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Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroups

Sukhendu Kar, Paltu Sarkar

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Abstract. In this paper, we introduce fuzzy quasi-ideal and fuzzy bi-ideal of a ternary semigroup and study some related properties of these two subsystems of ternary semigroups. We also characterize fuzzy quasi-ideal and fuzzy bi-ideal of a ternary semigroup.

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Corresponding Author : Sukhendu Kar (karsukhendu@yahoo.co.in)

1. Introduction

Fuzzy set has an important impact over the field of mathematical research in both theory and application. It has found manifold applications in mathematics and related areas. The introduction of fuzzy algebraic structures started by Rosenfeld in his pioneering paper [10]. He introduced the notion of fuzzy subgroup of a group. Kuroki introduced and studied the notion of fuzzy semigroups [6]. He also studied the concept of fuzzy quasi-ideals [5] and fuzzy bi-ideals [4] of semigroups. The notion of ternary algebraic system was first introduced by D. H. Lehmer [7] in 1932. The notion of ternary semigroups was introduced by S. Banach [8]. The ideal theory in ternary semigroups was studied by F. M. Sioson [12] in the year 1965. Recently, M. L. Santiago and S. Sri Bala [11] developed the theory of ternary semigroups. In [2], T. K. Dutta, S. Kar and B. K. Maity studied some properties of regular ternary semigroup, completely regular ternary semigroup, intra-regular ternary semigroup and characterized them by using various ideals of ternary semigroups. Many results in ordinary semigroups may be extended to n-ary semigroups for arbitrary n but the transition from n = 3 to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. For this reason, we shall confine ourselves in this paper wholly to ternary semigroups. In this paper we introduce and characterize the concept of fuzzy quasi-ideal and fuzzy bi-ideal in ternary semigroups and study their properties in ternary semigroups.

2. Preliminaries

In this section, we give some preliminary results of ternary semigroups and fuzzy ternary semigroups which will be required for our later discussions.

Definition 2.1. ([2]) A non-empty set S together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in S$.

Example 2.2. Let $\mathbb{Z}^-$ be the set of all negative integers. Then with the usual ternary multiplication, $\mathbb{Z}^-$ forms a ternary semigroup.

Definition 2.3. ([2])(A non-empty subset I of a ternary semigroup S is called
(i) a left ideal of S if $SSI \subseteq I$.
(ii) a lateral ideal of S if $SIS \subseteq I$.
(iii) a right ideal of S if $ISS \subseteq I$.
(iv) an ideal of S if I is a left ideal, a lateral ideal and a right ideal of S.
An ideal I of a ternary semigroup S is called a proper ideal if $I \neq S$.

Proposition 2.4. ([12]) *Let S be a ternary semigroup and $a \in S$. Then the principal*
(i) *left ideal generated by $'a'$ is given by $\langle a\rangle_L = \{a\} \cup SSa$.*
(ii) *right ideal generated by $'a'$ is given by $\langle a\rangle_R = \{a\} \cup aSS$.*
(iii) *lateral ideal generated by $'a'$ is given by $\langle a\rangle_M = \{a\} \cup SaS \cup SSaSS$.*
(iv) *ideal generated by $'a'$ is given by $\langle a\rangle = \{a\} \cup aSS \cup SaS \cup SSaSS \cup SSa$.*

Definition 2.5. ([12]) A non-empty subset Q of a ternary semigroup S is said to be a quasi-ideal of S if $(QSS \cap SQS \cap SSQ) \subseteq Q$ and $(QSS \cap SSQSS \cap SSQ) \subseteq Q$.

The above definition can be rewritten as follows :

Definition 2.6. A non-empty subset Q of a ternary semigroup S is said to be a quasi-ideal of S if $\big(QSS \cap (SQS \cup SSQSS) \cap SSQ\big) \subseteq Q$.

Definition 2.7. ([1]) A ternary subsemigroup B of a ternary semigroup S is said to be a bi-ideal of S if $BSBSB \subseteq B$.

Definition 2.8. ([13]) Let S be a non-empty set. A fuzzy subset of S is a function $f : S \longrightarrow [0, 1]$.

Definition 2.9. ([13]) Let f be a fuzzy subset of a non-empty set S. For any $t \in [0, 1]$; the subset $f_t = \{x \in S : f(x) \geq t\}$ of S is called a level subset of f.

Definition 2.10. ([13]) Let S be a non-empty set and $A \subseteq S$. Then the characteristic function $C_A : S \longrightarrow [0, 1]$ of A is a fuzzy subset of S, defined by, for any $x \in S$;

$$C_A(x) = \begin{cases} 1\,, & if\ x \in A\,; \\ 0\,, & if\ x \notin A\,. \end{cases}$$

We denote the characteristic function C_S of S as S i.e. $S = C_S$. Thus $S(x) = 1$ for all $x \in S$.

Definition 2.11. ([13]) For any two fuzzy subsets f and g of a non-empty set S, the union and the intersection of f and g, denoted by $f \cup g$ and $f \cap g$ are fuzzy subsets of S, defined as, for any $x \in S$, $(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$ and $(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$, where $\bigvee$ denotes maximum or supremum and $\bigwedge$ denotes minimum or infimum.

- Throughout this paper, $F(S)$ denotes the set of all non-empty fuzzy subsets (i.e. for any fuzzy subset f, there exists an $x \in S$ such that $f(x) \neq 0$) of the ternary semigroup S.

Definition 2.12. ([3]) Let $f, g, h \in F(S)$. Then the product of f, g, h; denoted by $f\ o\ g\ o\ h$, is defined as, for any $x \in S$,

$$(f\ o\ g\ o\ h)(x) = \begin{cases} \bigvee\limits_{x=pqr} \{f(p) \wedge g(q) \wedge h(r)\}, & \text{if } x \text{ can be expressed as } x = pqr \text{ for some } p, q, r \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.13. ([3]) *If* $f, g, h, k, l \in F(S)$, *then*
(*i*) $f \cap (g \cup h) \cap k = (f \cap g \cap k) \cup (f \cap h \cap k)$.
(*ii*) $(f \cup g)\ o\ h\ o\ k = (f\ o\ h\ o\ k) \cup (g\ o\ h\ o\ k)$.
(*iii*) $f\ o\ (g \cup h)\ o\ k = (f\ o\ g\ o\ k) \cup (f\ o\ h\ o\ k)$.
(*iv*) $f\ o\ g\ o\ (h \cup k) = (f\ o\ g\ o\ h) \cup (f\ o\ g\ o\ k)$.
(*v*) $(f\ o\ g\ o\ h)\ o\ k\ o\ l = f\ o\ (g\ o\ h\ o\ k)\ o\ l = f\ o\ g\ o\ (h\ o\ k\ o\ l)$.

Proposition 2.14. *If* $f, g \in F(S)$, *then*
(*i*) $\big((f \cap g)\ o\ S\ o\ S\big) \subseteq (f\ o\ S\ o\ S) \cap (g\ o\ S\ o\ S)$.
(*ii*) $\big(S\ o\ (f \cap g)\ o\ S\big) \subseteq (S\ o\ f\ o\ S) \cap (S\ o\ g\ o\ S)$.
(*iii*) $\big(S\ o\ S\ o\ (f \cap g)\big) \subseteq (S\ o\ S\ o\ f) \cap (S\ o\ S\ o\ g)$.

