topology, soft neighborhood, soft Hausdorff space etc. We define convergence of sequences in soft topological space, give some examples and see that a sequence converges to a unique point in soft Hausdorff space. However, the converse is not true in general and this is shown with the help of an example. We also give the definition of diagonal soft set and characterize the concept of Hausdorffness with this soft set. Finally, we show that compact soft set is closed in soft Hausdorff space.

2. Preliminaries

In this chapter we give some preliminaries about soft set most of them found in [5, 9, 19]. We make some small modifications to some of them in order to make theoretical study in detail. Throughout this paper, X refers to an initial universe, E is the set of all parameters for X and $A \subseteq E$.

Definition 2.1 ([5, 13])**.** A soft set F_A on the universe X is defined by the set of ordered pairs

$$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(X)\},$$

where $F_A : E \longrightarrow P(X)$, such that $F_A(e) \neq \varnothing$, if $e \in A \subseteq E$ and $F_A(e) = \varnothing$ if $e \notin A$.

The subscript A in the notation F_A indicates where the image of F_A is non-empty. The value $F_A(e)$ is a set called $e-$element of the soft set for all $e \in E$.

From now on, we will use $\mathcal{S}(X, E)$ instead of the all soft sets over X.

Definition 2.2 ([5])**.** The soft set $F_\varnothing \in \mathcal{S}(X, E)$ is called null soft set, denoted by Φ. Here, $F_\varnothing(e) = \varnothing$ for every $e \in E$.

Definition 2.3 ([5])**.** Let $F_A \in \mathcal{S}(X, E)$. If $F_A(e) = X$, $\forall e \in A$, then F_A is called $A-$absolute soft set, denoted by $\widetilde{A}$. If $A = E$, then the $A-$absolute soft set is called absolute soft set denoted by $\widetilde{E}$.

Definition 2.4 ([5])**.** Let $F_A, G_B \in \mathcal{S}(X, E)$. F_A is a soft subset of G_B, denoted $F_A \sqsubseteq G_B$ if $F_A(e) \subseteq G_B(e)$, for each $e \in E$.

Definition 2.5 ([5]). Let $F_A, G_B \in \mathcal{S}(X,E)$. F_A and G_B are soft equal, denoted by $F_A = G_B$ if $F_A \sqsubseteq G_B$ and $G_B \sqsubseteq F_A$.

Definition 2.6 ([5]). Let $F_A, G_B \in \mathcal{S}(X,E)$. Union of F_A and G_B is a soft set H_C defined by $H_C(e) = F_A(e) \cup G_B(e)$, $\forall e \in E$, where $C = A \cup B$.

That is, $H_C = F_A \sqcup G_B$.

Definition 2.7 ([5]). Let $F_A, G_B \in \mathcal{S}(X,E)$. Intersection of F_A and G_B is a soft set H_C defined by $H_C(e) = F_A(e) \cap G_B(e)$, $\forall e \in E$, where $C = A \cap B$.

That is, $H_C = F_A \sqcap G_B$.

Definition 2.8 ([5]). Let $F_A \in \mathcal{S}(X,E)$. The complement of F_A, denoted by F_A^c is defined by $F_A^c(e) = X - F(e)$.

Theorem 2.9 ([5, 10]). *Let J be an index set and $F_A,\ G_B,\ H_C, (F_A)_i, (G_B)_i \in \mathcal{S}(X,E)$ $\forall i \in J$. Then*

(1) $F_A \sqcap F_A = F_A, \quad F_A \sqcup F_A = F_A$.
(2) $F_A \sqcap G_B = G_B \sqcap F_A, \quad F_A \sqcup G_B = G_B \sqcup F_A$.
(3) $F_A \sqcup (G_B \sqcup H_C) = (F_A \sqcup G_B) \sqcup H_C, \quad F_A \sqcap (G_B \sqcap H_C) = (F_A \sqcap G_B) \sqcap H_C$.
(4) $F_A = F_A \sqcup (F_A \sqcap G_B), \quad F_A = F_A \sqcap (F_A \sqcup G_B)$.
(5) $F_A \sqcap \left(\bigsqcup_{i\in J} (G_B)_i\right) = \bigsqcup_{i\in J} (F_A \sqcap (G_B)_i)$.
(6) $F_A \sqcup \left(\bigsqcap_{i\in J} (G_B)_i\right) = \bigsqcap_{i\in J} (F_A \sqcup (G_B)_i)$
(7) $\Phi \sqsubseteq F_A \sqsubseteq \widetilde{A} \sqsubseteq \widetilde{E}$.
(8) $(F_A^c)^c = F_A$.
(9) $\left(\bigsqcap_{i\in J} (F_A)_i\right)^c = \bigsqcup_{i\in J} (F_A)_i^c$
(10) $\left(\bigsqcup_{i\in J} (F_A)_i\right)^c = \bigsqcap_{i\in J} (F_A)_i^c$
(11) *If $F_A \sqsubseteq G_B$, then $G_B^c \sqsubseteq F_A^c$.*
(12) $F_A \sqcup F_A^c = \widetilde{E}$, $F_A \sqcap F_A^c = \Phi$.

