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Some new generalized classes of sequences of fuzzy numbers defined by an Orlicz function

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Abstract. In this paper, we introduce some new classes of sequences of fuzzy numbers using Orlicz function. We also examine some properties of these classes of sequences.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [7] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [4] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [1], Nanda [6], Esi [2] and many others.

Let $C(R^n) = \{A \subset R^n : A \textit{ is compact and convex set}\}$. The space $C(R^n)$ has a linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\gamma A = \{\gamma a : a \in A\}$ for $A, B \in C(R^n)$ and $\gamma \in R$.

The Hausdorff distance between A and B in $C(R^n)$ is defined by

$$\delta_\infty(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a - b||, \sup_{b\in B}\inf_{a\in A} ||a - b||\right\}.$$

It is well-known that $(C(R^n), \delta_\infty)$ is a complete metric space.

A fuzzy number is a function X from R^n to $[0, 1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of $\{X \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α-level set

$$X^\alpha = \{X \in R^n : X(x) > \alpha\}$$

is a non-empty compact, convex subset of R^n with support X^0.

If R^n is replaced by R , then obviously the set $C(R^n)$ is reduced to the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on R, and also

$$\delta_\infty(A, B) = \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|).$$

Let $L(R)$ denote the set of all fuzzy numbers. The linear structure of $L(R)$ induces the addition $X + Y$ and the scalar multiplication λX in terms of α-level sets, by $[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda[X]^\alpha$ for each $0 \le \alpha \le 1$.

The set R of real numbers can be embedded in $L(R)$ if we define $\overline{r} \in L(R)$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r; \\ 0, & \text{if } t \ne r \end{cases}$$

The additive identity and multiplicative identity of $L(R)$ are denoted by $\overline{0}$ and $\overline{1}$, respectively.

For r in R and X in $L(R)$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \ne 0; \\ 0, & \text{if } r = 0 \end{cases}$$

Define a map $d : L(R) \times L(R) \to R$ by

$$d(X, Y) = \sup_{0 \le \alpha \le 1} \delta_\infty(X^\alpha, Y^\alpha).$$

For $X, Y \in L(R)$ define $X \le Y$ if and only if $X^\alpha \le Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(R), d)$ is complete metric space [4].

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ [4].

We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded [4].

We denote by $\ell_\infty(F)$ the set of all bounded sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\varepsilon > 0$ there is a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$ [4].

We denote by $c(F)$ the set of all convergent sequences $X = (X_k)$ of fuzzy numbers.

It is straightforward to see that $c(F) \subset \ell_\infty \subset w(F)$.

Nanda [6] studied the classes of bounded and convergent sequences of fuzzy numbers and showed that these are complete metric spaces.

A K-space of sequences for which the coordinate linear functionals are continuous.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy the $\Delta_2 - condition$ for all values of u, if there exists a constant $K > 0$, such that $M(2u) \le KM(u)$, $u \ge 0$. Note that, if $0 < \lambda < 1$, then $M(\lambda x) \le \lambda M(x)$, for all $x \ge 0$.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda \le 1$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x = (x_k) : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_k M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Let $p = (p_k) \in \ell_\infty$, then the following well-known inequality will be used in the paper: For sequences (a_k) and (b_k) of complex numbers we have

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}) \tag{1.1}$$

where $C = \max(1, 2^{H-1}), H = \sup_k p_k$.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma\left(\sigma^{k-1}(n)\right)$, $k = 1, 2, 3, ...,$ and let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k p_k = H < \infty$. We define the following classes of sequences of fuzzy numbers:

$$c_0^F(M, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, \overline{0})}{\rho}\right)\right]^{p_k} = 0, \right.$$
$$\left. uniformly\ in\ n\ for\ some\ \rho > 0, s \geq 0 \right\},$$

$$c^F(M, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \lim_k k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, X_0)}{\rho}\right)\right]^{p_k} = 0, \right.$$
$$\left. uniformly\ in\ n\ for\ some\ \rho > 0, s \geq 0 \right\},$$

$$\ell_\infty^F(M, p, \sigma, s) = \left\{ X = (X_k) \in w^F : \sup_{n,k} k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, \overline{0})}{\rho}\right)\right]^{p_k} < \infty, \right.$$
$$\left. for\ some\ \rho > 0, s \geq 0 \right\}.$$

If we take $s = 0$, $\sigma(n) = n+1$ and $M(x) = x$ then we obtain the classes c_0^F, c^F and ℓ_∞^F of ordinary null, convergent and bounded sequences of fuzzy numbers, respectively which were defined and studied by Matloka [4].

A metric d on $L(R)$ is said to be translation invariant if $d(X+Z, Y+Z) = d(X, Y)$ for $X, Y, Z \in L(R)$.

Lemma 1.2. [5] *If d is a translation invariant metric on $L(R)$ then*
(i) $d(X + Y, \overline{0}) \leq d(X, \overline{0}) + d(Y, \overline{0})$;
(ii) $d(\lambda X, \overline{0}) \leq |\lambda| d(X, \overline{0}), |\lambda| > 1$.

