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On characterization of timelike biharmonic D-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group H

Talat Körpinar, Essin Turhan

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Abstract. In this paper, we study timelike biharmonic D-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group **H**. Firstly, we characterize timelike biharmonic D-curves in terms of their curvature and torsion. Secondly, we obtain parametric equation of timelike biharmonic D-helices in the Lorentzian Heisenberg group **H**.

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Corresponding Author: Essin Turhan (essin.turhan@gmail.com)

1. Introduction

Moving frames are important in general relativity, where there is no privileged way of extending a choice of frame at an event p (a point in spacetime, which is a manifold of dimension four) to nearby points, and so a choice must be made. In contrast in special relativity, M is taken to be a vector space V (of dimension four). In that case a frame at a point p can be translated from p to any other point q in a well-defined way. Broadly speaking, a moving frame corresponds to an observer, and the distinguished frames in special relativity represent inertial observers.

The introduction of the trihedron, an invention of Darboux, allows for a conceptual simplification of the problem of moving frames on curves and surfaces by treating the coordinates of the point on the curve and the frame vectors in a uniform manner, [4, 10, 11].

On the other hand, let (N, h) and (M, g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where the section $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ, [1, 2, 3].

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi,$$

and called the bitension field of ϕ. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps, [5, 6, 7, 8, 12, 13].

In this paper, we study timelike biharmonic D-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group **H**. Firstly, we characterize timelike biharmonic D-curves in terms of their curvature and torsion. Secondly, we obtain parametric equation of timelike biharmonic D-helices in the Lorentzian Heisenberg group **H**.

2. Lorentzian Heisenberg group H

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\}$$

with the usual multiplication rule, [9].

We will use the following complex definition of the Heisenberg group $\mathbb{H}$:

$$\mathbb{H} = \mathbb{C} \times \mathbb{R} = \{(w, z) : w \in \mathbb{C}, z \in \mathbb{R}\}.$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$.

Let $a = (w_1, z_1)$, $b = (w_2, z_2)$ and $c = (w_3, z_3)$. The commutator of the elements $a, b \in$**H** is equal to

$$\begin{aligned} [a, b] &= a * b * a^{-1} * b^{-1} \\ &= (w_1, z_1) * (w_2, z_2) * (-w_1, -z_1) * (-w_2, -z_2) \\ &= (w_1 + w_2 - w_1 - w_2, z_1 + z_2 - z_1 - z_2) \\ &= (0, \alpha), \end{aligned}$$

where $\alpha \neq 0$ in general. For example

$$[(1, 0), (i, 0)] = (0, 2) \neq (0, 0).$$

Which shows that **H** is not abelian.

On the other hand, for any $a, b, c \in \mathbf{H}$, their double commutator is

$$\begin{aligned} [[a,b], c] &= [(0,\alpha), (w_3, z_3)] \\ &= (0,0). \end{aligned}$$

This implies that $\mathbf{H}$ is a nilpotent Lie group with nilpotency 2.

The left-invariant Lorentz metric on $\mathbf{H}$ is

$$\rho = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}.$$

The characterising properties of this algebra are the following commutation relations:

$$\rho(\mathbf{e}_1, \mathbf{e}_1) = \rho(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad \rho(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

For the covariant derivatives of the Levi-Civita connection of the left-invariant metric ρ, defined above the following is true:

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix},$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

3. Timelike biharmonic D-Helices according to Darboux frame on a non-degenerate timelike surface in the Lorentzian Heisenberg group

Let $\Pi \subset \mathbf{H}$ be a timelike surface with the unit normal vector $\mathbf{n}$ in the Lorentzian Heisenberg group $\mathbf{H}$. If γ is a timelike curve on $\Pi \subset \mathbf{H}$, then we have the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and Darboux frame $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ with spacelike vector $\mathbf{g} = \mathbf{T} \wedge \mathbf{n}$ along the curve γ. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbf{H}$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} \rho(\mathbf{T}, \mathbf{T}) &= -1,\ \rho(\mathbf{N}, \mathbf{N}) = 1,\ \rho(\mathbf{B}, \mathbf{B}) = 1, \\ \rho(\mathbf{T}, \mathbf{N}) &= \rho(\mathbf{T}, \mathbf{B}) = \rho(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

