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Absorbing maps and fixed point theorems in fuzzy metric spaces using implicit relation

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Abstract. In this paper, we prove a common fixed point in fuzzy metric space using absorbing maps for four self maps satisfying an implicit relation.

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1. Introduction

In 1965 Zadeh [13] introduced the notion of fuzzy sets. Later many authors have extensively developed the theory of fuzzy sets and application. The idea of fuzzy metric space introduced by Kramosil and Michalek [4] was modified by George and Veeramani [3]. In [8] Popa proved theorem for weakly compatible non-continuous mappings using implicit relation. Singh et. al. [10] introduced the notion of Semi-compatible maps in fuzzy metric space, and compared this notion of compatible map, compatible map of type (α), compatible map of type (β), and obtain some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [2]. In [12] Vasuki introduce the concept of R-weakly commuting map, and prove a fixed point theorem in fuzzy metric space. Ranadive et. al. [9] introduced the concept of absorbing mapping in metric space and prove the common fixed point theorem in this space. Moreover Ranadive et. al. [9] observed that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non-compatible maps. In [6] Mishra et. al. introduced absorbing maps in fuzzy metric space. In this paper, we obtain fixed point theorems in fuzzy metric space using absorbing maps, an implicit relation with reciprocal continuity and semi-compatible maps.

2. Preliminaries

In this section we recall some definitions and known results in fuzzy metric space.

Definition 2.1 (Zadeh [13]). Let fuzzy set A in X is a function with domain X and value in $[0, 1]$.

Definition 2.2 (Schweizer and Sklar [11]). A triangular norm $*$ (shortly $t-norm$) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied :

(1) $a * 1 = 1$,

(2) $a * b = b * a$,

(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,

(4) $a * (b * c) = (a * b) * c$.

Definition 2.3 (Kramosil and Michalek [4]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous $t-norm$ and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$.

(FM1) $M(x, y, 0) = 0$;

(FM2) $M(x, y, t) = 1$ for all $t > 0$, iff $x = y$,

(FM3) $M(x, y, t) = M(y, x, t)$,

(FM4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$,

(FM5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous ;

(FM6) $lim_{t\to\infty} M(x, y, t) = 1$.

In the definition of George and Veeramani [3] M is a fuzzy set on $X^2 \times [0, \infty)$ and $(FM1)$, $(FM2)$, $(FM5)$ are replaced respectively, with $(GV1)$, $(GV2)$, $(GV5)$ below :

(GV1) $M(x, y, 0) > 0$, for all $t > 0$,

(GV2) $M(x, x, t) = 1$, for all $t > 0$ and $x \neq y \Rightarrow M(x, y, t) < 1$ for all $t > 0$,

(GV5) $M(x, y, .) : (0, \infty) \to [0,\ 1]$ is continuous for all $x, y \in X$.

Example 2.4 (George and Veeramani [3]). Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = min[a, b]$) and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = \frac{t}{t+d(x,y)}$. Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.5 (George and Veeramani [3]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy if for each $\epsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. A fuzzy metric space $(X, M, *)$ is said to complete if every Cauchy sequence in X converges to a point in X. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $lim_{n\to\infty} M(x_n, x, t) > 1 - \epsilon$ for each $t > 0$, there exists $n_0 \in N$.

Definition 2.6 (Balasubramaniam et. al. [1]). A pair (A, B) of self maps of a fuzzy metric space $(X, M, *)$ is said to be reciprocal continuous if $lim_{n\to\infty} ABx_n = Ax$ and $lim_{n\to\infty} BAx_n = Bx$, whenever there exists a sequence $\{x_n\} \in X$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = x$ for some $x \in X$. If A and B are both continuous then they are obviously reciprocally continuous but the converse need not be true.

Definition 2.7 (Singh et. al. [10]). A pair (A, B) of self-maps of a fuzzy metric space $(X, M, *)$ is said to be semi-compatible if $lim_{n\to\infty} ABx_n = Bx$ whenever there exists a sequence $\{x_n\} \in X$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

Lemma 2.8 (Mishra et. al. [5]). *Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $t > 0$, $M(x, y, kt) \geq M(x, y, t)$, then $x = y$.*

Lemma 2.9 (Grabiec [2]). *$M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

The following proposition shows that in the concept of reciprocal continuity the notion of compatible and semi-compatibility of maps becomes equivalent.

Proposition 2.10 ([7]). *Let A and B be two self maps on a fuzzy metric space $(X, M, *)$. Assume that (A, B) is reciprocal continuous then (A, B) is semi-compatible if and only if (A, B) is compatible.*

Proof. Let $\{x_n\}$be a sequence in X such that $Ax_n \to z$ and $Bx_n \to z$. Since pair of maps (A, B) is reciprocally continuous then we have

$$lim_{n\to\infty} ABx_n = Az \quad \text{and} \quad lim_{n\to\infty} BAx_n = Bz.$$

Suppose that (A, B) is semi-compatible, then we have,

$$lim_{n\to\infty} M(ABx_n, Bz, t/2) = 1.$$

Now, we have

$$M(ABx_n, BAx_n, t) \geq M(ABx_n, Bz, t/2) * M(Bz, BAx_n, t/2)$$

Letting $n \to \infty$, we get

$$lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 * 1 = 1.$$

Thus A and B are compatible maps.