Definition 2.10 ([19]). Let $F_A \in \mathcal{S}(X,E)$ and $x \in X$. $x \widetilde{\in} F_A$ read as x belongs to the soft set F_A whenever $x \in F_A(e)$ for all $e \in A$.

For any $x \in X$, $x \widetilde{\notin} F_A$ if $x \notin F_A(e)$ for some $e \in A$.

Definition 2.11 ([19]). Let $x \in X$, then x_E denotes the soft set over X for which $x_E(e) = \{x\}$ for all $e \in E$.

Definition 2.12 ([9]). Let $\mathcal{S}(X,E)$ and $\mathcal{S}(Y,K)$ be the families of all soft sets over X and Y, respectively. The mapping (φ,ψ) is called a soft mapping from X to Y, where $\varphi : X \longrightarrow Y$ and $\psi : E \longrightarrow K$ are two mappings.

(1) Let $F_A \in \mathcal{S}(X,E)$, then the image of F_A under the soft mapping (φ,ψ) is the soft set over Y denoted by $(\varphi,\psi)(F_A)$ and defined by

$$(\varphi,\psi)(F_A)(k) = \begin{cases} \bigcup\limits_{e\in\psi^{-1}(k)\cap A} \varphi(F_A(e)), & \text{if } \psi^{-1}(k)\cap A \neq \varnothing; \\ \varnothing, & \text{otherwise.} \end{cases}.$$

(2) Let $G_B \in \mathcal{S}(Y,K)$, then the preimage of G_B under the soft mapping (φ,ψ) is the soft set over X denoted by $(\varphi,\psi)^{-1}(G_B)$, where

$$(\varphi,\psi)^{-1}(G_B)(e) = \begin{cases} \varphi^{-1}(G_B(\psi(e))), & \text{if } \psi(e) \in B; \\ \varnothing, & \text{otherwise.} \end{cases}.$$

If (φ, ψ) is a soft mapping from X to Y and $(\varphi^{'}, \psi^{'})$ is a soft mapping from Y to Z than the composition of (φ, ψ) and $(\varphi^{'}, \psi^{'})$ is denoted by $(\varphi^{'}, \psi^{'}) \circ (\varphi, \psi)$ and defined by $(\varphi^{'}, \psi^{'}) \circ (\varphi, \psi) := (\varphi^{'} \circ \psi, \varphi \circ \psi^{'})$.

The soft mapping (φ, ψ) is called injective, if φ and ψ are injective. The soft mapping (φ, ψ) is called surjective, if φ and ψ are surjective.

Theorem 2.13 ([9, 21])**.** *Let X and Y crisp sets F_A, $(F_A)_i \triangleq (F_i)_{A_i} \in \mathcal{S}(X, E)$ and G, $(G_B)_i \triangleq (G_i)_{B_i} \in \mathcal{S}(Y, K)$ $\forall i \in J$, where J is an index set.*

(1) *If $(F_A)_1 \sqsubseteq (F_A)_2$, then $(\varphi, \psi)((F_A)_1) \sqsubseteq (\varphi, \psi)((F_A)_2)$.*
(2) *If $G_1 \sqsubseteq G_2$, then $(\varphi, \psi)^{-1}((G_B)_1) \sqsubseteq (\varphi, \psi)^{-1}((G_B)_2)$.*
(3) *$F_A \sqsubseteq (\varphi, \psi)^{-1}\left((\varphi, \psi)(F_A)\right)$, the equality holds if (φ, ψ) is injective.*
(4) *$(\varphi, \psi)\left((\varphi, \psi)^{-1}(F_A)\right) \sqsubseteq F_A$, the equality holds if (φ, ψ) is surjective.*
(5) *$(\varphi, \psi)\left(\bigsqcup_{i \in J}(F_A)_i\right) = \bigsqcup_{i \in J}(\varphi, \psi)((F_A)_i)$.*
(6) *$(\varphi, \psi)\left(\bigsqcap_{i \in J}(F_A)_i\right) \sqsubseteq \bigsqcap_{i \in J}(\varphi, \psi)((F_A)_i)$, the equality holds if (φ, ψ) is injective.*
(7) *$(\varphi, \psi)^{-1}\left(\bigsqcup_{i \in J}(G_B)_i\right) = \bigsqcup_{i \in J}(\varphi, \psi)^{-1}((G_B)_i)$.*
(8) *$(\varphi, \psi)^{-1}\left(\bigsqcap_{i \in J}(G_B)_i\right) = \bigsqcap_{i \in J}(\varphi, \psi)^{-1}((G_B)_i)$*
(9) *$(\varphi, \psi)^{-1}\left(\widetilde{E}_Y\right) = \widetilde{E}_X, \quad (\varphi, \psi)^{-1}(\Phi_Y) = \Phi_X$.*
(10) *$(\varphi, \psi)\left(\widetilde{E}_X\right) = \widetilde{E}_Y$ if (φ, ψ) is surjective.*
(11) *$(\varphi, \psi)(\Phi_X) = \Phi_Y$.*