Proof. (*i*) Let $x \in S$. If $x \neq pqr$ for any $p, q, r \in S$, then

$$\big((f\ o\ S\ o\ S) \cap (g\ o\ S\ o\ S)\big)(x) = 0 = \big((f \cap g)\ o\ S\ o\ S\big)(x).$$

If $x = pqr$ for some $p, q, r \in S$, then

$$\begin{aligned}\big((f \cap g)\ o\ S\ o\ S\big)(x) &= \bigvee_{x=pqr} \{(f \cap g)(p) \wedge S(q) \wedge S(r)\} \\ &= \bigvee_{x=pqr} \{f(p) \wedge g(p)\} \\ &\qquad (\text{since } S(q) = S(r) = 1 \text{ for every } q, r \in S) \\ &\leq \Big\{\bigvee_{x=pqr} \{f(p)\}\Big\} \wedge \Big\{\bigvee_{x=pqr} \{g(p)\}\Big\} \\ &= \Big\{\bigvee_{x=pqr} \{f(p) \wedge S(q) \wedge S(r)\}\Big\} \wedge \Big\{\bigvee_{x=pqr} \{g(p) \wedge S(q) \wedge S(r)\}\Big\} \\ &= (f\ o\ S\ o\ S)(x) \wedge (g\ o\ S\ o\ S)(x) \\ &= \big((f\ o\ S\ o\ S) \cap (g\ o\ S\ o\ S)\big)(x).\end{aligned}$$

Therefore $\big((f \cap g)\ o\ S\ o\ S\big)(x) \leq \big((f\ o\ S\ o\ S) \cap (g\ o\ S\ o\ S)\big)(x)$.
Thus $\big((f \cap g)\ o\ S\ o\ S\big) \subseteq (f\ o\ S\ o\ S) \cap (g\ o\ S\ o\ S)$ and hence the result.
Similarly, we can prove the other results. ∎

Definition 2.15. ([3]) A non-empty fuzzy subset f of a ternary semigroup S is called a fuzzy ternary subsemigroup of S if $f(xyz) \geq f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z \in S$.

Definition 2.16. ([3]) A non-empty fuzzy subset f of a ternary semigroup S is called a fuzzy left (fuzzy lateral, fuzzy right) ideal of S if $f(xyz) \geq f(z)$ $\big($resp. $f(xyz) \geq f(y), f(xyz) \geq f(x)\big)$ for all $x, y, z \in S$.

If f is a fuzzy left ideal, a fuzzy lateral ideal and a fuzzy right ideal of S, then f is called a fuzzy ideal of S.

Lemma 2.17. ([3]) *Let A be a non-empty subset of a ternary semigroup S. Then*
(i) A is a ternary subsemigroup of S if and only if C_A is a fuzzy ternary subsemigroup of S.
(ii) A is a left ideal (lateral ideal, right ideal, ideal) of S if and only if C_A is a fuzzy left ideal (resp. fuzzy lateral ideal, fuzzy right ideal, fuzzy ideal) of S.

Lemma 2.18. ([3]) *Let $f \in F(S)$. Then*
(i) f is a fuzzy ternary subsemigroup of S if and only if $f \ o\ f \ o\ f \subseteq f$.
(ii) f is a fuzzy left ideal of S if and only if $S \ o\ S \ o\ f \subseteq f$.
(iii) f is a fuzzy lateral ideal of S if and only if $S \ o\ f \ o\ S \subseteq f$.
(iv) f is a fuzzy right ideal of S if and only if $f \ o\ S \ o\ S \subseteq f$.

Definition 2.19. ([9]) Let S be a non-empty set and $a \in S, t \in (0, 1]$. A fuzzy point a_t of S is a fuzzy subset of S, defined by,

$$a_t(x) = \begin{cases} t\,, & if\ x = a\,; \\ 0\,, & otherwise\,; \end{cases}$$

where $x \in S$.

Definition 2.20. ([9]) Let f be a non-empty fuzzy subset and x_t be a fuzzy point of a non-empty set S. The fuzzy point x_t of S is said to be contained in f or to belong to f (denoted by $x_t \in f$) if $f(x) \geq t$.

Proposition 2.21. ([3]) *Let x_t, y_s be two fuzzy points of a ternary semigroup S. Then for any $f \in F(S)$, we have the following results.*
(i) $x_t \in f$ if and only if $x_t \subseteq f$.
(ii) $x_t \subseteq y_s$ if and only if $x = y$ and $t \leq s$.

Definition 2.22. Let f be a non-empty fuzzy subset of a ternary semigroup S. Then the intersection of all fuzzy left ideals of S containing f is a fuzzy left ideal of S containing f, denoted by $\langle f \rangle_L$ and defined as

$$\langle f \rangle_L = \bigcap_{f \subseteq g \in FI_L(S)} \{\, g \,\}$$

where $FI_L(S)$ is the set of all fuzzy left ideals of S. This fuzzy left ideal is called fuzzy left ideal generated by the fuzzy subset f.

Similarly, we define $\langle f \rangle_M, \langle f \rangle_R$ and $\langle f \rangle$ as the fuzzy lateral ideal generated by the fuzzy subset f, fuzzy right ideal generated by the fuzzy subset f and fuzzy ideal generated by the fuzzy subset f respectively.

Note 2.23. *Here $\langle f\rangle_L$, $\langle f\rangle_M$ and $\langle f\rangle_R$ are respectively the smallest fuzzy left ideal, the smallest fuzzy lateral ideal and the smallest fuzzy right ideal of S containing the fuzzy subset f of S.*

Proposition 2.24. *Let f be a non-empty fuzzy subset of a ternary semigroup S. Then*
(i) $\langle f\rangle_L = f \cup (S\ o\ S\ o\ f)$.
(ii) $\langle f\rangle_R = f \cup (f\ o\ S\ o\ S)$.
(iii) $\langle f\rangle_M = f \cup (S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S)$.
(iv) $\langle f\rangle = f \cup (S\ o\ S\ o\ f) \cup (S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S) \cup (f\ o\ S\ o\ S)$.

Proof. (i) Since $\langle f\rangle_L = \bigcap_{f \subseteq g \in FI_L(S)} \{g\}$, $f \subseteq \langle f\rangle_L$. Therefore, $S\ o\ S\ o\ f \subseteq S\ o\ S\ o\ \langle f\rangle_L \subseteq \langle f\rangle_L$ (since $\langle f\rangle_L$ is a fuzzy left ideal of S). Thus $\big(f \cup (S\ o\ S\ o\ f)\big) \subseteq \langle f\rangle_L$. Again, $\big(f \cup (S\ o\ S\ o\ f)\big)$ is a fuzzy left ideal of S containing f and $\langle f\rangle_L$ is the smallest fuzzy left ideal of S containing f. Hence $\langle f\rangle_L \subseteq f \cup (S\ o\ S\ o\ f)$. Therefore $\langle f\rangle_L = f \cup (S\ o\ S\ o\ f)$. Similarly, we can prove the other results. □

Proposition 2.25. ([3]) *Let a_t be a fuzzy point of a ternary semigroup S. If $\langle a_t\rangle_L$, $\langle a_t\rangle_M$, $\langle a_t\rangle_R$ and $\langle a_t\rangle$ are respectively fuzzy left ideal, fuzzy lateral ideal, fuzzy right ideal and fuzzy ideal of S generated by the fuzzy point a_t, then*
(i) $\langle a_t\rangle_L = a_t \cup (S\ o\ S\ o\ a_t)$.
(ii) $\langle a_t\rangle_R = a_t \cup (a_t\ o\ S\ o\ S)$.
(iii) $\langle a_t\rangle_M = a_t \cup (S\ o\ a_t\ o\ S) \cup (S\ o\ S\ o\ a_t\ o\ S\ o\ S)$.
(iv) $\langle a_t\rangle = a_t \cup (S\ o\ S\ o\ a_t) \cup (S\ o\ a_t\ o\ S) \cup (S\ o\ S\ o\ a_t\ o\ S\ o\ S) \cup (a_t\ o\ S\ o\ S)$.