Let θ be the angle between $\mathbf{N}$ and $\mathbf{n}$. The relationships between $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T},\mathbf{n},\mathbf{g}\}$ are as follows:

$$\begin{aligned}
\mathbf{T} &= \mathbf{T},\\
\mathbf{N} &= \cos\theta\mathbf{n}+\sin\theta\mathbf{g},\\
\mathbf{B} &= \sin\theta\mathbf{n}-\cos\theta\mathbf{g},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{T} &= \mathbf{T},\\
\mathbf{g} &= \sin\theta\mathbf{N}-\cos\theta\mathbf{B},\\
\mathbf{n} &= \cos\theta\mathbf{N}+\sin\theta\mathbf{B}.
\end{aligned}$$

Using Frenet formulas we obtain

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= (\kappa\cos\theta)\,\mathbf{n}+(\kappa\sin\theta)\,\mathbf{g},\\
\nabla_{\mathbf{T}}\mathbf{g} &= (-\kappa\sin\theta)\,\mathbf{T}+\left(\tau+\frac{d\theta}{ds}\right)\mathbf{n},\\
\nabla_{\mathbf{T}}\mathbf{n} &= (-\kappa\cos\theta)\,\mathbf{T}-\left(\tau+\frac{d\theta}{ds}\right)\mathbf{g}.
\end{aligned}$$

If we represent $\kappa\cos\theta$, $\kappa\sin\theta$ and $\tau+\frac{d\theta}{ds}$ with the symbols κ_n, κ_g, and τ_g respectively, then the equations can be written as

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \kappa_g\mathbf{g}+\kappa_n\mathbf{n},\\
\nabla_{\mathbf{T}}\mathbf{g} &= -\kappa_g\mathbf{T}+\tau_g\mathbf{n},\\
\nabla_{\mathbf{T}}\mathbf{n} &= -\kappa_n\mathbf{T}-\tau_g\mathbf{g}.
\end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}
\mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3,\\
\mathbf{g} &= g_1\mathbf{e}_1+g_2\mathbf{e}_2+g_3\mathbf{e}_3.\\
\mathbf{n} &= n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3,
\end{aligned}$$

To separate a curve according to Darboux frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve as $\mathcal{D}$-curve.

Lemma 3.1. *Let $\gamma : I \longrightarrow \Pi \subset \boldsymbol{H}$ be a non-geodesic unit speed timelike curve on timelike surface Π in the Lorentzian Heisenberg group $\boldsymbol{H}$. γ is a unit speed timelike biharmonic curve on Π if and only if*

$$\begin{aligned}
\kappa_n^2+\kappa_g^2 &= \mathit{constant}\neq 0,\\
\kappa_n''-\kappa_n^3+\kappa_g\tau_g-\kappa_g^2\kappa_n+\kappa_g'\tau_g+\kappa_g\tau_g'-\tau_g^2\kappa_n &= \kappa_n(1-4g_1^2)-4\kappa_g n_1 g_1,\\
\kappa_g''-\kappa_g^3-2\kappa_n'\tau_g-\kappa_n^2\kappa_g-\kappa_n\tau_g'-\kappa_g\tau_g^2 &= 4\kappa_n n_1 g_1+\kappa_g(1-4n_1^2).
\end{aligned}$$