Definition 2.14 ([4])**. [Construction of the product]**
Let $F_A \in \mathcal{S}(X, E)$ and $G_B \in \mathcal{S}(Y, K)$. The cartesian product $F_A \times G_B$ is defined by $(F \times G)_{(A \times B)}$ where

$$(F \times G)_{(A \times B)}(e, k) = F_A(e) \times G_B(k), \forall (e, k) \in A \times B.$$

According to this definition the soft set $F_A \times G_B$ is a soft set over $X \times Y$ and its parameter universe is $E \times K$.

The pairs of projections $p_X : X \times Y \to X$, $q_E : E \times K \to E$ and $p_Y : X \times Y \to Y$, $q_K : E \times K \to K$ determine morphisms respectively (p_X, q_E) from $X \times Y$ to X and (p_Y, q_K) from $X \times Y$ to Y, where
$(p_X, q_E)(F_A \times G_B) = p_X(F \times G)_{q_E(A \times B)} = F_A$ and
$(p_Y, q_K)(F_A \times G_B) = p_Y(F \times G)_{q_K(A \times B)} = G_B$. [3]

3. Soft topological spaces

3.1. **Soft topological spaces.** In this section we give some basic results of soft topological spaces which we need next section.

Definition 3.1 ([19])**.** A soft topology $\mathcal{T}$ is a family of soft sets over X satisfying the following properties.

(1) $\Phi, \widetilde{E} \in \mathcal{T}$
(2) If $F_A,\ G_B \in \mathcal{T}$, then $F_A \sqcap G_B \in \mathcal{T}$
(3) If $(F_A)_\lambda \in \mathcal{T}, \forall \lambda \in \Lambda$, then $\bigsqcup_{\lambda \in \Lambda}(F_A)_\lambda \in \mathcal{T}$.

$(X, \mathcal{T})$ is called a soft topological space. Every member of $\mathcal{T}$ is called soft open.
A soft set G_B is called soft closed in $(X, \mathcal{T})$ if $G_B^c \in \mathcal{T}$.

Indiscrete soft topology, denoted by $\mathcal{T}^0$ contains only Φ and $\widetilde{E}$ while the discrete soft topology, denoted by $\mathcal{T}^1$ contains all soft sets over X.

Example 3.2 ([3]). Let $\mathbb{R}$ be the real numbers, $E = \mathbb{R}^+$ be the positive real numbers and $(F_E)_\lambda = \{(x, (x-\lambda, x+\lambda)) : x \in E\}$. If $\mathcal{T} = \{(F_E)_\lambda : \lambda \in E\} \cup \{\Phi, \widetilde{E}\}$, then the pair $(\mathbb{R}, \mathcal{T})$ is a soft topological space.

Example 3.3. Let $\mathbb{R}$ be the real numbers and E be a countable set. Consider the family $\mathcal{T}_* = \{F_A : \bigcup_{e\in E} X \setminus F_A(e)\ \textit{is countable}\} \cup \{\Phi\}$, then the pair $(\mathbb{R}, \mathcal{T}_*)$ is a soft topological space.

Definition 3.4 ([3]). Let $(X, \mathcal{T})$ be a soft topological space. A subcollection $\mathcal{B}$ of $\mathcal{T}$ is called a base for $\mathcal{T}$ if every member of $\mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$.

Definition 3.5 ([3]). Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two soft topological spaces. A soft mapping $(\varphi, \psi) : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is called soft continuous if $(\varphi, \psi)^{-1}(G_B) \in \mathcal{T}_1,\ \forall G_B \in \mathcal{T}_2$.

If $(\varphi, \psi) : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ and $(\varphi', \psi') : (Y, \mathcal{T}_2) \longrightarrow (Z, \mathcal{T}_3)$ are continuous then clearly $(\varphi', \psi') \circ (\varphi, \psi)$ is soft continuous.