3. Fuzzy Quasi-ideals of Ternary Semigroups

Definition 3.1. A non-empty fuzzy subset f of a ternary semigroup S is called a fuzzy quasi-ideal of S if

$$\big(f\ o\ S\ o\ S \cap \big((S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S)\big) \cap S\ o\ S\ o\ f\big) \subseteq f.$$

Example 3.2. Let $O = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), C = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $S = \{O, A, B, C, D\}$ becomes a ternary semigroup w.r.t. usual matrix multiplication and $Q = \{O, B\}$ is a quasi-ideal of S. Now we define a fuzzy subset f on S as $f(O) = f(B) = 0.85$ and $f(A) = f(C) = f(D) = 0$. Then f is a fuzzy quasi-ideal of S.

Note 3.3. *Using Proposition 2.13, the above definition can be rewritten in the following way.*

Definition 3.4. A non-empty fuzzy subset f of a ternary semigroup S is called a fuzzy quasi-ideal of S if

$$(f\ o\ S\ o\ S) \cap (S\ o\ f\ o\ S) \cap (S\ o\ S\ o\ f) \subseteq f$$

$$\text{and} \quad (f\ o\ S\ o\ S) \cap (S\ o\ S\ o\ f\ o\ S\ o\ S) \cap (S\ o\ S\ o\ f) \subseteq f.$$

Theorem 3.5. *Let Q be a non-empty subset of a ternary semigroup S. Then Q is a quasi-ideal of S if and only if the characteristic function C_Q of Q is a fuzzy quasi-ideal of S.*

Proposition 3.6. *A non-empty fuzzy subset f of a ternary semigroup S is a fuzzy quasi-ideal of S if and only if the level subset of f, f_t is a quasi-ideal of S for $t \in Im\, f$.*

Proposition 3.7. *Every fuzzy quasi-ideal of a ternary semigroup S is a fuzzy ternary subsemigroup of S.*

Proof. Let f be a fuzzy quasi-ideal of S. Then f is non-empty. Now $f\ o\ f\ o\ f \subseteq f\ o\ S\ o\ S$, $f\ o\ f\ o\ f \subseteq S\ o\ f\ o\ S \subseteq (S\ o\ f\ o\ S\ \cup\ S\ o\ S\ o\ f\ o\ S\ o\ S)$ and $f\ o\ f\ o\ f \subseteq S\ o\ S\ o\ f$. Therefore
$f\ o\ f\ o\ f \subseteq \big(f\ o\ S\ o\ S \cap (S\ o\ f\ o\ S \cup S\ o\ S\ o\ f\ o\ S\ o\ S) \cap S\ o\ S\ o\ f\big) \subseteq f$ (since f is a fuzzy quasi-ideal of S). Hence, by Lemma 2.18, f is a fuzzy ternary subsemigroup of S. □

Proposition 3.8. *Every fuzzy left ideal (fuzzy lateral ideal, fuzzy right ideal, fuzzy ideal) of a ternary semigroup S is a fuzzy quasi-ideal of S.*

Proof. Let f be a fuzzy left ideal of S. Then f is non-empty and $(S\ o\ S\ o\ f) \subseteq f$. Now

$$\begin{aligned} & \big(f\ o\ S\ o\ S \cap (S\ o\ f\ o\ S\ \cup\ S\ o\ S\ o\ f\ o\ S\ o\ S) \cap S\ o\ S\ o\ f\big) \\ \subseteq\ & (S\ o\ S\ o\ S) \cap (S\ o\ S\ o\ S) \cap f \\ \subseteq\ & (S \cap S \cap f) \subseteq f. \end{aligned}$$

Hence by Definition 3.1, f is a fuzzy quasi-ideal of S.
Similarly, we can prove the other cases. □

Note 3.9. *But the converse does not hold always. This follows from the following example.*

Example 3.10. In Example 3.2, $S = \{O, A, B, C, D\}$ is a ternary semigroup and f is a fuzzy quasi-ideal of S. But
$f(CAB) = f(D) = 0 \ngeq 0.85 = f(B) \Longrightarrow f$ is not a fuzzy left ideal of S.
$f(BDC) = f(A) = 0 \ngeq 0.85 = f(B) \Longrightarrow f$ is not a fuzzy right ideal of S.
$f(CBD) = f(D) = 0 \ngeq 0.85 = f(B) \Longrightarrow f$ is not a fuzzy lateral ideal of S.
Hence f is neither a fuzzy right ideal nor a fuzzy lateral ideal nor a fuzzy left ideal of S i.e. f is not a fuzzy ideal of S.

Proposition 3.11. *Intersection of any two fuzzy quasi-ideals of a ternary semigroup S is either empty fuzzy subset or a fuzzy quasi-ideal of S.*

Corollary 3.12. *Intersection of any two fuzzy ideals of a ternary semigroup S is a fuzzy quasi-ideal of S.*

Definition 3.13. ([2]) Let S be a ternary semigroup. An element $x \in S$ is called regular if there exists an element $a \in S$ such that $x = xax$. A ternary semigroup is called regular if all its elements are regular.

Theorem 3.14. ([3]) *The following conditions in a ternary semigroup S are equivalent :*
(i) S is regular.
(ii) For any fuzzy right ideal f, fuzzy lateral ideal g and fuzzy left ideal h of S,

$f \ o \ g \ o \ h = f \cap g \cap h$.
(iii) For $a, b, c \in S$ *and* $t \in (0, 1]$, $\langle a_t \rangle_R \ o \ \langle b_t \rangle_M \ o \ \langle c_t \rangle_L = \langle a_t \rangle_R \cap \langle b_t \rangle_M \cap \langle c_t \rangle_L$.
(iv) For $a \in S$ *and* $t \in (0, 1]$, $\langle a_t \rangle_R \ o \ \langle a_t \rangle_M \ o \ \langle a_t \rangle_L = \langle a_t \rangle_R \cap \langle a_t \rangle_M \cap \langle a_t \rangle_L$.

Proposition 3.15. *If* f *be a fuzzy quasi-ideal of a ternary semigroup* S*, then*

$$(f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq f.$$

Proof. Let f be a fuzzy quasi-ideal of S. Now

$(f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq f \ o \ S \ o \ S$,
$(f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq S \ o \ S \ o \ f$,
$(f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq (S \ o \ S \ o \ f \ o \ S \ o \ S \cup S \ o \ f \ o \ S)$.