Conversely, suppose (A, B) is compatible and reciprocal continuous, then for $t > 0$, we have

$$lim_{n\to\infty} M(ABx_n, BAx_n, t/2) = 1.$$

for all $\{x_n\} \in X$. Now,

$$\begin{aligned} lim_{n\to\infty} M(ABx_n, Bz, t) &\geq lim_{n\to\infty}(M(ABx_n, BAx_n, t/2) * M(BAx_n, Bz, t/2)) \\ &= 1 * 1 = 1. \end{aligned}$$

i.e. $lim_{n\to\infty} M(ABx_n, Bz, t) = 1$. Thus A and B are semi-compatible. This complete the proof. □

Definition 2.11 ([6]). Let A and B be two self-maps on a fuzzy metric space $(X, M, *)$ then A is called B-absorbing if there exists a positive integer $R > 0$ such that $M(Bx, BAx, t) \geq M(Bx, Ax, \frac{t}{R})$ for all $x \in X$. Similarly B is called A-absorbing if there exists a positive integer $R > 0$ such that $M(Ax, ABx, t) \geq M(Ax, Bx, \frac{t}{R})$ for all $x \in X$.

3. A class of implicit relation

Let Φ be the set of all real continuous functions $F : (R^+)^5 \to R$ non-decreasing in first argument satisfying the following conditions :

(i) For $u, v \geq 0, F(u, v, v, u, 1) \geq 0$ implies that $u \geq v$.

(ii) $F(u, 1, 1, u, 1) \geq 0$ or $F(u, 1, u, 1, u) \geq 0$, or $F(u, u, 1, 1, u) \geq 0$ implies that $u \geq 1$.

Example 3.1. Define $F(t_1, t_2, t_3, t_4, t_5) = 16t_1 - 12t_2 - 8t_3 + 4t_4 + t_5 - 1$.Then $F \in \Phi$.

(i) $F(u, v, v, u, 1) = 20(u - v) \geq 0 \Rightarrow u \geq v$.

(ii) $F(u, 1, 1, u, 1) = 20(u - 1) \geq 0 \Rightarrow u \geq 1$ or

$$F(u, 1, u, 1, u) = 9(u - 1) \geq 0 \Rightarrow u \geq 1$$

or

$$F(u, u, 1, 1, u) = 5(u - 1) \geq 0 \Rightarrow u \geq 1$$

A characterization of Φ in linear form : Define

$$F(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3 + dt_4 + t_5 - 1,$$

where $a, b, c, d \in R$, with $a+b+c+d = 0$, $a > 0$, $b+d > 0$, $c+d > 0$, and $a+d > 0$. Then $F \in \Phi$.

Proof. For $u, v \geq 0$;

(i) $F(u, v, v, u, 1) \geq 0 = (a + d)u + (b + c)v \geq 0,$

$$\Rightarrow (a + d)u \geq (a + d)v, \quad i.e.\ u \geq v,$$

(ii) $F(u, 1, 1, u, 1) \geq 0 = (a + d)u + (b + c) \geq 0,$

$$\Rightarrow (a + d)u \geq (a + d), \quad i.e.\ u \geq 1$$

Again

$$F(u, 1, u, 1, u) = u[1 - (b + d)] - [1 - (b + d)] \geq 0 \quad i.e.\ u \geq 1$$

and or

$$F(u, u, 1, 1, u) = u[1 - (c + d)] - [1 - (c + d)] \geq 0, \quad i.e.\ u \geq 1$$

as $a+b+c+d = 0$. Hence $u \geq 1$. As $a \geq 0$, F is non-decreasing in the first argument and the result follows. □

4. Main result

In this paper, we prove a fixed point theorem by using reciprocal continuity and employing absorbing mapping and semi-compatibility.

Theorem 4.1. *Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with $t-norm$ defined by $a * b = min\ \{a\ b\}$, satisfying :*

(1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(2) *B is T absorbing ,*

(3) *for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$F(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, kt), M(Ax, Ty, t)) \geq 0.$$

If the pair $\{A, S\}$ is reciprocal continuous, semi-compatible maps. Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$, be any arbitrary point. From (1), there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, ...$ Putting $x = x_{2n}, y = x_{2n+1}$ for $t > 0$ in (3); we get

$$F\{M(Ax_{2n}, Bx_{2n+1}, kt), M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\ M(Bx_{2n+1}, Tx_{2n+1}, kt), M(Ax_{2n}, Tx_{2n+1}, t)\} \geq 0.$$

that is,

$$F\{M(y_{2n}, y_{2n+1}, kt), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n+1}, y_{2n}, kt), M(y_{2n}, y_{2n}, t)\} \geq 0.$$

