Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 5, No. 1, (January 2013), pp. 1–13
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
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Dislocated fuzzy quazi metric spaces and common fixed points

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Received 19 November 2011; Revised 27 March 2012; Accepted 4 April 2012

Abstract. Weakly compatible and occassionally weakly compatible mappings are discussed in detail and generalised common fixed point theorems for two mappings $f : X \to X$ and $T : X^2 \to X$ are proved in Dislocated fuzzy metric space (in short $DFM - Space$) and dislocated fuzzy quasi metric space (in short $DFqM - Space$). Our results extends and generalises many well known results.

2010 AMS Classification: 54E40, 54E50, 47H10, 54H25

Keywords: Dislocated fuzzy metric space, Dislocated fuzzy quasi metric space, Coincidence and common fixed points, Weakly compatible maps, Occasionally weakly compatible mappings.

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1. Introduction

Zadeh [30] introduction of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. In the last two decades many fixed point theorems for contractions in fuzzy metric spaces and quasi fuzzy metric spaces appeared (see [3], [7], [8], [10], [12], [17], [18], [21], [24], [25], [27], [28], [29]). The role of topology in logic programming has come to be recognized in recent years. In particular topological methods are employed in order to obtain fixed point semantics for logic programs. In classical approach to logic programming semantics in which positive or definite positive programs are considered (those in which negation does not occur) Knaster - Tarski fixed point theorem can be applied to obtain a least fixed point of an operator called the single step or immediate consequence operator. However when the syntax is enhanced in the sense that negation is allowed, the approach using Knaster - Tarski theorem does not work. In such cases the Banach contraction mapping theorem for complete metric spaces is an alternative to Knaster - Tarski fixed point theorem. However topological spaces which arise in the area of denotational semantics are often not Hausdorff and so spaces which are weaker than metric spaces in a topological sense had to be cosidered. Motivated by this fact Hitzler and Seda [13] introduced the concept of dislocated metric space and studied the dislocated topologies which is a generalisation of the conventinal topologies and can be thought of as underlying the notion of dislocated metrics. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. Later George and Khan [22] introduced the concept of dislocated fuzzy metric spaces and studied the associated topologies. In [1] Alaca introduced the concept of Dislocated Fuzzy Quasi Metric Space ($DFqM-Space$)in the sense of Kramosil and Michalek as well as George and Veeramani and discussed the topologies associated with it which is conventional in nature. In this paper we have discussed the dislocated fuzzy topologies associated with a $DFqM-Space$ and also proved a common fixed point theorem of Presic type which extends and generalises the well known Banach contraction principle and also fuzzyfies other known results.

2. Preliminaries

Let (X, d) be a metric space and $T : X \to X$ and $f : X \to X$ be mappings. Let $C(T, f)$ denote the set of all coincident points of the mappings f and T, that is $C(T, f) = \{u : fu = Tu\}$.

Definition 2.1 ([14])**.** The mappings f and T are said to be weakly compatible if and only if they commute at their coincidence points.

Remark 2.2. Clearly if $C(T, f) = \phi$ then f and T are weakly compatible.

Definition 2.3 ([2])**.** The mappings f and T are said to be occasionally weakly compatible (*owc*) if and only if they commute at some coincidence point of f and T, i.e. $fTu = Tfu$ for some $u \in C(T, f)$.

Remark 2.4. Occasionally weakly compatible pairs of mappings requires the set of coincident points of the mappings under consideration to be non empty. In other words if $C(T, f) = \phi$ then f and T cease to be *owc* and so *owc* pair of mappings cannot be seen as a generalisation of weakly compatible pair. Hence we modify the definition of *owc* pair of mappings as follows:

Definition 2.5. The mappings f and T are said to be occasionally weakly compatible (*owc*) if and only if $fTu = Tfu$ for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

In [6] Doric et al has shown that if the point of coincidence is unique then occasionally weakly compatible mappings are weakly compatible.

Lemma 2.6 ([6])**.** *Let f and T be occasionally weakly compatible mappings of X. If f and T have a unique point of coincidence then f and T are weakly compatible.*

Proof. Let v be the unique point of coincidence of f and T. Since f and T be occasionally weakly compatible mappings there exists $u \in X$ such that $v = fu = Tu$ and $fTu = Tfu$, i.e $fv = Tv$. Let $u^* \in C(f, T)$. Then $fu^* = Tu^* = v$ and $fTu^* = fv = Tv = Tfu^*$. Thus f and T are weakly compatible. □

Thus if mappings f and T have a unique point of coincidence then the pair (f, T) are weakly compatible iff they are occasionally weakly compatible.

The following lemma appears in Jungc and Rhoades [15].