2. MAIN RESULTS

In this section we investigate linear topological structures of the spaces $c_0^F(M,p,\sigma,s)$, $c^F(M,p,\sigma,s)$ and $\ell_\infty^F(M,p,\sigma,s)$ and find out some relations related to the these spaces.

Theorem 2.1. *(a) $c_0^F(M,p,\sigma,s)$, $c^F(M,p,\sigma,s)$ and $\ell_\infty^F(M,p,\sigma,s)$ are closed under the operations of addition and scalar multiplication if d is a translation invariant metric.*

(b) $c_0^F(M,p,\sigma,s) \subset c^F(M,p,\sigma,s) \subset \ell_\infty^F(M,p,\sigma,s)$.

Proof. (a) If d is translation metric, then

$$d(X_{\sigma^k(n)} + Y_{\sigma^k(n)}, X_0 + Y_0) \leq d(X_{\sigma^k(n)}, X_0) + d(Y_{\sigma^k(n)}, Y_0) \tag{2.1}$$

and

$$d(\lambda X_{\sigma^k(n)}, \lambda X_0) \leq |\lambda| d(X_{\sigma^k(n)}, X_0) \tag{2.2}$$

where λ is a scalar with $0 < \lambda \leq 1$. It is easy to see that the classes of $c_0^F(M,p,\sigma,s)$, $c^F(M,p,\sigma,s)$ and $\ell_\infty^F(M,p,\sigma,s)$ are closed under the operations of addition and scalar multiplication.

(b) The first inclusion in clear. For the second, using by the triangle inequality

$$k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)},\bar{0})}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)},X_0)}{\rho}\right)\right]^{p_k} + k^{-s}\left[M\left(\frac{d(X_0,\bar{0})}{\rho}\right)\right]^{p_k}$$

$$\leq k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)},X_0)}{\rho}\right)\right]^{p_k} + \max\left(1, k^{-s}\left[M\left(\frac{|X_0|}{\rho}\right)\right]^{p_k}\right).$$

So, $X = (X_k) \in c^F(M,p,\sigma,s)$ implies that $X = (X_k) \in \ell_\infty^F(M,p,\sigma,s)$. This completes the proof. □

Theorem 2.2. *$c^F(M,p,\sigma,s)$ is a complete metric space with the metric*

$$\delta(X,Y) = \inf\left\{\rho > 0 : \sup_{n,k} k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, Y_{\sigma^k(n)})}{\rho}\right)\right]^{p_k} \leq 1\right\}.$$

The proof of the Theorem 2.2 is straightforward. So we omit it.

Theorem 2.3. *The spaces $c^F(M,p,\sigma,s)$ and $c_0^F(M,p,\sigma,s)$ are nowhere dense subsets of $\ell_\infty^F(M,p,\sigma,s)$.*

Proof. It is obvious in view of Theorem 2.1 and Theorem 2.2. □

The proof of the following result is a routine work in view of the techniques used for establishing the above result.

Theorem 2.4. *The spaces $c^F(M,p,\sigma,s)$ and $c_0^F(M,p,\sigma,s)$ and $\ell_\infty^F(M,p,\sigma,s)$ are K-spaces.*

Theorem 2.5. *(a) $c^F(M,q,\sigma,s) \subset c^F(M,p,\sigma,s)$ if $\liminf(p_k q_k^{-1}) > 0$.*

(b) $\ell_\infty^F(M,q,\sigma,s)$ is closed subset of $\ell_\infty^F(M,p,\sigma,s)$ if $0 < p_k \leq q_k \leq 1$.

Proof. (a) Suppose that $\liminf(p_k q_k^{-1}) > 0$ holds and $X = (X_k) \in c^F(M, q, \sigma, s)$. Then there is a $\beta > 0$ such that $p_k > \beta q_k$ for large $k \in N$. Hence for large k

$$k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, X_0)}{\rho}\right)\right]^{p_k} \leq k^{-s}\left(\left[M\left(\frac{d(X_{\sigma^k(n)}, X_0)}{\rho}\right)\right]^{q_k}\right)^{\beta}.$$

Since

$$k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, X_0)}{\rho}\right)\right]^{q_k} < 1$$

for each k, n and for some $\rho > 0$. Hence $X = (X_k) \in c^F(M, p, \sigma, s)$.

