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On category of interval valued fuzzy topological spaces

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Abstract. In this paper, we investigate the relations between continuous, open and closed maps in interval valued fuzzy topological spaces (IVFTSs for short) and those of some induced fuzzy topological spaces (FTSs for short). Then to provide a categorical framework for interval valued fuzzy topology, we construct some of its subcategories, define many functors between those subcategories and investigate some of their properties. Moreover, the relationship between the category of IVFTSs and some of its subcategories is studied. We proved that the category of FTSs is a bireflective full subcategory of that of IVFTSs.

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1. Introduction

Since Zadeh has introduced fuzzy sets[13], many new approaches and theories treating imprecision and uncertainty have been proposed (e.g.[1], [2], [3],[4]). Some of these theories are extensions of the fuzzy set theory. It is well-known that a generalization of a fuzzy set is an interval valued fuzzy set, which is originally proposed by Zadeh [14], attributed to Gorzalczany [5] and Turksen [12], has been regarded as an important mathematical tool to deal with vagueness and the relation between objects in fuzzy information systems. Many topologists have shown interest in several categories such as Top, Bitop, Unif, Prox[10], etc. The striking similarities among these categories have led categorical topologists to establish the theory of topological categories in fuzzy setting and its generalizations. Lowen [7] proved that the category of FTSs is topological. Then Mondal, et. al [9] defined the topology of IVFSs and studied some of its properties, they proved that the category of IVFTSs with continuous maps is topological. The main aim of this paper is to provide a categorical framework for an interval valued fuzzy topology. In section 2, we recall some preliminary results on IVFSs and IVFTSs which will be used in this paper. In section 3, we studied the relation between continuity in IVFTSs and that of some induced FTSs. In the final section, we defined many subcategories of the category of IVFTSs, some functors on these categories are constructed and investigated some of their properties. In addition, we examine the relationship between the category of IVFTSs and some of its subcategories.

2. Preliminaries

In this section, we recall some definitions and results which will be used in this sequel. For detail we refer to (e.g.[5], [6], [8],[9],[11],[12]). Throughout this paper let $I = [0,1]$ be the closed unit interval and let us denote the family of all closed subintervals of I as $[I]$, where $[I] = \{[a,b] : a \leq b, a, b \in I\}$. For any $a \in I$, we define $\underline{a} = [a,a]$, then $\underline{a} \in [I]$.

Definition 2.1. [9] Let X be an ordinary set. Then the mapping $A : X \to [I]$ is called an interval valued fuzzy set(IVFSs, briefly) on X. For any IVFS $\underline{A}$ denote $A(x) = [A_1(x), A_2(x)]$, where $A_1(x) \leq A_2(x)$, $x \in X$. Then the two fuzzy sets $A_1 : X \to [I]$, $A_2 : X \to [I]$ are called the lower fuzzy set and the upper fuzzy set of $\underline{A}$, respectively.

Kandil, et al.[6] reformed the above definition for simplicity as follows:
An interval valued fuzzy set is an ordered pair $\underline{A} = (A_1, A_2) \in I^X \times I^X$ such that $A_1 \subseteq A_2$, where I^X is the family of all fuzzy sets on X. And so the set of all IVFSs is denoted by II^X and defined by $II^X = \{\underline{A} = (A_1, A_2) : A_1, A_2 \in I^X, A_1 \subseteq A_2\}$. The IVFS $\underline{X} = (X, X)$ is called the universal-IVFS and the IVFS $\underline{\phi} = (\phi, \phi)$ is called the empty-IVFS. Obviously any fuzzy set A on X is an IVFS of the form $\underline{A} = (A, A)$.

Definition 2.2. [6] Let $\underline{A} = (A_1, A_2)$ and $\underline{B} = (B_1, B_2)$ are two IVFSs. Then:
1) $\underline{A} = \underline{B} \Leftrightarrow A_i = B_i$ where, $i = 1, 2$.
2) $\underline{A} \subseteq \underline{B} \Leftrightarrow A_i = B_i$ where, $i = 1, 2$.
3) $(\underline{A} \cup \underline{B}) = (A_1 \cup B_1, A_2 \cup B_2)$.
4) $(\underline{A} \cap \underline{B}) = (A_1 \cap B_1, A_2 \cap B_2)$.
5) $\underline{A}^C = (1 - A_2, 1 - A_1)$ where, $\underline{A}^C$ is the complement of $\underline{A}$.

Definition 2.3. [6] Let X be a nonempty set and $x \in X$ be a fixed element. An interval valued fuzzy point(IVFP, briefly) is an ordered pair (x_α, x_β) of two fuzzy points with $(x_\alpha \leq x_\beta)$ where, $\alpha, \beta \in I$, $\beta > 0$ and is denoted by $x_{(\alpha,\beta)} = (x_\alpha, x_\beta)$. An IVFP $x_{(\alpha,\beta)} \in \underline{A}$ if and only if $(x_\alpha \in A_1$ and $x_\beta \in A_2)$.