Lemma 2.7. *If a weakly compatible pair (f, T) of self maps has a unique point of coincidence, then the point of coincidence is a unique common fixed point of f and T.*

The next example shows that if the point of coincidence is not unique then occasionally weakly compatible mappings are more general than weakly compatible mappings.

Example 2.8. Take $X = [0, 1]$, $fx = x^2$, $Tx = \frac{x}{2}$. It is obvious that $C(f, T) = \{0, \frac{1}{2}\}$, $fT0 = Tf0$ but $fT\frac{1}{2} \neq Tf\frac{1}{2}$ and so f and T are occasionally weakly compatible but not weakly compatible. Note that 0 and $\frac{1}{4}$ are two point of coincidence and 0 is the unique common fixed point.

Definition 2.9. f is said to be coincidentally idempotent with respect to T if and only if f is idempotent at the coincidence points of f and T.

Definition 2.10. The mapping f is said to be occasionally coincidentally idempotent $(oci)$with respect to T, if and only if $ffu = fu$ for some $u \in C(T, f)$ whenever $C(T, f) \neq \phi$.

Clearly if f and T are coincidentally idempotent then they are *oci*. However Example above shows that the converse is not necessarily true.

3. Dislocated Fuzzy Quasi Metric Space

In this section we will define Dislocated Fuzzy Quasi Metric Space and discuss the topologies associated with it.

Definition 3.1 ([26])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $([0, 1], *)$ is an abelian monoid with unit one such that, for all a,b,c,d in [0,1], $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$.

Definition 3.2. Let X be any non empty set, * be a continuous t-norm and $M : X^2 \times [0, \infty) \to [0, 1]$ be a fuzzy set. For all $x, y, z \in X$ and $t, s \in [0, \infty)$, consider the following conditions:

$FM1$ $M(x, y, 0) = 0$
$FM2$ $M(x, x, t) = 1$
$FM3$ $M(x, y, t) = 1$ and $M(y, x, t) = 1 \Rightarrow x = y$
$FM4$ $M(x, y, t) = M(y, x, t)$
$FM5$ $M(x, y, t + s) \geq M(x, z, t) + M(z, y, s)$
$FM6$ $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous
$FM7$ $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous

If M satisfies conditions $FM1$ to $FM6$ then $(X, M, *)$ is called a *Fuzzy Metric Space* [16]. If M satisfies conditions $FM1$ and $FM3$ to $FM6$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Metric Space in the sense of Kramosil and Michalek (in short D_{KM}FM-Space)* [22]. If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions $FM1$ and $FM3$ to $FM5$ and $FM7$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy*

Metric Space in the sense of George and Veeramani (in short D_{GV}FM-Space) [22]. If M satisfies conditions $FM1, FM3, FM5$ and $FM6$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Quasi Metric Space in the sense of Kramosil and Michalek (in short D_{KM}FqM-Space)* [1]. If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions $FM3, FM5$ and $FM7$ then we say that $(X, M, *)$ is a *Dislocated Fuzzy Quasi Metric Space in the sense of George and Veeramani (in short D_{GV}FqM-Space)* [1].

Example 3.3. Let $X = R$; Define $a * b = ab$, M(x,y,t) $= \left[exp^{\frac{|x-y|+2|x|+|y|}{t}}\right]^{-1}$ for all $(x, y) \in X \times X$, $t \in (0, \infty)$. Then $(X, M, *)$ is a $D_{GV}FqM - Space$.

For all $(x, y) \in X \times X$, $t \in (0, \infty)$ let $M^{\ddagger}(x, y, t) = min\{M(x, y, t), M(y, x, t)\}$. Clearly if $(X, M, *)$ is a D_{GV}FqM-Space (or D_{KM}FqM-Space) then $(X, M^{\ddagger}, *)$ is a D_{GV}FM-Space (or D_{KM}FM-Space). Obviously each D_{GV}FqM-Space can be cosidered as a D_{KM}FqM-Space by defining $M(x, y, 0) = 0$ for all $x, y \in X$ (see [11]). Hereafter by a Dislocated Fuzzy Quasi Metric Space ($DFqM - Space$) we mean a D_{GV}FqM-Space or a D_{KM}FqM-Space.

Definition 3.4. Let $(X, M, *)$ be a $DFqM - Space$). We define a left open ball (L-open ball) with centre x and radius r $(0 < r < 1)$ in X as $B_L(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$, for all $t \in (0, \infty)$. We define a right open ball(R-open ball) with centre x and radius r $(0 < r < 1)$ in X as $B_R(x, r, t) = \{y \in X : M(y, x, t) > 1 - r\}$, for all $t \in (0, \infty)$. We define an open ball with centre x and radius r $(0 < r < 1)$ in X as $B(x, r, t) = \{y \in X : M^{\ddagger}(x, y, t) > 1 - r\}$, for all $t \in (0, \infty)$.