A soft mapping $(\varphi, \psi) : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is called soft open if $(\varphi, \psi)(F_A) \in \mathcal{T}_2,\ \forall F_A \in \mathcal{T}_1$.

Definition 3.6 ([3]). Let $(X, \mathcal{T})$ be a soft topological space. A subcollection $\mathcal{S}$ of $\mathcal{T}$ is said to be a subbase for $\mathcal{T}$ if the family of all finite intersections of members of $\mathcal{S}$ forms a base for $\mathcal{T}$.

Theorem 3.7 ([3]). *Let $\mathcal{S}$ be a family of soft set over X such that $\Phi, \widetilde{E} \in \mathcal{S}$. Then $\mathcal{S}$ is a base for the topology $\mathcal{T}$, whose members are of the form* $\bigsqcup_{i\in J}\left(\bigsqcap_{\lambda\in\Lambda_i}(F_A)_{i,\lambda}\right)$ *where J is arbitrary index set and for each $i \in J$, Λ_i is a finite index set, $(F_A)_{i,\lambda} \in \mathcal{S}$ for $i \in J$ and $\lambda \in \Lambda_i$.*

Definition 3.8 ([3]). Let $\{(\varphi, \psi)_i : \mathcal{S}(X, E) \longrightarrow (Y_i, \mathcal{T}_i)\}_{i\in J}$ be a family of soft mappings and $\{(Y_i, \mathcal{T}_i)\}_{i\in J}$ is a family of soft topological spaces. Then the topology $\mathcal{T}$ generated from the subbase $\mathcal{S} = \{(\varphi, \psi)_i^{-1}(F_A) : F_A \in \mathcal{T}_i, i \in J\}$ is called the soft topology (or initial soft topology) induced by the family of soft mappings $\{(\varphi, \psi)_i\}_{i\in J}$.

Theorem 3.9 ([3]). *The initial soft topology $\mathcal{T}$ on X induced by the family $\{(\varphi, \psi)_i\}$ from X to Y_i, respectively, is the coarsest soft topology making $(\varphi, \psi)_i : (X, \mathcal{T}) \longrightarrow (Y_i, \mathcal{T}_i)$ continuous, $\forall i \in J$.*

Definition 3.10 ([3]). Let $\{(X, \mathcal{T}_i)\}_{i\in J}$ be a family of soft topological spaces. Then the initial soft topology on X $(= \prod_{i\in J} X_i)$ generated by the family $\{(p, q)_i\}_{i\in J}$ is called product soft topology on X.

The product soft topology is denoted by $\prod_{i\in J} \mathcal{T}_i$.

Definition 3.11 ([21]). Let $(X, \mathcal{T})$ be a soft topological space and $F_A \in \mathcal{S}(X, E)$. The soft interior of F_A is the soft set $(F_A)^o = \sqcup\{G_B : G_B\ \textit{is soft open set and}\ G_B \sqsubseteq F_A\}$.

Proposition 3.12 ([21]). *Let $(X, \mathcal{T})$ be a soft topological space and $F_A \in \mathcal{S}(X, E)$. F_A is soft open set iff $F_A = (F_A)^o$.*

Definition 3.13 ([19])**.** Let $(X, \mathcal{T})$ be a soft topological space, F_A be a soft set over X and $x \in X$. Then x is called a soft interior point of F_A if there exists a soft open set G_B such that $x \widetilde{\in} G_B \sqsubseteq F_A$.

Definition 3.14 ([19])**.** Let $(X, \mathcal{T})$ be a soft topological space, F_A be a soft set over X and $x \in X$. Then F_A is called a soft neighborhood of x if there exists a soft open set G_B such that $x \widetilde{\in} G_B \sqsubseteq F_A$.

The neighborhood system of a point x, denoted by $\mathcal{N}_{\mathcal{T}}(x)$, is the family of all its neighborhoods.

Proposition 3.15 ([19])**.** *Let $(X, \mathcal{T})$ be a soft topological space. The neighborhood system $\mathcal{N}_{\mathcal{T}}(x)$ in $(X, \mathcal{T})$ has the following properties.*

(1) *Each $x \in X$ has a soft neighborhood.*
(2) *If $F_A, G_B \in \mathcal{N}_{\mathcal{T}}(x)$, then $F_A \sqcap G_B \in \mathcal{N}_{\mathcal{T}}(x)$.*
(3) *If $F_A \in \mathcal{N}_{\mathcal{T}}(x)$ and $F_A \sqsubseteq G_B$, then $G_B \in \mathcal{N}_{\mathcal{T}}(x)$.*

Definition 3.16 ([19])**.** Let $(X, \mathcal{T})$ be a soft topological space and M be a non-empty subset of X. The set $\mathcal{T}_M = \{\widetilde{E}_M \sqcap F_A : F_A \in \mathcal{T}\}$ is called the soft relative topology on M and $(M, \mathcal{T}_M)$ is called soft subspace of $(X, \mathcal{T})$.