Therefore

$$\begin{aligned} & (f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \\ \subseteq \ & \big(f \ o \ S \ o \ S \ \cap \ (S \ o \ f \ o \ S \ \cup \ S \ o \ S \ o \ f \ o \ S \ o \ S) \ \cap \ S \ o \ S \ o \ f\big) \subseteq f \end{aligned}$$

(by assumption) and hence the result. □

Note 3.16. *The converse of the above Proposition 3.15, in general, not always true. But in case of regular ternary semigroup, the converse is true always.*

Proposition 3.17. *If* f *be a non-empty fuzzy subset of a regular ternary semigroup* S *such that* $(f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq f$*, then* f *is a fuzzy quasi-ideal of* S*.*

Proof. Let f be a non-empty fuzzy subset of a regular ternary semigroup S such that $(f \ o \ S \ o \ f \ o \ S \ o \ f \cup f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq f$. Now we have to show that f is a fuzzy quasi-ideal of S. Since S is regular and $f \ o \ S \ o \ S$, $(S \ o \ f \ o \ S \cup S \ o \ S \ o \ f \ o \ S \ o \ S)$ and $S \ o \ S \ o \ f$ are respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S, so

$$\begin{aligned} & (f \ o \ S \ o \ S) \ \cap \ (S \ o \ f \ o \ S \ \cup \ S \ o \ S \ o \ f \ o \ S \ o \ S) \ \cap \ (S \ o \ S \ o \ f) \\ = \ & (f \ o \ S \ o \ S) \ o \ \big((S \ o \ f \ o \ S) \ \cup \ (S \ o \ S \ o \ f \ o \ S \ o \ S)\big) \ o \ (S \ o \ S \ o \ f), \\ & \qquad \text{by Theorem 3.14} \\ = \ & \big(f \ o \ (S \ o \ S \ o \ S) \ o \ f \ o \ (S \ o \ S \ o \ S) \ o \ f\big) \\ & \quad \cup \big(f \ o \ S \ o \ (S \ o \ S \ o \ S) \ o \ f \ o \ (S \ o \ S \ o \ S) \ o \ S \ o \ f\big), \\ & \qquad \text{by Proposition 2.13} \\ \subseteq \ & (f \ o \ S \ o \ f \ o \ S \ o \ f \ \cup \ f \ o \ S \ o \ S \ o \ f \ o \ S \ o \ S \ o \ f) \subseteq f \end{aligned}$$

(by assumption). Hence by Definition 3.1, f is a fuzzy quasi-ideal of S. □

Proposition 3.18. *For any three fuzzy quasi-ideals* f_1, f_2, f_3 *of a regular ternary semigroup* S, $f_1 \ o \ f_2 \ o \ f_3$ *is fuzzy quasi-ideal of* S*.*

Proof. Since f_1, f_2, f_3 are fuzzy quasi-ideals of S, then $f_1\ o\ f_2\ o\ f_3$ is a non-empty fuzzy subset of S. Now

$$\begin{aligned}
& ((f_1\ o\ f_2\ o\ f_3)\ o\ S\ o\ (f_1\ o\ f_2\ o\ f_3)\ o\ S\ o\ (f_1\ o\ f_2\ o\ f_3)) \\
& \qquad \cup ((f_1\ o\ f_2\ o\ f_3)\ o\ S\ o\ S\ o\ (f_1\ o\ f_2\ o\ f_3)\ o\ S\ o\ S\ o\ (f_1\ o\ f_2\ o\ f_3)) \\
= & \Big((f_1\ o\ (f_2\ o\ f_3\ o\ S)\ o\ f_1\ o\ (f_2\ o\ f_3\ o\ S)\ o\ f_1)\ o\ f_2\ o\ f_3\Big) \\
& \qquad \cup \Big((f_1\ o\ (f_2\ o\ f_3\ o\ S)\ o\ S\ o\ f_1\ o\ (f_2\ o\ f_3\ o\ S)\ o\ S\ o\ f_1)\ o\ f_2\ o\ f_3\Big) \\
\subseteq & \Big((f_1\ o\ (S\ o\ S\ o\ S)\ o\ f_1\ o\ (S\ o\ S\ o\ S)\ o\ f_1)\ o\ f_2\ o\ f_3\Big) \\
& \qquad \cup \Big((f_1\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ f_1\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ f_1)\ o\ f_2\ o\ f_3\Big) \\
\subseteq & ((f_1\ o\ S\ o\ f_1\ o\ S\ o\ f_1)\ o\ f_2\ o\ f_3) \cup ((f_1\ o\ S\ o\ S\ o\ f_1\ o\ S\ o\ S\ o\ f_1)\ o\ f_2\ o\ f_3) \\
= & (f_1\ o\ S\ o\ f_1\ o\ S\ o\ f_1) \cup (f_1\ o\ S\ o\ S\ o\ f_1\ o\ S\ o\ S\ o\ f_1)\ o\ f_2\ o\ f_3 \\
\subseteq & f_1\ o\ f_2\ o\ f_3 \text{ (since } f_1 \text{ is a fuzzy quasi-ideal of } S \text{ and by Proposition 3.15)}
\end{aligned}$$

Since S is regular, it follows from Proposition 3.17 that $f_1\ o\ f_2\ o\ f_3$ is a fuzzy quasi-ideal of S. □

Corollary 3.19. *The set of all fuzzy quasi-ideals of a regular ternary semigroup S forms a ternary semigroup under the operation $'o'$, defined in Definition 2.12.*

Theorem 3.20. *A non-empty fuzzy subset f of a ternary semigroup S is a fuzzy quasi-ideal of S if and only if f is the intersection of a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of S.*

Proof. Let $f = g \cap h \cap k$ where g, h, k are respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S. Then

$$\begin{aligned}
& (f\ o\ S\ o\ S) \cap (S\ o\ f\ o\ S) \cap (S\ o\ S\ o\ f) \\
= & ((g \cap h \cap k)\ o\ S\ o\ S\big) \cap \big(S\ o\ (g \cap h \cap k)\ o\ S\big) \cap \big(S\ o\ S\ o\ (g \cap h \cap k)) \\
\subseteq & (g\ o\ S\ o\ S) \cap (S\ o\ h\ o\ S) \cap (S\ o\ S\ o\ k) \\
\subseteq & g \cap h \cap k = f
\end{aligned}$$

and

$$\begin{aligned}
& (f\ o\ S\ o\ S) \cap (S\ o\ S\ o\ f\ o\ S\ o\ S) \cap (S\ o\ S\ o\ f) \\
= & ((g \cap h \cap k)\ o\ S\ o\ S) \cap (S\ o\ S\ o\ (g \cap h \cap k)\ o\ S\ o\ S) \cap (S\ o\ S\ o\ (g \cap h \cap k)) \\
\subseteq & (g\ o\ S\ o\ S) \cap (S\ o\ S\ o\ h\ o\ S\ o\ S) \cap (S\ o\ S\ o\ k)\ \subseteq\ g \cap h \cap k = f.
\end{aligned}$$

Hence, by Definition 3.4, $f = g \cap h \cap k$ is a fuzzy quasi-ideal of S.

Conversely, let f be a fuzzy quasi-ideal of S. Then $f \subseteq \langle f\rangle_R \cap \langle f\rangle_M \cap \langle f\rangle_L$

$$= \big(f \cup (f\ o\ S\ o\ S)\big) \cap \big(f \cup (S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S)\big) \cap \big(f \cup (S\ o\ S\ o\ f)\big)$$

(by Proposition 2.24)

$$= f \cup \Big((f\ o\ S\ o\ S) \cap \big((S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S)\big) \cap (S\ o\ S\ o\ f)\Big)$$

(by Proposition 2.13)

$$\subseteq f \cup f \text{ (by assumption)} = f.$$

It implies that $f = \langle f\rangle_R \cap \langle f\rangle_M \cap \langle f\rangle_L$. Hence f is the intersection of a fuzzy right ideal, a fuzzy lateral ideal and a fuzzy left ideal of S. □

Corollary 3.21. *In a regular ternary semigroup S, a non-empty-fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if $f = g\ o\ h\ o\ k$ where g, h and k are respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S.*

Proof. Let f be a fuzzy quasi-ideal of regular ternary semigroup S. Then for any fuzzy right ideal g, fuzzy lateral ideal h and fuzzy left ideal k of S, by Theorem 3.20, $f = g \cap h \cap k = g\ o\ h\ o\ k$ (since S is regular and by Theorem 3.14).