Obviously $B(x, r, t) = B_L(x, r, t) \bigcap B_R(x, r, t)$ and its not necessary that $x \in B(x, r, t)$ for all $x \in X$.

For more details on topologies associated with $DFqM - Space$ refer to [23].

Definition 3.5. A sequence x_n in a $DFqM - Space$ $(X, M, *)$ is said to be bi-convergent to a point $x \in X$ if and only if $Lim_{n\to\infty} M^{\ddagger}(x_n, x, t) = 1$ for all $t > 0$. In this case we say that limit of the sequence x_n is x.

Definition 3.6. A sequence x_n in a $DFqM - Space$ $(X, M, *)$ is said to be Left (Right) Cauchy sequence if and only if

$$Lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \ (Lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1)$$

for all $t > 0$, $p > 0$.

Definition 3.7. A sequence x_n in a $DFqM-Space$ $(X, M, *)$ is said to be bi-Cauchy if and only if $Lim_{n\to\infty} M^{\ddagger}(x_n, x_{n+p}, t) = 1$ for all $t > 0$, $p > 0$.

Definition 3.8. A $DFqM - Space$ is said to be Left (or Right) complete if and only if every Left (or Right)Cauchy sequence in it is bi-convergent.

Definition 3.9. A $DFqM - Space$ is said to be bi-complete if and only if every bi-Cauchy sequence in it is bi-convergent.

Remark 3.10. Clearly a sequence x_n in a $DFqM - Space$ $(X, M, *)$ is bi- Cauchy sequence if and only if sequence x_n is a Cauchy sequence in the $DFM - Space$ $(X, M^{\ddagger}, *)$. A $DFqM - Space$ $(X, M, *)$ is bi-Complete if and only if the $DFM - Space$ $(X, M^{\ddagger}, *)$ is complete.

Proposition 3.11. *Limit of a sequence in a* $DFqM - Space$ $(X, M, *)$ *is unique.*

Proof. Let x_n be a sequence in X and suppose u and v are two limits of x_n. Then we have $M^{\ddagger}(u, v, t) \geq M^{\ddagger}(u, x_n, t/2) * M^{\ddagger}(x_n, v, t/2)$. Taking the limit as $n \to \infty$ we have $M^{\ddagger}(u, v, t) \geq 1 * 1 = 1$. Hence $u = v$. □

Proposition 3.12. *Let* $(X, M, *)$ *be a* $DFqM - Space$*(or* $DFM - Space$*) and* x_n *be a sequence in X. If sequence* x_n *bi-converges (or converges) to* $x \in X$ *then* $M(x, x, t) = 1$ *for all* $t > 0$*.*

Proof. We have $M(x, x, t) \geq M(x, x_n, t/2) * M(x_n, x, t/2)$ for all n. Taking the limit as $n \to \infty$ we have $M(x, x, t) \geq 1 * 1 = 1$. □

Proposition 3.13. *Let* $(X, M, *)$ *be a* $DFqM - Space$ *(or* $DFM - Space$*),* $f, g : X \to X$ *be mappings. If* $fz = gz$ *and* $M^{\ddagger}(fgz, gfz, t) = 1$ *(or* $M(fgz, gfz, t) = 1$*) for some* $z \in X$ *and* $t \in [0, \infty)$*, then* $M(ffz, ffz, t) = 1$ *for all* $t \in [0, \infty)$*.*

Proof. Since $M^{\ddagger}(fgz, gfz, t) = 1$ we have $fgz = gfz$. Therefore $M(ffz, ffz, t) = M(fgz, fgz, t) = M(fgz, gfz, t) = 1$. □

4. Major Section

Let $(X, M, *)$ be a $DFqM - Space$(or $DFM - Space$), $T : X^2 \to X$ and $f : X \to X$ be mappings. A point $z \in X$ is said to be a *coincidence point* of f and T if $T(z, z) = fz$. z is said to be a *common fixed point* of f and T if $T(z, z) = fz = z$. Let $C(T, f)$ denote the set of all coincidence points of the mappings f and T. Clearly if z is a coincidence point of f and T, then $M^{\ddagger}(fz, fz, t) = 1$ and $M^{\ddagger}(T(z, z), T(z, z), t) = 1$

Definition 4.1. The mappings f and T in a $DFqM - Space$ (or $DFM - Space$) are said to be weakly compatible if and only if $M^{\ddagger}(T(fz, fz), f(T(z, z), t) = 1$ (or $M(T(fz), f(T(z, z), t) = 1)$ for all $z \in C(T, f)$ and $t \in [0, \infty)$.