Proof. Using Darboux frame, we have a third-order differential equation with respect to γ, yield

$$\begin{aligned}
\tau_2(\gamma) &= \nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T},\nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} \\
&= (-3\kappa_n\kappa_n' - 3\kappa_g\kappa_g')\mathbf{T} + (\kappa_n'' - \kappa_n^3 + \kappa_g\tau_g - \kappa_g^2\kappa_n + \kappa_g'\tau_g + \kappa_g\tau_g' - \tau_g^2\kappa_n)\mathbf{n} \\
&\quad +(\kappa_g'' - \kappa_g^3 - 2\kappa_n'\tau_g - \kappa_n^2\kappa_g - \kappa_n\tau_g' - \kappa_g\tau_g^2)\mathbf{g} \\
&\quad -\kappa_n R(\mathbf{T},\mathbf{n})\mathbf{T} - \kappa_g R(\mathbf{T},\mathbf{g})\mathbf{T} \\
&= 0.
\end{aligned}$$

By above system, we see $-\kappa_n\kappa_n' - \kappa_g\kappa_g' = 0$. Since

$$\kappa_n'' - \kappa_n^3 + \kappa_g\tau_g - \kappa_g^2\kappa_n + \kappa_g'\tau_g + \kappa_g\tau_g' - \tau_g^2\kappa_n = \kappa_n R(\mathbf{T},\mathbf{n},\mathbf{T},\mathbf{n}) + \kappa_g R(\mathbf{T},\mathbf{g},\mathbf{T},\mathbf{n}),$$

we have $\kappa_g'' - \kappa_g^3 - 2\kappa_n'\tau_g - \kappa_n^2\kappa_g - \kappa_n\tau_g' - \kappa_g\tau_g^2 = \kappa_n R(\mathbf{T},\mathbf{n},\mathbf{T},\mathbf{g}) + \kappa_g R(\mathbf{T},\mathbf{g},\mathbf{T},\mathbf{g})$. A direct computation using curvatures, yields

$$\begin{aligned}
R(\mathbf{T},\mathbf{n},\mathbf{T},\mathbf{n}) &= 1 - 4g_1^2, \\
R(\mathbf{T},\mathbf{g},\mathbf{T},\mathbf{n}) &= -4n_1g_1, \\
R(\mathbf{T},\mathbf{n},\mathbf{T},\mathbf{g}) &= 4n_1g_1, \\
R(\mathbf{T},\mathbf{g},\mathbf{T},\mathbf{g}) &= 1 - 4n_1^2.
\end{aligned}$$

Substituting now the above expression, complete the proof of the Lemma 3.1. □

Theorem 3.2. *Let $\gamma : I \longrightarrow \Pi \subset \boldsymbol{H}$ be a non-geodesic unit speed timelike biharmonic D-helix on timelike surface Π in the Lorentzian Heisenberg group $\boldsymbol{H}$. Then parametric equations of timelike biharmonic D- helix are*

$$\begin{aligned}
x(s) &= \frac{\cosh\aleph}{(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)}(\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s]\sinh[\wp] \\
&\quad + \cosh[\wp]\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s]) + \wp_1,
\end{aligned}$$

$$\begin{aligned}
y(s) &= \frac{\cosh\aleph}{(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)}(\cosh[\wp]\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s] \\
&\quad + \sinh[\wp]\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s]) + \wp_2,
\end{aligned}$$

$$
\begin{aligned}
z(s) &= \sinh\aleph s - \frac{(-\wp - (\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s)\cosh^2\aleph}{2(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)^2} \\
&\quad - \frac{\wp_1\cosh^2\aleph\cosh[\wp]\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s]}{(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)} \\
&\quad - \frac{\wp_1\cosh^2\aleph\sinh[\wp]\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s]}{(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)} \\
&\quad - \frac{\cosh^2\aleph\sinh[2(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s + 2\wp]}{4(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)^2} + \wp_3,
\end{aligned}
$$

where $\wp$ is constant of integration.

Proof. Assume that γ *b*e a non-geodesic unit speed timelike biharmonic D- helix on timelike surface Π in the Lorentzian Heisenberg group $\mathbf{H}$. So, without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$\rho(\mathbf{T}, \mathbf{e}_1) = T_1 = \sinh A,$$

where $\aleph$ is constant angle.