Conversely, let f be a non-empty fuzzy subset of S such that $f = g\ o\ h\ o\ k$ where g, h and k are respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S. Then for regular ternary semigroup S, $f = g\ o\ h\ o\ k = g \cap h \cap k$, by Theorem 3.14. So from Theorem 3.20, it follows that f is a fuzzy quasi-ideal of S. □

Definition 3.22. A fuzzy ideal (fuzzy quasi-ideal) f of a ternary semigroup S is said to be a minimal fuzzy ideal (minimal fuzzy quasi-ideal) of S if there is no fuzzy ideal (fuzzy quasi-ideal) of S strictly contained in f.

Theorem 3.23. *A non-empty fuzzy subset f of a ternary semigroup S is minimal fuzzy quasi-ideal of S if and only if f is the intersection of a minimal fuzzy right ideal, a minimal fuzzy lateral ideal and a minimal fuzzy left ideal of S.*

Proof. Let $f = g \cap h \cap k$ where g, h, k are respectively, a minimal fuzzy right ideal, a minimal fuzzy lateral ideal and a minimal fuzzy left ideal of S. Then f is a fuzzy quasi-ideal of S, by Theorem 3.20. Let f_1 be another fuzzy quasi-ideal of S such that $f_1 \subseteq f$. Then $f_1\ o\ S\ o\ S \subseteq f\ o\ S\ o\ S = (g \cap h \cap k)\ o\ S\ o\ S \subseteq g\ o\ S\ o\ S \subseteq g$ (since g is a fuzzy right ideal of S). Also, $f_1\ o\ S\ o\ S$ is a fuzzy right ideal of S. But g is a minimal fuzzy right ideal of S. So $g = f_1\ o\ S\ o\ S$. In the same way, we can show that $h = (S\ o\ f_1\ o\ S) \cup (S\ o\ S\ o\ f_1\ o\ S\ o\ S)$ and $k = S\ o\ S\ o\ f_1$. Therefore $f = g \cap h \cap k = (f_1\ o\ S\ o\ S) \cap \big((S\ o\ f_1\ o\ S) \cup (S\ o\ S\ o\ f_1\ o\ S\ o\ S)\big) \cap (S\ o\ S\ o\ f_1)$ $\subseteq f_1$ (since f_1 is a fuzzy quasi-ideal of S). So $f = f_1$ and hence f is minimal fuzzy quasi-ideal of S.

Conversely, let f be a minimal fuzzy quasi-ideal of S. Consider a fuzzy point a_t of S such that $a_t \in f$. Then $a_t\ o\ S\ o\ S$, $\big((S\ o\ a_t\ o\ S) \cup (S\ o\ S\ o\ a_t\ o\ S\ o\ S)\big)$ and $S\ o\ S\ o\ a_t$ are respectively, fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S. Therefore $(a_t\ o\ S\ o\ S) \cap \big((S\ o\ a_t\ o\ S) \cup (S\ o\ S\ o\ a_t\ o\ S\ o\ S)\big) \cap (S\ o\ S\ o\ a_t)$ is a fuzzy quasi-ideal of S, by Theorem 3.20. Also

$$\begin{aligned} & (a_t\ o\ S\ o\ S) \cap \big((S\ o\ a_t\ o\ S) \cup (S\ o\ S\ o\ a_t\ o\ S\ o\ S)\big) \cap (S\ o\ S\ o\ a_t) \\ \subseteq\ & (f\ o\ S\ o\ S) \cap \big((S\ o\ f\ o\ S) \cup (S\ o\ S\ o\ f\ o\ S\ o\ S)\big) \cap (S\ o\ S\ o\ f) \subseteq f, \end{aligned}$$

since f is a fuzzy quasi-ideal of S. But f is minimal, so
$f = (a_t \ o \ S \ o \ S) \cap \big((S \ o \ a_t \ o \ S) \cup (S \ o \ S \ o \ a_t \ o \ S \ o \ S)\big) \cap (S \ o \ S \ o \ a_t)$..........(1).
Let g be any fuzzy right ideal of S such that $g \subseteq a_t \ o \ S \ o \ S$. Then $g \ o \ S \ o \ S \subseteq g \subseteq a_t \ o \ S \ o \ S$ where $g \ o \ S \ o \ S$ is a fuzzy right ideal of S. Therefore

$$(g \ o \ S \ o \ S) \cap \big((S \ o \ a_t \ o \ S) \cup (S \ o \ S \ o \ a_t \ o \ S \ o \ S)\big) \cap (S \ o \ S \ o \ a_t)$$

is a fuzzy quasi- ideal of S and

$$\begin{aligned} & (g \ o \ S \ o \ S) \cap \big((S \ o \ a_t \ o \ S) \cup (S \ o \ S \ o \ a_t \ o \ S \ o \ S)\big) \cap (S \ o \ S \ o \ a_t) \\ \subseteq \ & (a_t \ o \ S \ o \ S) \cap \big((S \ o \ a_t \ o \ S) \cup (S \ o \ S \ o \ a_t \ o \ S \ o \ S)\big) \cap (S \ o \ S \ o \ a_t) = f, \end{aligned}$$

from (1). Now, since f is a minimal fuzzy quasi-ideal of S, it follows from the above that $f = (g \, o \, S \, o \, S) \cap \big((S \, o \, a_t \, o \, S) \cup (S \, o \, S \, o \, a_t \, o \, S \, o \, S)\big) \cap (S \, o \, S \, o \, a_t)$. It implies that $f \subseteq g \, o \, S \, o \, S$. Therefore $a_t \, o \, S \, o \, S \subseteq f \, o \, S \, o \, S \subseteq (g \, o \, S \, o \, S) \, o \, S \, o \, S \subseteq g \, o \, S \, o \, S \subseteq g$. Thus $g = a_t \ o \ S \ o \ S$ and hence $(a_t \ o \ S \ o \ S)$ is a minimal fuzzy right ideal of S. Similarly, we can show that $\big((S \ o \ a_t \ o \ S) \cup (S \ o \ S \ o \ a_t \ o \ S \ o \ S)\big)$ and $(S \ o \ S \ o \ a_t)$ are respectively, a minimal fuzzy lateral ideal and a minimal fuzzy left ideal of S. Thus it follows from (1) that f is the intersection of a minimal fuzzy right ideal, a minimal fuzzy lateral ideal and a minimal fuzzy left ideal of S . □

4. Fuzzy Bi-ideals of Ternary Semigroups

Definition 4.1. A fuzzy ternary subsemigroup f of a ternary semigroup S is called a fuzzy bi-ideal of S if $f(uvwxy) \geq f(u) \wedge f(w) \wedge f(y)$ for all $u, v, w, x, y \in S$.