Definition 4.2. The mappings f and T in a $DFqM - Space$ (or $DFM - Space$) are said to be occasinally weakly compatible (*owc*) if and only if

$$M^{\ddagger}(T(fz, fz), f(T(z, z), t) = 1 \text{ (or } M(T(fz, fz), f(T(z, z), t) = 1)$$

for some $z \in C(T, f)$ and $t \in [0, \infty)$, whenever $C(T, f) \neq \phi$.

Consider a function $\phi : [0, 1]^2 \to [0,1]$ such that
(a) ϕ is an increasing function, *i.e* $x_1 \leq y_1, x_2 \leq y_2$ implies $\phi(x_1, x_2) \leq \phi(y_1, y_2)$.
(b) $\phi(t, t) \geq t$, for all $t \in [0, 1]$
(c) ϕ is continuous in both variables.

Now we present our main results as follows:

Theorem 4.3. *Let* $(X, M, *)$ *be a* $DFM - Space$*,* $f : X \longrightarrow X$ *and* $T : X^2 \longrightarrow X$ *be mappings, such that*

$$T(X^2) \subseteq f(X) \tag{4.1}$$

$$M(T(x_1, x_2), T(x_2, x_3), qt) \geq \phi\{M(fx_1, fx_2, t), M(fx_2, fx_3, t))\}, \tag{4.2}$$

where x_1, x_2, x_3 are arbitrary elemants in X, $0 < q < \frac{1}{2}$ *and* $t \in [0, \infty)$

(4.3) *f(X) is complete.*

Then the sequence $< y_n >$ *defined by*

(4.4) $$y_{n+2} = f(x_{n+2}) = T(x_n, x_{n+1})$$

for arbitrary elements x_1, x_2 *in* X *and* $n = 1, 2, ...,$ *converges to a point of coincidence of* f *and* T.

Proof. Let $\alpha_n = M(y_n, y_{n+1}, qt)$. By the method of mathematical induction we will prove that

(4.5) $$\alpha_n \geq \left(\frac{K - \theta^n}{K + \theta^n}\right)^2$$

where $\theta = \frac{1}{q}$, $K = Min\{\frac{\theta(1+\sqrt{\alpha}_1)}{(1-\sqrt{\alpha}_1)}, \frac{\theta^2(1+\sqrt{\alpha}_2)}{(1-\sqrt{\alpha}_2)}\}$. Clearly from the definition of K, we see that (4.5) is true for $n = 1, 2$. Let the 2 inequalities

$$\alpha_n \geq \left(\frac{K - \theta^n}{K + \theta^n}\right)^2, \ \alpha_{n+1} \geq \left(\frac{K - \theta^{n+1}}{K + \theta^{n+1}}\right)^2$$

be the induction hypothesis. Then we have

$$\alpha_{n+2} = M(y_{n+2}, y_{n+3}, qt)$$

$$= M(T(x_n, x_{n+1}), T(x_{n+1}, x_{n+2}), qt)$$

$$\geq \phi\{M(fx_n, fx_{n+1}, t), M(fx_{n+1}, fx_{n+2}, t)\}$$

$$= \phi\{\alpha_n, \alpha_{n+1}\}$$

$$\geq \phi\{\left(\tfrac{K-\theta^n}{K+\theta^n}\right)^2, \left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2\}$$

$$\geq \phi\{\left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2, \left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2\}$$

$$\geq \left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2 \geq \left(\tfrac{K-\theta^{n+2}}{K+\theta^{n+2}}\right)^2.$$

Thus inductive proof of (4.5) is complete. Now for $p \in N$ and $t \in [0, \infty)$, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, \tfrac{t}{2}) \star M(y_n, y_{n+1}, \tfrac{t}{2^2}) \star ... \star M(y_{n+p-1}, y_{n+p}, \tfrac{t}{2^p})$$

$$\geq \left(\tfrac{K-2^n}{K+2^n}\right)^2 \star \left(\tfrac{K-2^{2n}}{K+2^{2n}}\right)^2 \star ... \star \left(\tfrac{K-2^{np}}{K+2^{np}}\right)^2$$

$$\to 1 \star 1 \star ... \star 1 = 1, \text{ as } n \to \infty.$$

Hence $< y_n >$ is a Cauchy sequence in $f(X)$ and since $f(X)$ is complete, there will exist v in $f(X)$ such that $lim_{n\to\infty} y_n = v$. Let $v = f(u)$ for some $u \in X$. Then we have

$M(T(u,u), fu, t) = lim_{n\to\infty} M(T(u,u), y_{n+2}, t)$

$= lim_{n\to\infty} M(T(u,u), T(x_n, x_{n+1}), t)$

$\geq lim_{n\to\infty} M(T(u,u), T(u,x_n), \frac{t}{2}) \star M(T(u,x_n), T(x_n, x_{n+1}), \frac{t}{2})$

$\geq lim_{n\to\infty} \phi\{M(fu, fu, t), M(fu, fx_n, t)\} \star \phi\{(M(fu, fx_n, t), M(fx_n, fx_{n+1}, t))$

$= 1$, i.e.