Using (i), we get

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n}, y_{2n-1}, t).$$

Again we put $x = x_{2n+2}$ and $y = x_{2n+3}$ in (3); we have

$$F\{M(Ax_{2n+2}, Bx_{2n+3}, kt), M(Sx_{2n+2}, Tx_{2n+3}, t), M(Ax_{2n+2}, Sx_{2n+2}, t), \\ M(Bx_{2n+3}, Tx_{2n+3}, kt), M(Ax_{2n+2}, Tx_{2n+3}, kt)\} \geq 0,$$

that is

$$F\{M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, t), \\ M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+2}, y_{2n+2}, t)\} \geq 0.$$

Using (i); we get

$$M(y_{2n+3}, y_{2n+2}, kt) \geq M(y_{2n+2}, y_{2n+1}, t).$$

Thus for any n and t, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t).$$

Hence by Lemma 2.9 $\{y_n\}$ is a Cauchy sequence in X. Since X is complete therefore $\{y_n\} \to z$ in X and its subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}, \{Sx_{2n}\}$ also converges to z. (A, S) is reciprocally continuous mapping then we have,

$$lim_{n\to\infty} ASx_{2n} = Az, \quad lim_{n\to\infty} SAx_{2n} = Sz$$

And Semi-compatibility of (A, S) gives;

$$lim_{n\to\infty} ASx_{2n} = Sz.$$

Hence Az = Sz.

Step (1) : By putting $x = z, y = x_{2n+1}$ in (3), we get

$$F\{M(Az, Bx_{2n+1}, kt), M(Sz, Tx_{2n+1}, t), M(Az, Sz, t), \\ M(Bx_{2n+1}, Tx_{2n+1}, kt), M(Az, Tx_{2n+1}, t)\} \geq 0.$$

Letting $n \to \infty$; we get

$$F\{M(Az, z, kt), M(Az, z, t), M(Az, Az, t), M(z, z, t), M(Az, z, t)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M(Az, z, t), M(Az, z, t), 1, 1, M(Az, z, t)\} \geq 0.$$

that is

$$M(Az, z, t) \geq 1.$$

Therefore $Az = z = Sz$.

Step (2) : As $A(X) \subseteq T(X)$, there exists $u \in X$ such that z = Az = Tu , putting $x = x_{2n}$ and $y = u$ in (3), we get

$$F\{M(Ax_{2n}, Bu, kt), M(Sx_{2n}, Tu, t), M(Ax_{2n}, Sx_{2n}, t), \\ M(Bu, Tu, kt), M(Ax_{2n}, Tu, t)\} \geq 0$$

Letting $n \to \infty$; we get

$$F\{M(z, Bu, kt), M(z, z, t), , M(z, z, t)M(Bu, z, kt), M(z, z, t)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M(z, Bu, t), 1, , 1, M(Bu, z, t), 1\} \geq 0$$

that is

$$M(z, Bu, t) \geq 1$$

Therefore $z = Bu = Tu$. Since B is T-absorbing then we have;

$$M(Tu, TBu, t) \geq M(Tu, Bu, t/R) \geq 1;$$

i.e., $Tu = TBu \Rightarrow z = Tz$.

putting $x = z$ and $y = z$ in (3), we get

$$F\{M(Az, Bz, kt), M(Sz, Tz, t), M(Az, Sz, t), \\ M(Bz, Tz, kt), M(Az, Tz, t)\} \geq 0$$

we get

$$F\{M(z, Bz, kt), M(z, z, t), , M(z, z, t)M(Bz, z, kt), M(z, z, t)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M(z, Bz, kt), 1, , 1, M(Bz, z, t), 1\} \geq 0$$

that is

$$M(z, Bz, t) \geq 1.$$

Therefore $z = Bz = Tz$. Hence $Az = Bz = Sz = Tz = z$.
Uniqueness : Let w be another fixed point of A, B, S and T ; therefore putting $x = z$ and $y = w$ in (3), we have

$$F\{M(Az, Bw, kt), M(Sz, Tw, t), M(Az, Sz, t), M(Bz, Tz, kt), M(Az, Tw, t)\} \geq 0$$
$$F\{M(z, w, kt), M(z, w, t), M(z, z, kt), M(z, z, t), M(z, w, t)\} \geq 0.$$

As F is non decreasing in the first argument, we have

$$F\{M(z, w, t), M(z, w, t), 1, 1, M(z, w, t)\} \geq 0,$$

i.e., $z = w$. Hence z is a unique fixed point in X. □

Corollary 4.2. *Let A, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with $t - norm$ defined by $a * b = min\{a, b\}$, satisfying :*

(4) $A(X) \subseteq S(X) \cap T(X)$;

(5) *A is T absorbing ;*

(6) *for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$F\{(M(Ax, Ay, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(Ay, Ty, kt), M(Ax, Ty, t)\} \geq 0,$$

If the pair $\{A, S\}$ is reciprocal continuous, semi-compatible maps. Then A, S and T have a unique common fixed point in X.

Proof. If we take $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Ax_{2n+1} = Sx_{2n+2} = y_{2n+1}$, we get the required result. □

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