Proposition 3.17 ([19])**.** *Let $(M, \mathcal{T}_M)$ be a soft subspace of $(X, \mathcal{T})$ and $F_A \in \mathcal{S}(X, E)$. Then F_A is soft open in M if and only if $F_A = \widetilde{E}_M \sqcap G_B$, for some $G_B \in \mathcal{T}$.*

Definition 3.18. Let $(X, \mathcal{T})$ be a soft topological space, $(x_n) \subset X$ be a sequence and $x_0 \in X$. (x_n) is called converges to x_0 in $(X, \mathcal{T})$ if for all $F_A \in \mathcal{N}(x_0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \widetilde{\in} F_A$ for all $n \geq n_0$.

Example 3.19. Let $(\mathbb{R}, \mathcal{T}^0)$ be an indiscrete soft topological space and (x_n) converges to x_0 in $(\mathbb{R}, \mathcal{T}^0)$. In this topological space $\mathcal{N}(x_0) = \{\widetilde{E}\}$, so $x_n \widetilde{\in} \widetilde{E}$ for all $n \in \mathbb{N}$. Hence, every sequence converges to every point in $(\mathbb{R}, \mathcal{T}^0)$.

Example 3.20. Let $(\mathbb{R}, \mathcal{T}_*)$ be a soft topological space which is defined in Example 3.3 and let (x_n) converges to x_0 in $(\mathbb{R}, \mathcal{T}_*)$. Then for all $G_B \in \mathcal{N}(x_0)$ there exists $n_0 \in \mathbb{N}$ such that $x_n \widetilde{\in} G_B$ for all $n \geq n_0$. Let define the soft set $F_A : E \to P(X)$ by

$$F_A(e) = \{x_n : x_n \neq x_0 \text{ and } n \in \mathbb{N}\}, \ \forall e \in E.$$

From here, $x_0 \notin F_A(e)$ and $x_0 \in X \setminus F_A(e)$. So, $x_0 \widetilde{\in} F_A^c \in \mathcal{T}_*$. Let take $G_B := F_A^c \in \mathcal{N}(x_0)$. Then $x_n \widetilde{\in} F_A^c$, $\forall n \geq n_0$ and we obtain $x_n \widetilde{\notin} F_A$, $\forall n \geq n_0$. Hence, $x_n = x_0$, $\forall n \geq n_0$.

3.2. **Soft Hausdorff spaces.**

Definition 3.21 ([19])**.** Let $(X, \mathcal{T})$ be a soft topological space and $x, y \in X$ such that $x \neq y$. $(X, \mathcal{T})$ is called soft Hausdorff space or soft T_2 space if there exist soft open sets F_A and G_B such that $x \widetilde{\in} F_A$, $y \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$.

Example 3.22. Let $\mathbb{R}$ be the real numbers, $E = \mathbb{R}$ and

$$(F_E)_y = \{(x, (x, y]) : x, y \in E \text{ and } x < y\}.$$

If $\mathcal{T} = \{(F_E)_y : y \in E\} \cup \{\Phi, \widetilde{E}\}$, then the pair $(\mathbb{R}, \mathcal{T})$ is a soft topological space. Moreover, $(\mathbb{R}, \mathcal{T})$ is a soft Hausdorff space.

Definition 3.23. Let $F_A \in \mathcal{S}(X,E)$, $x \in X$ and $A \subseteq E$. Then $(F_A)_\Delta$ denotes the soft set over $X \times X$ for which $(F_A)_\Delta : E \to P(X \times X)$ and

$$(F_A)_\Delta(e) = \Delta = \{(x,x) : x \in X\}$$

if $e \in A$, and $(F_A)_\Delta(e) = \varnothing$ if $e \notin A$. $(F_A)_\Delta$ is called $A-$diagonal soft set. If $A = E$, then it is called diagonal soft set.