$M(T(u,u), fu, t) = 1$ and so $C(f,t) \neq \varnothing$ and v is a point of coincidence of f and T. ☐

Theorem 4.4. *Let $(X, M, *)$ be a $DFM-Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be weakly compatible mappings satisfying (4.1),(4.2), (4.3) and*

$$lim_{t\to\infty} M(x, y, t) = 1. \tag{4.6}$$

Then the sequence $< y_n >$ defined by (4.4) converges to a unique common fixed point of f and T.

Proof. Proceeding on the same lines as in the proof of Theorem (4.3), we see that sequence $< y_n >$ converges to v which is a point of coincidence of f and T. Then we have

$M(fu, fu, qt) = M(T(u,u), T(u,u), qt)$

$\geq \phi\{M(fu, fu, t), M(fu, fu, t)\}$

$\geq M(fu, fu, t) = M(Tu, u), T(u,u), t)$

$\geq \phi\{M(fu, fu, \frac{t}{q}), M(fu, fu, \frac{t}{q})\}$

$\geq M(fu, fu, \frac{t}{q}) \geq ...... \geq M(fu, fu, \frac{t}{q^{n-1}})$.

As $n \to \infty$ we get $M(fu, fu, qt) = 1$. Suppose there exists $v^* \in X$ such that $f(u^*)$ $= T(u^*, u^*) = v^*$ for some u^* in $C(f,T)$. Then

$M(v, v^*, q\frac{t}{2}) = M(T(u,u), T(u^*, u^*), q\frac{t}{2})$

$\geq M(T(u,u), T(u,u^*), \frac{qt}{4}) \star M(T(u,u^*), T(u^*,u^*), \frac{qt}{4})$

$\geq \phi\{M(fu, fu, \frac{t}{4}), M(fu, fu^*, \frac{t}{4})\} \star \phi\{M(fu, fu^*, \frac{t}{4}), M(fu^*, fu^*, \frac{t}{4})\}$

$\geq \phi\{M(fu, fu^*, \frac{t}{4}), M(fu, fu^*, \frac{t}{4})\} \star \phi\{M(fu, fu^*, \frac{t}{4}), M(fu, fu^*, \frac{t}{4})\}$

$\geq M(fu, fu^*, \frac{t}{4}) \star M(fu, fu^*, \frac{t}{4})$.

$= M(T(u,u), T(u^*,u^*), \frac{t}{4}) \star M(T(u,u), T(u^*,u^*), \frac{t}{4})$

$\geq M(T(u,u), T(u,u^*), \frac{t}{2^3}) \star M(T(u,u^*), T(u^*,u^*), \frac{t}{2^3}) \star M(T(u,u), T(u,u^*), \frac{t}{2^3})$
$\star M(T(u,u^*), T(u^*,u^*), \frac{t}{2^3})$

$\geq \phi\{M(fu, fu, \frac{t}{2^3 q}), M(fu, fu^*, \frac{t}{2^3 q})\} \star \phi\{M(fu, fu, \frac{t}{2^3 q}), M(fu, fu^*, \frac{t}{2^3 q})\}$

$\geq \phi\{M(fu, fu^*, \frac{t}{2^3 q}), M(fu, fu^*, \frac{t}{2^3 q})\} \star \phi\{M(fu, fu^*, \frac{t}{2^3 q}), M(fu, fu^*, \frac{t}{2^3 q})\}$

$\geq M(fu, fu^*, \frac{t}{2^3 q}) \star M(fu, fu^*, \frac{t}{2^3 q})$.

Repeating the above process n times we get

$$M(v, v^*, qt) \geq M(fu, fu^*, \frac{t}{2^{n+1} q^{n-1}}) \star M(fu, fu^*, \frac{t}{2^{n+1} q^{n-1}}).$$

Taking the limit as $n \to \infty$ we get $M(v, v^*, qt) \geq 1$ and so $v = v^*$, i.e. v is the unique point of coincidence of f and T. Hence by lemma (4.3) v is a unique common fixed point of f and T. □

Note that the condition $lim_{t\to\infty} M(x,y,t) = 1$ ensures the uniqueness of the point of coincidence. However in the next result we will remove the condition $lim_{t\to\infty} M(x,y,t) = 1$ and also increase the range of q.