Example 4.2. Consider

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} : a, b, c, d, e \in {Z_0}^- \right\}$$

and

$$B = \left\{ \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} : p \in {Z_0}^- \right\}$$

where ${Z_0}^-$ is the set of all non-positive integers. Then S is a ternary semigroup with respect to the usual matrix multiplication and B is a bi-ideal of S. Now we define a fuzzy subset f of S by

$$f(X) = \begin{cases} 0.85, & if \ X \in B\,; \\ 0, & otherwise\,. \end{cases}$$

Then f is a fuzzy bi-ideal of S.

Theorem 4.3. *Let B be a non-empty subset of a ternary semigroup S. Then B is a bi-ideal of S if and only if the characteristic function C_B of B is a fuzzy bi-ideal of S.*

Theorem 4.4. *A fuzzy ternary subsemigroup f of a ternary semigroup S is a fuzzy bi-ideal of S if and only if $(f \ o \ S \ o \ f \ o \ S \ o \ f) \subseteq f$.*

Proof. Let f be a fuzzy bi-ideal of a ternary semigroup S and $x \in S$.
Case I: If $x \neq pqr$ for any $p, q, r \in S$, then $(f\ o\ S\ o\ f\ o\ S\ o\ f)(x) = 0 \leq f(x)$.
Case II: If such exists, let $x = pqr$ for some $p, q, r \in S$. Then

$$\begin{aligned}
(f\ o\ S\ o\ f\ o\ S\ o\ f)(x) &= \bigvee_{x=pqr} \{(f\ o\ S\ o\ f)(p) \wedge S(q) \wedge f(r)\} \\
&= \bigvee_{x=pqr} \Big\{\{ \bigvee_{p=kmn} \{f(k) \wedge S(m) \wedge f(n)\}\} \wedge S(q) \wedge f(r)\Big\} \\
&= \bigvee_{x=pqr} \Big\{\{ \bigvee_{p=kmn} \{f(k) \wedge f(n)\}\} \wedge f(r)\Big\} \\
&\quad \text{(since } S(m) = 1 = S(q)) \\
&\leq \Big\{ \bigvee_{x=kmnqr} \{f(k) \wedge f(n) \wedge f(r)\}\Big\} \\
&\leq \Big\{ \bigvee_{x=kmnqr} \{f(kmnqr)\}\Big\} \text{ (since } f \text{ is a fuzzy bi-ideal of } S) \\
&= \bigvee \{f(x)\} = f(x).
\end{aligned}$$

Hence $(f\ o\ S\ o\ f\ o\ S\ o\ f) \subseteq f$.

Conversely, let f be a fuzzy ternary subsemigroup of S such that $(f\ o\ S\ o\ f\ o\ S\ o\ f) \subseteq f$. Let $u, v, w, x, y \in S$. Then $uvwxy \in S$. Let $a = (uvw)xy$. Then

$$\begin{aligned}
f(uvwxy) &\geq (f\ o\ S\ o\ f\ o\ S\ o\ f)(a) \\
&= \bigvee_{a=pqr} \Big\{(f\ o\ S\ o\ f)(p) \wedge S(q) \wedge f(r)\Big\} \text{ for some } p, q, r \in S \\
&\geq (f\ o\ S\ o\ f)(uvw) \wedge S(x) \wedge f(y) \\
&= \bigvee_{uvw=kmn} \Big\{f(k) \wedge S(m) \wedge f(n)\Big\} \wedge f(y) \text{ (since } S(x) = 1) \\
&\geq f(u) \wedge S(v) \wedge f(w) \wedge f(y) \\
&= f(u) \wedge f(w) \wedge f(y) \text{ (since } S(v) = 1).
\end{aligned}$$

Hence, by Definition 4.1, f is a fuzzy bi-ideal of S. □

Theorem 4.5. *A fuzzy ternary subsemigroup f of a semigroup S is a fuzzy bi-ideal of S if and only if the level subset of f, f_t is a bi-ideal of S for $t \in Im\, f$.*

Proposition 4.6. *Let f and g be two non-empty fuzzy subsets of a ternary semigroup S. Then $f\ o\ S\ o\ g$ is a fuzzy bi-ideal of S.*

Proof. Since f and g are non-empty fuzzy subsets of S, then $f\ o\ S\ o\ g$ is also non-empty. Now

$$\begin{aligned}
(f\ o\ S\ o\ g)\ o\ (f\ o\ S\ o\ g)\ o\ (f\ o\ S\ o\ g) &\subseteq f\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ (S\ o\ S\ o\ S)\ o\ g \\
&\subseteq f\ o\ (S\ o\ S\ o\ S)\ o\ g \subseteq f\ o\ S\ o\ g.
\end{aligned}$$

Therefore, by Lemma 2.18, $f\ o\ S\ o\ g$ is a fuzzy ternary subsemigroup of S. Again,

$$\begin{aligned} & (f\ o\ S\ o\ g)\ o\ S\ o\ (f\ o\ S\ o\ g)\ o\ S\ o\ (f\ o\ S\ o\ g) \\ \subseteq\ & f\ o\ (S\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ S)\ o\ g \\ \subseteq\ & f\ o\ S\ o\ g. \end{aligned}$$

Hence, by Theorem 4.4, $f\ o\ S\ o\ g$ is a fuzzy bi-ideal of S. □

Proposition 4.7. *Let f and h be two non-empty fuzzy subsets of a ternary semigroup S and g be a fuzzy left ideal (fuzzy lateral ideal, fuzzy right ideal) of S. Then $f\ o\ g\ o\ h$ is a fuzzy bi-ideal of S.*

Proof. Let g be a fuzzy left ideal of S. Then $f\ o\ g\ o\ h$ is non-empty. Again,

$$\begin{aligned} (f\ o\ g\ o\ h)\ o\ (f\ o\ g\ o\ h)\ o\ (f\ o\ g\ o\ h) &\subseteq f\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ (S\ o\ S\ o\ g)\ o\ h \\ &\subseteq f\ o\ (S\ o\ S\ o\ g)\ o\ h \subseteq f\ o\ g\ o\ h. \end{aligned}$$

Therefore, by Lemma 2.18, $f\ o\ g\ o\ h$ is a fuzzy ternary subsemigroup of S.
Now

$$\begin{aligned} & (f\ o\ g\ o\ h)\ o\ S\ o\ (f\ o\ g\ o\ h)\ o\ S\ o\ (f\ o\ g\ o\ h) \\ \subseteq\ & f\ o\ (S\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ g)\ o\ h \\ \subseteq\ & f\ o\ (S\ o\ S\ o\ g)\ o\ h \subseteq f\ o\ g\ o\ h. \end{aligned}$$

Hence, by Theorem 4.4, $f\ o\ g\ o\ h$ is a fuzzy bi-ideal of S.
Similarly, we can prove the other cases. □

Proposition 4.8. *Let f be a fuzzy ternary subsemigroup of a ternary semigroup S. If g, h, k be respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S such that $g\ o\ h\ o\ k \subseteq f \subseteq g \cap h \cap k$, then f is a fuzzy bi-ideal of S.*

Proof. For fuzzy right ideal g, fuzzy lateral ideal h and fuzzy left ideal k of S,

$$\begin{aligned} f\ o\ S\ o\ f\ o\ S\ o\ f &\subseteq (g \cap h \cap k)\ o\ S\ o\ (g \cap h \cap k)\ o\ S\ o\ (g \cap h \cap k) \\ &\subseteq g\ o\ (S\ o\ h\ o\ S)\ o\ k \\ &\subseteq g\ o\ h\ o\ k \subseteq f. \end{aligned}$$