Theorem 3.24. *$(X,\mathcal{T})$ is soft Hausdorff space if and only if the soft diagonal set $(F_A)_\Delta$ is soft closed.*

Proof. Let X be a soft Hausdorff space. We must show that $(F_A)^c_\Delta$ is soft open. Suppose that $(x_1,x_2)\widetilde{\in}(F_A)^c_\Delta$. Then $(x_1,x_2)\widetilde{\notin}(F_A)_\Delta$ and for some $e \in E, (x_1,x_2) \notin (F_A)_\Delta(e)$. Thus, we have $x_1 \neq x_2$. Since X is soft Hausdorff space, there exist $G_B, H_C \in \mathcal{T}$ such that $x_1\widetilde{\in}G_B$, $x_2\widetilde{\in}H_C$ and $G_B \sqcap H_C = \Phi$. So, for each $e \in E$, $x_1 \in G_B(e)$, $x_2 \in H_C(e)$ and $G_B(e) \cap H_C(e) = \varnothing$. This implies that $(x_1,x_2) \in G_B(e) \times H_C(e)$ and $(G_B(e) \times H_C(e)) \cap (F_A)_\Delta(e) = \varnothing$. Hence, $(x_1,x_2)\widetilde{\in}G_B \times H_C$ and $(G_B \times H_C) \sqcap (F_A)_\Delta = \Phi$.

Conversely, let $(F_A)_\Delta$ is soft closed set. Let $x,y \in X$ and $x \neq y$. Then $(x,y)\widetilde{\notin}(F_A)_\Delta$. So $(x,y)\widetilde{\in}(F_A)^c_\Delta$. By the definition of soft base there exist G_B and $H_C \in \mathcal{S}(X,E)$ which is element of soft base such that $(x,y)\widetilde{\in}G_B \times H_C \sqsubseteq (F_A)^c_\Delta$. Hence, $x\widetilde{\in}G_B$, $y\widetilde{\in}H_C$, $G_B, H_C \in \mathcal{T}$ and $G_B \sqcap H_C = \Phi$. □

Theorem 3.25. *If $(X,\mathcal{T})$ is soft Hausdorff space and $(\varphi,\psi) : (X,\mathcal{T}) \to (Y,\mathcal{T}^*)$ is injective, surjective and soft open, then $(Y,\mathcal{T}^*)$ is soft Hausdorff space.*

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since (φ,ψ) is surjective there exist $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2$ and $x_1 \neq x_2$. From hypothesis $(X,\mathcal{T})$ is soft Hausdorff space, so there exist $F_A, G_B \in \mathcal{T}$ such that $x_1\widetilde{\in}F_A, x_2\widetilde{\in}G_B$ and $F_A \sqcap G_B = \Phi$. So, for each $e \in E$, $x_1 \in F_A(e)$, $x_2 \in G_B(e)$ and $F_A(e) \cap G_B(e) = \varnothing$. This implies that $\varphi(x_1) = y_1 \in \varphi(F_A(e))$, $\varphi(x_2) = y_2 \in \varphi(G_B(e))$. Hence, $y_1\widetilde{\in}(\varphi,\psi)(F_A)$, $y_2\widetilde{\in}(\varphi,\psi)(G_B)$. Since (φ,ψ) is open, then $(\varphi,\psi)(F_A)$, $(\varphi,\psi)(G_B) \in \mathcal{T}^*$ and since (φ,ψ) is injective $(\varphi,\psi)(F_A) \sqcap (\varphi,\psi)(G_B) = (\varphi,\psi)(F_A \sqcap G_B) = \Phi$. Thus, $(Y,\mathcal{T}^*)$ is soft Hausdorff space. □

Theorem 3.26 ([19])**.** *The property of being a soft Hausdorff space is hereditary.*

Definition 3.27. Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be two soft topological spaces. A soft function (φ,ψ) from X to Y is called homeomorphism if (φ,ψ) is one-one, onto, continuous and open.

Lemma 3.28. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be two soft topological spaces. Then, X and Y are homeomorphic to a subspace of $X \times Y$.*

Proof. Let $(a_1,a_2) \in X \times Y$ and $(e',k') \in E \times K$ fixed. We need to show that a soft function (φ,ψ) from X to $X \times \{a_2\} \subseteq X \times Y$ is a homeomorphism. Here, $\varphi : X \to X \times \{a_2\}$ and $\psi : E \to E \times \{k'\}$. φ and ψ are one-one and onto mappings, then the soft mapping (φ,ψ) is one-one and onto.

Now we show that (φ,ψ) is continuous. Let F_A be a soft set which is element of base on subspace $X \times \{a_2\}$. By the definition of subspace, there exists $G_B \times H_C \in \mathcal{S}(X \times Y, E \times K)$ open such that $F_A = (G_B \times H_C) \sqcap \widetilde{E}_{X \times \{a_2\}}$.