Theorem 4.5. *Let $(X, M, *)$ be a $DFM - Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be mappings satisfying (4.1), (4.2) (with $0 < q < 1$) and (4.3). Then f and T has a common fixed point if one of the following two conditions are satisfied :*
(i) f is oci with respect to T and the pair (f, T) is weakly compatible,
(ii) f is coincidentally idempotent with respect to T and the pair (f, T) is owc.

Proof. Proceeding as in the proof of Theorem 4.4, we can show that $C(T, f) \neq \phi$. Now suppose f is *oci* with respect to T and the pair (f, T) is weakly compatible. Then there will exist $z \in C(f, T)$ such that $ffz = fz$ and also $f(T(z,z)) = T(fz, fz)$. Thus we have $fz = ffz = f(T(z,z)) = T(fz, fz)$, i.e. fz is a common fixed point of f and T. The proof follows on the same lines in the other case also. □

Theorem 4.6. *Let $(X, M, *)$ be a $DFqM - Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be weakly compatible mappings satisfying (4.1), (4.6) and the following:*

(4.7) $$M(T(x_1, x_2), T(x_3, x_1), qt) \geq \phi\{M(fx_1, fx_3, t), M(fx_2, fx_1, t)\}$$

where x_1, x_2, x_3 are arbitrary elemants in X, $0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

(4.8) *f(X) is R-complete.*

Then the sequence $< y_n >$ defined by (4.4) converges to a unique common fixed point of f and T.

Proof. Let $\alpha_n = M(y_{n+1}, y_n, qt)$. By the method of mathematical induction we will prove that

(4.9) $$\alpha_n \geq \left(\frac{K - \theta^n}{K + \theta^n}\right)^2.$$

where $\theta = \frac{1}{q}$, $K = Min\{\frac{\theta(1+\sqrt{\alpha}_1)}{(1-\sqrt{\alpha}_1)}, \frac{\theta^2(1+\sqrt{\alpha}_2)}{(1-\sqrt{\alpha}_2)}\}$. Clearly from the definition of K, we see that (4.9) is true for $n = 1, 2$. Let the 2 inequalities

$$\alpha_n \geq \left(\frac{K-\theta^n}{K+\theta^n}\right)^2, \alpha_{n+1} \geq \left(\frac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2$$

be the induction hypothesis. Then we have

$$\alpha_{n+2} = M(y_{n+3}, y_{n+2}, qt)$$

$$= M(T(x_{n+1}, x_{n+2}), T(x_n, x_{n+1}), qt)$$

$$\geq \phi\{M(fx_{n+1}, fx_n, t), M(fx_{n+2}, fx_{n+1}, t)\}$$

$$\geq \phi\{\alpha_n, \alpha_{n+1}\}$$

$$\geq \phi\{\left(\tfrac{K-\theta^n}{K+\theta^n}\right)^2, \left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2\}$$

$$= \left(\tfrac{K-\theta^{n+1}}{K+\theta^{n+1}}\right)^2$$

$$\geq \left(\tfrac{K-\theta^{n+2}}{K+\theta^{n+2}}\right)^2.$$

Thus inductive proof of (4.9) is complete. Now for $p \in N$ and $t \in [0, \infty)$, we have

$$M(y_{n+p}, y_n, t) \geq M(y_{n+p}, y_{n+p-1}, \tfrac{t}{2}) \star M(y_{n+p-1}, y_{n+p-2}, \tfrac{t}{2^2}) \star \ldots \star M(y_{n+1}, y_n, \tfrac{t}{2^p})$$

$$\geq \left(\tfrac{K-2^{n+p-1}}{K+2^{n+p-1}}\right)^2 \star \left(\tfrac{K-2^{2n+p-2}}{K+2^{2n+p-2}}\right)^2 \star \ldots \star \left(\tfrac{K-2^{np}}{K+2^{np}}\right)^2$$

$$\to 1 \star 1 \star \ldots \star 1 = 1, \text{ as } n \to \infty.$$

Hence $< y_n >$ is a R-Cauchy sequence in $f(X)$ and since $f(X)$ is R-complete, there will exist z in $f(X)$ such that $lim_{n\to\infty} y_n = z$. Let $z = f(u)$ for some $u \in X$. Then we have