On the other hand, the tangent vector $\mathbf{T}$ is a unit timelike vector, we get

$$
\begin{aligned}
T_2 &= \cosh\aleph\sinh\Lambda, \\
T_3 &= \cosh\aleph\cosh\Lambda,
\end{aligned}
$$

where Λ is an arbitrary function of s. So, substituting the components T_1, T_2 and T_3 in $\mathbf{T}$, we have the following equation

$$\mathbf{T} = \sinh\aleph\mathbf{e}_1 + \cosh\aleph\sinh\Lambda\mathbf{e}_2 + \cosh\aleph\cosh\Lambda\mathbf{e}_3.$$

The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = T_1'\mathbf{e}_1 + (T_2' + 2T_1T_3)\mathbf{e}_2 + (T_3' + 2T_1T_2)\mathbf{e}_3.$$

Using Darboux frame in above equation with our metric, we obtain

$$\rho(\nabla_{\mathbf{T}}\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}) = \kappa_n^2 + \kappa_g^2.$$

Thus, we have

$$(T_1')^2 + (T_2' + 2T_1T_3)^2 - (T_3' + 2T_1T_2)^2 = \kappa_n^2 + \kappa_g^2.$$

It is apparent that

$$\Lambda(s) = (\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s + \wp,$$

where $\wp$ is constant of integration. Since,

$$
\begin{aligned}
\mathbf{T} &= \sinh\aleph\mathbf{e}_1 + \cosh\aleph\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s + \wp_1]\mathbf{e}_2 \\
&\quad + \cosh\aleph\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph} - 2\sinh\aleph)s + \wp_1]\mathbf{e}_3.
\end{aligned}
$$

Combining above equations, we have

$$\mathbf{T} = (\cosh\aleph\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp],$$

$$\cosh\aleph\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp],$$

$$\sinh\aleph - x\cosh\aleph\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp]).$$

Also, from above equation, we get

$$\begin{aligned}
\frac{dx}{ds} &= \cosh\aleph\cosh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp],\\
\frac{dy}{ds} &= \cosh\aleph\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp],\\
\frac{dz}{ds} &= \sinh\aleph - x\cosh\aleph\sinh[(\frac{\sqrt{\kappa_n^2+\kappa_g^2}}{\cosh\aleph}-2\sinh\aleph)s+\wp].
\end{aligned}$$

If we take integrate above system we have theorem. The proof is completed. $\square$

Corollary 3.3. *Let $\gamma : I \longrightarrow \Pi \subset \boldsymbol{H}$ be a non-geodesic unit speed timelike biharmonic D-curve on timelike surface Π in the Lorentzian Heisenberg group $\boldsymbol{H}$. Then*

$$\kappa = constant \neq 0.$$

Proof. By the formula of the curvature, we have

$$\kappa_n^2 + \kappa_g^2 = constant.$$

We will prove that κ is a constant. Because

$$\kappa_n = \kappa\cos\theta, \quad \kappa_g = \kappa\sin\theta,$$

then we have

$$\kappa^2(\cos^2\theta + \sin^2\theta) = constant.$$

Finally, $\kappa^2 =$*constant,* we express the desired result. $\square$

4. Conclusion

Consider a curve in the Lorentzian Heisenberg group **H** suppose that the curve is sufficiently smooth so that the Darboux frame adapted to it is defined the curvatures then provide a complete characterization of the curve.

In this article, we study timelike biharmonic D-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group **H**. Firstly, we characterize timelike biharmonic D-curves in terms of their curvature and torsion. Secondly, we obtain parametric equation of timelike biharmonic D-helices in the Lorentzian Heisenberg group **H**.

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Talat Körpinar (talatkorpinar@gmail.com)
Fi rat University, Department of Mathematics, 23119, Elazığ, Turkey

Essin Turhan (essin.turhan@gmail.com)
Fırat University, Department of Mathematics, 23119, Elazığ, Turkey