For $\psi(e) = (e^{'}, k^{'})$,

$$\begin{aligned}(\varphi,\psi)^{-1}(F_A)(e^{'},k^{'}) &= (\varphi,\psi)^{-1}\left((G_B \times H_C) \sqcap \widetilde{E}_{X\times\{a_2\}}\right)(e^{'},k^{'})\\ &= (\varphi,\psi)^{-1}\left(\left((G_B \times H_C) \sqcap \widetilde{E}_{X\times\{a_2\}}\right)(\psi(e))\right)\\ &= \varphi^{-1}\left((G_B(e) \times H_C(k)) \cap X \times \{a_2\}\right)\\ &= \begin{cases}\varphi^{-1}\left(G_B(e)\times\{a_2\}\right), & \text{if } a_2 \in H_C(k);\\ \varnothing, & \text{otherwise.}\end{cases}\\ &= \begin{cases}G_B(e), & a_2 \in H_C(k);\\ \varnothing, & \text{otherwise.}\end{cases}\end{aligned}$$

Then, $(\varphi,\psi)^{-1}(F_A) = \begin{cases}G_B, & a_2 \widetilde{\in} H_C;\\ \Phi, & \text{otherwise.}\end{cases}$ Hence, $(\varphi,\psi)^{-1}(F_A)$ is soft open, so (φ,ψ) is soft continuous.

Now we show that (φ,ψ) is open. Let F_A be a soft open set on X. For $k \in K$,

$$\begin{aligned}(\varphi,\psi)(F_A)(k) &= \begin{cases}\bigcup_{e\in\psi^{-1}\cap A}\varphi(F_A(e)), & \psi^{-1}(k)\cap A \neq \varnothing;\\ \varnothing, & \text{otherwise.}\end{cases}\\ &= \begin{cases}F_A(e)\times\{a_2\}, & \psi^{-1}(k)\cap A \neq \varnothing;\\ \varnothing, & \text{otherwise.}\end{cases}\\ &= \begin{cases}(F_A(e)\times Y)\cap(X\times\{a_2\}), & \psi^{-1}(k)\cap A \neq \varnothing;\\ \varnothing, & \text{otherwise.}\end{cases}\end{aligned}$$

Then, $(\varphi,\psi)(F_A) = (F_A \times \widetilde{E}_Y) \sqcap \widetilde{E}_{X\times Y}$ is open on subspace. Consequently, the soft mapping (φ,ψ) is open. □

Theorem 3.29. *X and Y are soft Hausdorff spaces if and only if $X \times Y$ is soft Hausdorff space.*

Proof. Let X and Y be soft Hausdorff spaces. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$. So we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. Since X is soft Hausdorff space there exist $F_A, G_B \in \mathcal{T}$ such that $x_1 \widetilde{\in} F_A$, $x_2 \widetilde{\in} G_B$ and $F_A \sqcap G_B = \Phi$. Then $F_A \times \widetilde{E}_Y$ and $G_B \times \widetilde{E}_Y$ are soft open set on $X \times Y$. Hence, $(x_1, y_1) \widetilde{\in} F_A \times \widetilde{E}_Y$, $(x_2, y_2) \widetilde{\in} G_B \times \widetilde{E}_Y$ and $(F_A \times \widetilde{E}_Y) \sqcap (G_B \times \widetilde{E}_Y) = \Phi$.

Conversely, let $X \times Y$ be soft Hausdorff space. By the Theorem 3.26 and Lemma 3.28, it is obvious. □

Theorem 3.30. *$(X, \mathcal{T})$ is a soft Hausdorff space if and only if*

$$x_E = \sqcap\{F_E : F_E \in \mathcal{N}(x) \text{ and } F_E \text{ is soft closed}\}.$$

Proof. Let $(X, \mathcal{T})$ be a soft Hausdorff space. Suppose that $x_E \neq \sqcap\{F_E : F_E \in \mathcal{N}(x)$ *and* F_E *is soft closed*$\}$. Then, there exists $y \in X$ such that $x \neq y$ and $y \widetilde{\in} \sqcap \{F_E : F_E \in \mathcal{N}(x)$ *and* F_E *is soft closed*$\}$ $(*)$.

This implies that $y \in \cap F_E(e), \forall e \in E$. Since X is soft Hausdorff space, there exist soft open sets G_E, H_E such that $x \widetilde{\in} G_E$, $y \widetilde{\in} H_E$ and $G_E \sqcap H_E = \Phi$ and so

$x \in G_E(e) \subset X \setminus H_E(e)$. Hence, $H_E^c \in \mathcal{N}(x)$ and H_E^c is soft closed. By the (*), we have $y \widetilde{\in} H_E^c$ and so $y \widetilde{\notin} H_E$. This is a contradiction and this completes the proof.