$$M(T(u, u), fu, t) = lim_{n\to\infty} M(T(u, u), y_{n+2}, t)$$

$$= lim_{n\to\infty} M(T(u, u), T(x_n, x_{n+1}), t)$$

$$\geq lim_{n\to\infty} M(T(u, u), T(x_{n+1}, u), \tfrac{t}{2}) \star M(T(x_{n+1}, u), T(x_n, x_{n+1}), \tfrac{t}{2})$$

$$\geq lim_{n\to\infty}\{\phi\{M(fu, fx_{n+1}, t), M(fu, fu, t)\} \star \phi\{M(fx_{n+1}, fx_n, t), (M(fu, fx_{n+1}, t))\}$$

$$= 1.$$

Similarly it can be shown that $M(fu, T(u,u), t) = 1$, and so $C(f,t) \neq \varnothing$ and v is a point of coincidence of f and T. Suppose there exists $v^* \in X$ such that $f(u^*) = T(u^*, u^*, u^*) = v^*$ for some u^* in $C(f,T)$. Then

$$M(v, v^*, qt) = M(T(u,u), T(u^*, u^*), qt)$$

$$\geq M(T(u,u), T(u^*,u), \tfrac{qt}{2}) \star M(T(u^*,u), T(u^*,u^*), \tfrac{qt}{2})$$

$$\geq \phi\{M(fu, fu^*, \tfrac{t}{2}), M(fu, fu, \tfrac{t}{2}))\} \star \phi\{M(fu^*, fu^*, \tfrac{t}{2}), M(fu, fu^*, \tfrac{t}{2})\}$$

$$\geq \phi\{M(fu, fu^*, \tfrac{t}{2}), M(fu, fu^*, \tfrac{t}{2})\} \star \phi\{M(fu, fu^*, \tfrac{t}{2}), M(fu, fu^*, \tfrac{t}{2})\}$$

$$\geq M(fu, fu^*, \tfrac{t}{2}) \star M(fu, fu^*, \tfrac{t}{2})$$

$$= M(T(u,u), T(u^*, u^*), \tfrac{t}{2}) \star M(T(u,u), T(u^*,u^*), \tfrac{t}{2})$$

$$\geq M(T(u,u), T(u^*,u), \tfrac{t}{2^2}) \star M(T(u^*,u), T(u^*,u^*), \tfrac{t}{2^2})$$

$$\geq \phi\{M(fu, fu^*, \tfrac{t}{2^2 q}), M(fu, fu, \tfrac{t}{2^2 q})\} \star \phi\{M(fu^*, fu^*, \tfrac{t}{2^2 q}), M(fu, fu^*, \tfrac{t}{2^2 q})\}.$$

$$\geq \phi\{M(fu, fu^*, \tfrac{t}{2^2 q}), M(fu, fu^*, \tfrac{t}{2^2 q})\} \star \phi\{M(fu, fu^*, \tfrac{t}{2^2 q}), M(fu, fu^*, \tfrac{t}{2^2 q})\}.$$

$$\geq M(fu, fu^*, \tfrac{t}{2^q}) \star M(fu, fu^*, \tfrac{t}{2^q})$$

Repeating as above n times we get

$$M(v, v^*, qt) \geq M(fu, fu^*, \tfrac{t}{2^n q^{n-1}}) \star M(fu, fu^*, \tfrac{t}{2^n q^{n-1}}) \to 1 \text{ as } n \to \infty.$$

Similarly it can be shown that $M(v^*, v, qt) \to 1$ and so $v = v^*$, i.e. v is the unique point of coincidence of f and T. Hence by lemma (4.3) v is a unique common fixed point of f and T. □

Proof of the following theorems follows on the same lines as that of the previous theorems:

Theorem 4.7. *Let $(X, M, *)$ be a $DFqM-Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be mappings satisfying (4.1), (4.7) and (4.8). Then f and T has a common fixed point if one of the following two conditions are satisfied:*
(i) f is oci with respect to T and the pair (f,T) is weakly compatible,
(ii) f is coincidentally idempotent with respect to T and the pair (f,T) is owc.

Theorem 4.8. *Let $(X, M, *)$ be a $DFqM-Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be weakly compatible mappings satisfying (4.1), (4.2), (4.6) and the following :*

(4.10) *f(X) is L-complete*

Then the sequence $< y_n >$ defined by (4.4) converges to a unique common fixed point of f and T.

Theorem 4.9. *Let $(X, M, *)$ be a $DFqM - Space$, $f : X \longrightarrow X$ and $T : X^2 \longrightarrow X$ be mappings satisfying (4.1),(4.2) (with $0 < q < 1$) and (4.10). Then f and T has a common fixed point if one of the following two conditions are satisfied:*
(i) f is oci with respect to T and the pair (f, T) is weakly compatible,
(ii) f is coincidentally idempotent with respect to T and the pair (f, T) is owc.