Hence, by Theorem 4.4, f is a fuzzy bi-ideal of S. □

Proposition 4.9. *If any one of the fuzzy subsets g, h, k of a ternary semigroup S be a fuzzy left (fuzzy lateral, fuzzy right) ideal of S and the remaining two be fuzzy ternary subsemigroup of S, then $g\ o\ h\ o\ k$ is a fuzzy bi-ideal of S.*

Proof. Let g be a fuzzy left ideal of S and h, k be the fuzzy ternary subsemigroups of S. Then

$$\begin{aligned} (g\ o\ h\ o\ k)\ o\ (g\ o\ h\ o\ k)\ o\ (g\ o\ h\ o\ k) &\subseteq (S\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ S)\ o\ (g\ o\ h\ o\ k) \\ &\subseteq (S\ o\ S\ o\ g)\ o\ h\ o\ k \subseteq g\ o\ h\ o\ k. \end{aligned}$$

Therefore, by Lemma 2.18, $g\ o\ h\ o\ k$ is a fuzzy ternary subsemigroup of S. Now

$$(g\ o\ h\ o\ k)\ o\ S\ o\ (g\ o\ h\ o\ k)\ o\ S\ o\ (g\ o\ h\ o\ k) \subseteq (S\ o\ S\ o\ g)\ o\ h\ o\ k \subseteq g\ o\ h\ o\ k.$$

Hence, by Theorem 4.4, $g\ o\ h\ o\ k$ is a fuzzy bi-ideal of S.
Similar result for the other cases. $\square$

Corollary 4.10. *For fuzzy right ideal g, fuzzy lateral ideal h and fuzzy left ideal k of a ternary semigroup S, $g\ o\ h\ o\ k$ is a fuzzy bi-ideal of S.*

Proposition 4.11. *Every fuzzy quasi-ideal of a ternary semigroup S is a fuzzy bi-ideal of S.*

Proof. Let f be a fuzzy quasi-ideal of S. Then f is a fuzzy ternary subsemigroup of S, by Proposition 3.7. Now

$$\begin{aligned} f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq f\ o\ (S\ o\ S\ o\ S)\ o\ S &\subseteq f\ o\ S\ o\ S\,, \\ f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq S\ o\ (S\ o\ S\ o\ S)\ o\ f &\subseteq S\ o\ S\ o\ f\,, \\ f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq S\ o\ (S\ o\ f\ o\ S)\ o\ S &\subseteq \big((S\ o\ S\ o\ f\ o\ S\ o\ S) \cup (S\ o\ f\ o\ S)\big). \end{aligned}$$

Therefore,

$$f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq (f\ o\ S\ o\ S) \cap \big((S\ o\ S\ o\ f\ o\ S\ o\ S) \cup (S\ o\ f\ o\ S)\big) \cap (S\ o\ S\ o\ f) \subseteq f,$$

by assumption. Thus by Theorem 4.4, f is a fuzzy bi-ideal of S. $\square$

Corollary 4.12. *Every fuzzy right ideal (fuzzy lateral ideal, fuzzy left ideal) of a ternary semigroup S is a fuzzy bi-ideal of S.*

Lemma 4.13. *For a ternary semigroup S, the following conditions are equivalent.*
(i) S is regular.
(ii) $f = f\ o\ S\ o\ f\ o\ S\ o\ f$ for every fuzzy bi-ideal f of S.
(iii) $f = f\ o\ S\ o\ f\ o\ S\ o\ f$ for every fuzzy quasi-ideal f of S.

Proof. Let S be a regular ternary semigroup and consider f be a fuzzy bi-ideal of S. Then $f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq f$. Let $x \in S$. Since S is regular, there exists an element $a \in S$ such that $x = xax$ i.e. $x = xaxax$. Now

$$\begin{aligned} (f\ o\ S\ o\ f\ o\ S\ o\ f)(x) &= \bigvee_{x=pqr} \big\{(f\ o\ S\ o\ f)(p) \wedge S(q) \wedge f(r)\big\} \text{ for some } p, q, r \in S \\ &\geq\ (f\ o\ S\ o\ f)(xax) \wedge S(a) \wedge f(x) \\ &=\ \bigvee_{xax=mnk} \big\{f(m) \wedge S(n) \wedge f(k)\big\} \wedge f(x) \text{ (since } S(a) = 1) \\ &\geq\ f(x) \wedge S(a) \wedge f(x) \wedge f(x) = f(x). \end{aligned}$$

Therefore $f \subseteq f\ o\ S\ o\ f\ o\ S\ o\ f$. Hence $f = f\ o\ S\ o\ f\ o\ S\ o\ f$ and so $(i) \Longrightarrow (ii)$.
Clearly $(ii) \Longrightarrow (iii)$.
Let (iii) holds and consider f, g, h be respectively, fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S. Then $f \cap g \cap h$ is a fuzzy quasi-ideal of S. Therefore, by (iii),

$$\begin{aligned} f \cap g \cap h &= (f \cap g \cap h)\ o\ S\ o\ (f \cap g \cap h)\ o\ S\ o\ (f \cap g \cap h) \\ &\subseteq f\ o\ (S\ o\ g\ o\ S)\ o\ h \subseteq f\ o\ g\ o\ h \end{aligned}$$

i.e. $f \cap g \cap h \subseteq f\ o\ g\ o\ h$. But $f\ o\ g\ o\ h \subseteq f\ o\ S\ o\ S \subseteq f$, $f\ o\ g\ o\ h \subseteq S\ o\ g\ o\ S \subseteq g$ and $f\ o\ g\ o\ h \subseteq S\ o\ S\ o\ h \subseteq h$. So $f\ o\ g\ o\ h \subseteq f \cap g \cap h$. Therefore $f\ o\ g\ o\ h = f \cap g \cap h$ and hence by Theorem 3.14, S is regular. Thus $(iii) \Longrightarrow (i)$. $\square$

Theorem 4.14. *For a ternary semigroup S, the following conditions are equivalent.*
(i) *S is regular.*
(ii) *$f = f\ o\ S\ o\ f$ for every fuzzy bi-ideal f of S.*
(iii) *$f = f\ o\ S\ o\ f$ for every fuzzy quasi-ideal f of S.*

Proof. $(i) \Longrightarrow (ii)$. Let S be a regular ternary semigroup and consider f be a fuzzy bi-ideal of S. Then $f = f\ o\ S\ o\ f\ o\ S\ o\ f$. Now $f = f\ o\ S\ o\ f\ o\ S\ o\ f \subseteq f\ o\ (S\ o\ S\ o\ S)\ o\ f \subseteq f\ o\ S\ o\ f$ and $f\ o\ S\ o\ f \subseteq f\ o\ S\ o\ (f\ o\ S\ o\ f\ o\ S\ o\ f) \subseteq f\ o\ S\ o\ f\ o\ (S\ o\ S\ o\ S)\ o\ f \subseteq f\ o\ S\ o\ f\ o\ S\ o\ f = f$. Therefore, $f = f\ o\ S\ o\ f$. Clearly $(ii) \Longrightarrow (iii)$. Let (iii) holds and let f be a fuzzy quasi-ideal of S such that $f = f\ o\ S\ o\ f$. Then $f = f\ o\ S\ o\ f = f\ o\ S\ o\ f\ o\ S\ o\ f$ and hence, by Lemma 4.13, S is regular. □

Remark 4.15. The converse of the above Proposition 4.11 and Corollary 4.12 are not always true in general.