(b) Suppose that $0 < p_k \leq q_k \leq 1$ holds and $X = (X_k) \in \ell_\infty^F(M, q, \sigma, s)$. Then there is a constant $T > 1$ such that

$$k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, \overline{0})}{\rho}\right)\right]^{q_k} \leq T$$

for each k, n and for some $\rho > 0$. This implies that

$$k^{-s}\left[M\left(\frac{d(X_{\sigma^k(n)}, \overline{0})}{\rho}\right)\right]^{p_k} \leq T$$

for each k and n. Hence $X = (X_k) \in \ell_\infty^F(M, p, \sigma, s)$. To show that $\ell_\infty^F(M, q, \sigma, s)$ is closed, suppose that $X^i = (X_k^i) \in \ell_\infty^F(M, q, \sigma, s), X^i \to X_0$ and $X_0 \in \ell_\infty^F(M, p, \sigma, s)$. Then for every $\varepsilon, 0 < \varepsilon < 1$ there is $i_0 \in N$ such that for all k, n and for some $\rho > 0$

$$k^{-s}\left[M\left(\frac{d(X^i_{\sigma^k(n)} - X_0, \overline{0})}{\rho}\right)\right]^{p_k} < \varepsilon \ \ for \ \ i > i_0.$$

Now

$$k^{-s}\left[M\left(\frac{d(X^i_{\sigma^k(n)} - X_0, \overline{0})}{\rho}\right)\right]^{q_k} < k^{-s}\left[M\left(\frac{d(X^i_{\sigma^k(n)} - X_0, \overline{0})}{\rho}\right)\right]^{p_k} < \varepsilon \ for \ i > i_0.$$

Therefore $X = (X_k) \in \ell_\infty^F(M, q, \sigma, s)$ i.e. $\ell_\infty^F(M, q, \sigma, s)$ is closed subset of $\ell_\infty^F(M, p, \sigma, s)$. □

Theorem 2.6. *Let $0 < h = \inf p_k \leq \sup_k = H < \infty$. For any Orlicz function M which satisfies Δ_2-condition, then $c^F(p, \sigma, s) \subset c^F(M, p, \sigma, s)$, where*

$$c^F(p, \sigma, s) = \left\{X = (X_k) \in w^F : \lim_k k^{-s}\left[d(X_{\sigma^k(n)}, X_0)\right]^{p_k} = 0, \ uniformly \ in \ n\ , s \geq 0\right\}.$$

Proof. Let $X = (X_k) \in c^F(p, \sigma, s)$, so that $\lim_k k^{-s}\left[d(X_{\sigma^k(n)}, X_0)\right]^{p_k} = 0$, uniformly in n. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. We can write $y_k = \frac{d(X_{\sigma^k(n)}, X_0)}{\rho}$ and we consider

$$k^{-s}[M(y_k)]^{p_k} = k^{-s}[M(y_k)]^{p_k} \mid_{y_k \leq \delta, k \in N} + k^{-s}[M(y_k)]^{p_k} \mid_{y_k > \delta, k \in N}.$$

For $y_k \leq \delta$, we have

$$k^{-s}[M(y_k)]^{p_k} < k^{-s}\max(\varepsilon, \varepsilon^h)$$

by using the continuity of M.

For $y_k > \delta$, we have
$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$
Since M is non-decreasing and convex, it follows that
$$M(y_k) < M\left(1 + \frac{y_k}{\delta}\right) \le \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_k}{\delta}\right).$$
Since M satisfies the Δ_2-condition, we can write
$$M(y_k) \le \frac{K}{2}\frac{y_k}{\delta}M(2) + \frac{K}{2}\frac{y_k}{\delta}M(2) = K\frac{y_k}{\delta}M(2).$$
We get the following estimates
$$k^{-s}[M(y_k)]^{p_k} \le k^{-s}\max\left(1, [KM(2)\delta^{-1}]^H[y_k]^{p_k}\right)$$
$$k^{-s}[M(y_k)]^{p_k} \le k^{-s}\max(\varepsilon, \varepsilon^h) + k^{-s}\max\left(1, [KM(2)\delta^{-1}]^H[y_k]^{p_k}\right).$$
Taking $\varepsilon \to 0$ and $k \to \infty$, it follows that $X = (X_k) \in c^F(M, p, \sigma, s)$. □

The proofs of the following results are a routine work in view of the techniques used for establishing the above result.

Theorem 2.7. *Let M, M_1 and M_2 be Orlicz functions. Then*
(a) $Z(M_1, p, \sigma, s) \subset Z(M.M_1, p, \sigma, s)$.
(b) $Z(M_1, p, \sigma, s) \cap Z(M_2, p, \sigma, s) \subset Z(M_1 + M_2, p, \sigma, s)$, where $Z = c_0^F, c^F, \ell_\infty^F$.

Theorem 2.8. *Let M_1 and M_2 be two Orlicz functions such that $M_1 \cong M_2$. Then $Z(M_1, p, \sigma, s) = Z(M_2, p, \sigma, s)$, where $Z = c_0^F, c^F, \ell_\infty^F$.*

References

[1] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets, Fuzzy Sets and Systems 35 (1990) 241–249.
[2] A. Esi, On some new paranormed sequence spaces of fuzzy numbers defined by Orlicz functions and statistical convergence, Math. Model. Anal. 11(4) (2006) 379–388.
[3] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971) 345–355.
[4] M. Matloka, Sequences of fuzzy numbers, BUSEFAL 28 (1986) 28–37.
[5] M. Mursaleen and M. Basarir, On some new sequence spaces of fuzzy numbers, Indian J. Pure Appl. Math. 34(9) (2003) 1351–1357.
[6] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123–126.
[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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