Taking $X^2 = X$ in the Theorems 4.6 and 4.8, we get the following.

Corollary 4.10. *Let $(X, M, *)$ be a $DFqM - Space$, $f : X \longrightarrow X$ and $T : X \longrightarrow X$ be mappings, such that*

$$T(X) \subseteq f(X) \tag{4.11}$$

$$M(Tx, Ty, qt) \geq \phi\{M(fx, fy, t)\}, \tag{4.12}$$

for all $x, y \in X$, $0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

(4.13) *f(X) is L-complete or R-complete*

$$lim_{t\to\infty} M(x, y, t) = 1. \tag{4.14}$$

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$. Further f and T has a common fixed point provided the pair (f, T) is weakly compatible .

Taking $X^2 = X$ in the Theorems 4.7 and 4.9, we get the following.

Corollary 4.11. *Let $(X, M, *)$ be a $DFqM - Space$, $f : X \longrightarrow X$ and $T : X \longrightarrow X$ be mappings, such that*

$$T(X) \subseteq f(X) \tag{4.15}$$

$$M(Tx, Ty, qt) \geq \phi\{M(fx, fy, t)\}, \tag{4.16}$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$

(4.17) *f(X) is L-complete or R-complete*

Then f and T has a coincidence point, i.e. $C(f, T) \neq \phi$. Further f and T has a common fixed point if one of the following two conditions are satisfied:
(i) f is oci with respect to T and the pair (f, T) is weakly compatible,
(ii) f is coincidentally idempotent with respect to T and the pair (f, T) is owc.

If we take f to be the identity mapping in the above corollaries, we get the following

Corollary 4.12. *Let $(X, M, *)$ be a L-complete or R-complete $DFqM - Space$, $T : X \longrightarrow X$ be mappings, such that*

$$M(Tx, Ty, qt) \geq M(x, y, t), \tag{4.18}$$

where $x, y \in X$, $0 < q < 1$ and $t \in [0, \infty)$. Then T has a fixed point.

Example 4.13. Let $X = [0,2]$ and $a*b = ab$ for all $a, b \in [0,\infty)$. Let $d : X\times X \to X$ be given by

$$d(x,y) =| x - y | + | x | + | y | .$$

Then d is a dislocated metric on X and $(X, M, *)$ is a dislocated fuzzy metric space where M is the dislocated fuzzy metric induced by d. Let $T : X^2 \to X$ and $f : X \to X$be defined by $T(x,y)=\frac{(x^2+y^2)}{16}$ and $f(x) = \frac{x^2}{2}$. Then we have

$$M((T(x,y),T(y,z),qt))= \frac{qt}{qt+d(T(x,y),T(y,z))}$$

$$= \frac{qt}{qt+|\frac{x^2-z^2}{16}|+|\frac{x^2+y^2}{16}|+|\frac{y^2+z^2}{16}|}$$

$$\geq \frac{qt}{qt+|\frac{x^2-y^2}{16}|+|\frac{x^2}{16}|+|\frac{y^2}{16}|+|\frac{y^2-z^2}{16}|+|\frac{y^2}{16}|+|\frac{z^2}{16}|}$$

$$\geq \frac{t}{t+|\frac{x^2-y^2}{4}|+|\frac{x^2}{4}|+|\frac{y^2}{4}|+|\frac{y^2-z^2}{4}|+|\frac{y^2}{4}|+|\frac{z^2}{4}|} \text{ (Taking } q = \tfrac{1}{4})$$

$$\geq Min\{\frac{t}{t+|\frac{x^2-y^2}{2}|+|\frac{x^2}{2}|+|\frac{y^2}{2}|}, \frac{t}{t+|\frac{y^2-z^2}{2}|+|\frac{y^2}{2}|+|\frac{z^2}{2}|}\}$$

$$= Min\{M(fx, fy, t), M(fy, fz, t)\}.$$

Thus f and T satisfy condition (4.2) with $\phi(t_1, t_2) = Min\{t_1, t_2\}$ and $q = \frac{1}{4}$. We see that $C(f, T) = \{0\}$, f and T commute at 0. Finally 0 is the unique common fixed point of f and T.

Remark 4.14. Corollary 4.12 is generalised fuzzy version of Banach Contraction Principle proved in [9]. Corollary 4.10 and 4.11 are generalised and extended version of the result proved in [22]. Theorems 4.3,4.4,4.5,4.6, 4.7,4.8 and 4.9 are generalised and extended fuzzy version of the results proved in [4], [5], [19] and [20] for $k = 2$.

Acknowledgements. The authors are thankful to the learned referee for the valuable comments which helped in bringing this paper to its present form.

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