Example 4.16. Consider
$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} : a, b, c, d, e \in {Z_0}^- \right\} \text{ and } B = \left\{ \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} : p \in {Z_0}^- \right\}$$
where ${Z_0}^-$ is the set of all non-positive integers. Then S is a ternary semigroup with respect to usual matrix multiplication and B is a bi-ideal of S. But B is not a quasi-ideal of S because
in SSB,
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P \text{ (say)}$$
in SBS,
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P$$
and in BSS,
$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = P.$$

Therefore $P \in (SSB \cap (SBS \cup SSBSS) \cap BSS)$. But $P \notin B$ and hence $(SSB \cap (SBS \cup SSBSS) \cap BSS) \nsubseteq B$. Now we define a fuzzy subset f of S as follows :
$$f(x) = \begin{cases} 0.75\,, & if\ x \in B\,; \\ 0, & if\ x \notin B\,. \end{cases}$$
Then f is a fuzzy bi-ideal of S, but not a fuzzy quasi-ideal of S.

Example 4.17. Consider
$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & g \end{pmatrix} : a, b, c, d, e, g \in {Z_0}^- \right\}$$

and

$$B = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix} : p, q \in {Z_0}^- \right\}$$

where ${Z_0}^-$ is the set of all non-positive integers. Then S is a ternary semigroup with respect to usual matrix multiplication and B is a bi-ideal of S. But B is not an ideal of S because
in SSB,

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{pmatrix} \notin B$$

in SBS,

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{pmatrix} \notin B$$

and in BSS,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin B.$$

Now we define a fuzzy subset f of S as follows :

$$f(x) = \begin{cases} 0.75, & if\ x \in B\,; \\ 0, & otherwise\,. \end{cases}$$

Then f is a fuzzy bi-ideal of S, but not a fuzzy ideal of S.

Remark 4.18. The converse of the Proposition 4.11 is true when S is regular.

Proposition 4.19. *Every fuzzy bi-ideal of a regular ternary semigroup S is a fuzzy quasi-ideal of S.*

Proof. Let f be a fuzzy bi-ideal of a regular ternary semigroup S. Then $f = f\ o\ S\ o\ f$, by Theorem 4.14. Again $f\ o\ S\ o\ S$, $(S\ o\ f\ o\ S \cup S\ o\ S\ o\ f\ o\ S\ o\ S)$ and $S\ o\ S\ o\ f$ are respectively fuzzy right ideal, fuzzy lateral ideal and fuzzy left ideal of S. Therefore

$$\begin{aligned}
& (f\ o\ S\ o\ S) \cap (S\ o\ f\ o\ S \cup S\ o\ S\ o\ f\ o\ S\ o\ S) \cap (S\ o\ S\ o\ f) \\
= \ & (f\ o\ S\ o\ S)\ o\ (S\ o\ f\ o\ S \cup S\ o\ S\ o\ f\ o\ S\ o\ S)\ o\ (S\ o\ S\ o\ f) (\text{by Theorem } 3.14) \\
= \ & \big(f\ o\ (S\ o\ S\ o\ S)\ o\ f\ o\ (S\ o\ S\ o\ S)\ o\ f\big) \\
& \quad \cup \big(f\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ f\ o\ S\ o\ (S\ o\ S\ o\ S)\ o\ f\big) (\text{by Proposition } 2.13) \\
\subseteq \ & (f\ o\ S\ o\ S\ o\ S\ o\ f) \cup (f\ o\ (S\ o\ S\ o\ S)\ o\ S\ o\ S\ o\ f) \\
\subseteq \ & (f\ o\ S\ o\ f) \cup (f\ o\ S\ o\ f) = f\ o\ S\ o\ f = f.
\end{aligned}$$

Hence by Definition 3.1, f is a fuzzy quasi- ideal of S. □

Corollary 4.20. *The intersection of any two fuzzy bi-ideals of a ternary semigroup S is either empty or a fuzzy bi-ideal of S.*

Corollary 4.21. *Let f be a fuzzy ideal and g be a fuzzy quasi-ideal of a ternary semigroup S. Then $f \cap g$ is a fuzzy bi-ideal of S.*

Corollary 4.22. *For any three fuzzy bi-ideals f_1, f_2, f_3 of a regular ternary semigroup S, $f_1\ o\ f_2\ o\ f_3$ is a fuzzy bi-ideal of S.*

Proposition 4.23. *If S be a regular ternary semigroup, then*
(i) $f^3 = f^5$ for any fuzzy bi-ideal f of S.
(ii) $f^3 = f^5$ for any fuzzy quasi-ideal f of S.

Proof. (i) Let f be a fuzzy bi-ideal of a regular ternary semigroup S. Then $(f\ o\ f\ o\ f) \subseteq f$. Therefore, $f^5 = (f\ o\ f\ o\ f)\ o\ f\ o\ f \subseteq f\ o\ f\ o\ f = f^3$ i.e. $f^5 \subseteq f^3$. Now since S is regular and $f\ o\ f\ o\ f$ is a fuzzy bi-ideal of S (by Corollary 4.22), $f^3 = f^3\ o\ S\ o\ f^3$ (by Theorem 4.14) $= f\ o\ f\ o\ (f\ o\ S\ o\ f)\ o\ f\ o\ f \subseteq f\ o\ f\ o\ f\ o\ f\ o\ f = f^5$. Hence $f^3 = f^5$.
(ii) Let f be a fuzzy quasi-ideal of a regular ternary semigroup S. Then f is a fuzzy bi-ideal of S and hence $f^3 = f^5$. □

Theorem 4.24. *For a ternary semigroup S, the following conditions are equivalent.*

(i) S is regular.
(ii) $f \cap g = f\ o\ g\ o\ f$ for every fuzzy quasi-ideal f and fuzzy lateral ideal g of S.
(iii) $f \cap g = f\ o\ g\ o\ f$ for every fuzzy bi-ideal f and fuzzy lateral ideal g of S.

Proof. $(i) \Longrightarrow (iii)$. Let S be regular and f, g be respectively fuzzy bi-ideal and fuzzy lateral ideal of S. Therefore by Theorem 4.14, $f = f\ o\ S\ o\ f$. Now $f\ o\ g\ o\ f \subseteq f\ o\ S\ o\ f = f$ and $f\ o\ g\ o\ f \subseteq S\ o\ g\ o\ S \subseteq g$. Therefore $f\ o\ g\ o\ f \subseteq f \cap g$. Now let $x \in S$. Then by assumption, there exists an element $a \in S$ such that $x = xax$ i.e. $x = xaxax$. Therefore $(f\ o\ g\ o\ f)(x) = (f\ o\ g\ o\ f)(xaxax) \geq f(x) \wedge g(axa) \wedge f(x) \geq f(x) \wedge g(x)$ (since g is a fuzzy lateral ideal of S) $= (f \cap g)(x)$. So $(f \cap g) \subseteq (f\ o\ g\ o\ f)$. Hence $f \cap g = f\ o\ g\ o\ f$.
Clearly, $(iii) \Longrightarrow (ii)$.
$(ii) \Longrightarrow (i)$. Let (ii) holds and f be a fuzzy quasi-ideal of S. Since S is itself a fuzzy lateral ideal of S, $f = f \cap S = f\ o\ S\ o\ f$, by (ii). Hence by Theorem 4.14, S is regular. □

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Sukhendu Kar (karsukhendu@yahoo.co.in) – Department of mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.
Paltu Sarkar (pal2.ju.math@gmail.com) – Department of mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.