Conversely, let $x, y \in X$ with $x \neq y$. Then

$$y \widetilde{\notin} x_E = \sqcap\{F_E : F_E \in \mathcal{N}(x) \ and \ F_E \ is \ soft \ closed\}.$$

So, there exist $G_E \in \mathcal{N}(x)$ and G_E is soft closed such that $y \widetilde{\notin} G_E$. This implies that $y \notin G_E(e)$ for some $e \in E$. Then $y \widetilde{\in} G_E^c$ and G_E^c is soft open. Since $G_E \in \mathcal{N}(x)$ there exists $H_E \in \mathcal{T}$ such that $x \widetilde{\in} H_E \sqsubseteq G_E$. Hence, $x \widetilde{\in} H_E$, $y \widetilde{\in} G_E^c$ and $H_E \sqcap G_E^c = \Phi$. Consequently, $(X, \mathcal{T})$ is a soft Hausdorff space. □

Theorem 3.31. *In soft Hausdorff space, a sequence converges to a unique point.*

Proof. Suppose that (x_n) converges to x and y and $x \neq y$. Since $(X, \mathcal{T})$ is soft Hausdorff space there exist $G_B, H_C \in \mathcal{T}$ such that $x \widetilde{\in} G_B, y \widetilde{\in} H_C$ and $G_B \sqcap H_C = \Phi$. This implies that for all $e \in E$, $x \in G_B(e)$, $y \in H_C(e)$ and $G_B(e) \cap H_C(e) = \varnothing$. Since x_n converges to x and G_B is soft neighborhood of x, then there exists $n_1 \in \mathbb{N}$ such that $x_n \widetilde{\in} G_B$ for all $n \geq n_1$. Since x_n converges to y and H_C is soft neighborhood of y, then there exists $n_2 \in \mathbb{N}$ such that $x_n \widetilde{\in} H_C$ for all $n \geq n_2$. Let $n_0 = max(n_1, n_2)$, then for all $n \geq n_0$, $x_n \widetilde{\in} G_B$ and $x_n \widetilde{\in} H_C$. This implies that $x_n \in G_B(e)$ and $x_n \in H_C(e)$ for all $e \in E$. Then $G_B(e) \cap H_C(e) \neq \varnothing$. Hence, $G_B \sqcap H_C \neq \Phi$. This is a contradiction. □

Remark 3.32. The converse of the Theorem 3.31 is not true in general. For instance, in soft topological space $(\mathbb{R}, \mathcal{T}_*)$ every sequence converges to a unique point, but this soft topological space is not Hausdorff.

Definition 3.33 ([3])**.** Let (X, τ) be a soft topological space and $W \subseteq X$.

(1) A family $\mathcal{C} = \{(F_A)_i \mid i \in J\}$ of open soft sets is called an open cover of X, if it satisfies $\bigsqcup_{i \in J}(F_A)_i = \widetilde{E}$, for each $e \in E$. A finite subfamily of a soft open cover $\{(F_A)_i \mid i \in J\}$ of W is called a finite subcover of $\{(F_A)_i \mid i \in J\}$.

(2) W is called soft compact if every soft open cover of W has a finite subcover.

Theorem 3.34. *Let $(X, \mathcal{T})$ be a soft Hausdorff space. If F_A is soft compact on X, then F_A is soft closed.*

Proof. We must show that F_A^c is soft open. Let $x \widetilde{\in} F_A^c$. So, for each $e \in E, x \in F_A^c(e) = X \setminus F_A(e)$ and $x \notin F_A(e)$. Then for all $y \in F_A(e), x \neq y$. Since $(X, \mathcal{T})$ is soft Hausdorff space there exist $(G_B)_y, (H_C)_y \in \mathcal{T}$ such that $x \widetilde{\in} (G_B)_y, y \widetilde{\in} (H_C)_y$ and $(G_B)_y \sqcap (H_C)_y = \Phi$. This implies, for all $e \in E, x \in (G_B)_y(e)$, $y \in (H_C)_y(e)$ and $(G_B)_y(e) \cap (H_C)_y(e) = \varnothing$. Then $F_A(e) \subset (H_C)_y(e)$, so $F_A \sqsubseteq (H_C)_y$. The family $\mathcal{C} = \{(H_C)_y : y \widetilde{\in} F_A\}$ is a soft open cover of F_A. Since F_A is soft compact, F_A has a finite subcover. So, $F_A \sqsubseteq \sqcup_{i=1}^n (H_C)_{y_i}$. Then $\sqcup_{i=1}^n (H_C)_{y_i}$ and $\sqcap_{i=1}^n (G_B)_{y_i} \in \mathcal{T}$ and $(\sqcup_{i=1}^n (H_C)_{y_i}) \sqcap (\sqcap_{i=1}^n (G_B)_{y_i}) = \Phi$. Since $x \widetilde{\in} (G_B)_y$, then $x \widetilde{\in} (G_B)_{y_i} \sqsubseteq (\sqcup_{i=1}^n (H_C)_{y_i})^c \sqsubseteq F_A^c$. Hence, F_A^c is soft open. □

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