Let X and Y be two nonempty sets and $f : X \to Y$ be a map. If $\underline{A} = (A_1, A_2)$ is an IVFS in X, then $f(\underline{A})$ is an IVFS in Y defined by $f(\underline{A}) = (f(A_1), f(A_2))$ where, $f(A_i)(y) = sup_{x \in f^{-1}(y)} A_i(x)$ if $f^{-1}(y) \neq \phi$ and equal ϕ otherwise $\forall y \in Y$, $i = 1, 2$. And if $\underline{B} = (B_1, B_2)$ is an IVFS in Y, then $f^{-1}(\underline{B})$ is an IVFS in X defined by $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2))$ where, $f^{-1}(B_i)(x) = B_i(f(x))$ $\forall x \in X, i = 1, 2$.
For the basic properties of the image and primage of the map f we refer to [9].

Definition 2.4. [9] The family η of IVFSs in X is called an interval valued fuzzy topology on X iff η contains $\underline{X}, \underline{\phi}$ and it is closed under finite intersection and arbitrary union. Then the pair (X, τ) is called an interval valued fuzzy topological space. Every element of η is called an interval valued fuzzy open set (IVFOS for short) in X. The complement $\underline{A}^C$ of an IVFOS in (X, τ) is called an interval valued fuzzy closed set (IVFCS, for short) in X and the set of all IVFCSs is denoted by η^C.

Definition 2.5. [6] Let $\beta_1, \beta_2 \subseteq I^X$ are two families of fuzzy sets. Then the IVF-product of β_1, β_2 is denoted by $\beta_1 \hat{\times} \beta_2$ and defined by $\beta_1 \hat{\times} \beta_2 = \{(A_1, A_2) \in \beta_1 \hat{\times} \beta_2 : A_1 \subseteq A_2\}$.

Theorem 2.6. [11] *Let (X, η) be an IVFTS on X. Then the following collections are fuzzy topologies on X induced by η:*
i) $\pi_1 = \{A_1 : (A_1, A_2) \in \eta\}$,
ii) $\pi_2 = \{A_2 : (A_1, A_2) \in \eta\}$,
iii) $\pi_3 = \{A_1 : (A_1, X) \in \eta\} \cup \{\phi\}$,
iv) $\pi_4 = \{A_2 : (\phi, A_2) \in \eta\} \cup \{X\}$,
v) $\pi_\Delta = \{A : (A, A) \in \eta\}$. *Moreover* $\pi_3 \subseteq \pi_1, \pi_4 \subseteq \pi_2$ *and* $\pi_\Delta \subseteq \pi_1 \cap \pi_2$.

Definition 2.7. [6] An interval fuzzy topological space (X, τ) is called a G-IVFTS iff $\eta = \pi_1 \hat{\times} \pi_2$ which is the greatest IVFT constructed by IVF-product of π_1, π_2 where, in general $\eta \subseteq \pi_1 \hat{\times} \pi_2$.

Note: It is clear that every a G-IVFT is an IVFT. The Example (2.4.2) in [11] shows that the converse may not be true in general.

Theorem 2.8. [11] *Let (X, τ) be a fuzzy topological space. Then the following families are G-IVFTSs on X induced by τ :*
1) $\eta_1 = \tau \hat{\times} I^X = \{(A_1, A_2) : A_1 \in \tau\}$,
2) $\eta_2 = I^X \hat{\times} \tau = \{(A_1, A_2) : A_2 \in \tau\}$,
3) $\eta_3 = \tau \hat{\times} i(X) = \{(A_1, X) : A_1 \in \tau\} \cup \{\underline{\phi}\}$, *where* $i(X) = \{X, \phi\}$ *is the indiscrete fuzzy topology on X.*
4) $\eta_4 = i(X) \hat{\times} \tau = \{(\phi, A_2) : A_2 \in \tau\} \cup \{\underline{X}\}$,
5) $\eta_\Delta = \{(A, A) : A \in \tau\}$. *Moreover for* $\eta_i (i = 1, 2, 3, 4)$ *we have* $\eta_3 \subseteq \eta_1, \eta_4 \subseteq \eta_1$ *and* $\eta_\Delta \subseteq \eta_1 \cap \eta_2$.

3. Some relations between IVF-Continuity and F-Continuity

Definition 3.1. [6] Let $(X, \eta), (Y, \eta^*)$ be IVFTSs. Then a map $f : X \to Y$ is called:
1) IVF-continuous if $f^{-1}(\underline{B})$ is IVFOS of X for all IVFOS $\underline{B}$ of Y, [or equivalently, $f^{-1}(\underline{B})$ is IVFCS of X for each IVFCS $\underline{B}$ of Y],
2) IVF-open function if $f(\underline{A})$ is IVFOS of Y for each IVFOS $\underline{A}$ of X,
3) IVF-closed function if $f(\underline{A})$ is IVFCS of Y for each IVFCS of X,
4) IVF-homeomorphism if f is bijective and f, f^{-1} are IVF-continuous.

Theorem 3.2. *Let $(X, \eta), (Y, \eta^*)$ be any two IVFTSs. If the map $f : (X, \eta) \to (Y, \eta^*)$ is IVF-continuous, then the maps $f : (X, \pi_i) \to (Y, \pi_i^*), i = 1, 2$ are F-continuous, where $(\pi_i, \pi_i^*), i = 1, 2$ are defined as in Theorem (2.6).*

Proof. Let $f : (X, \eta) \rightarrow (Y, \eta^*)$ be an IVF-continuous map and $B_1 \in \pi_1^*$. Then there is $B_2 \in \pi_2^*$. Since $\underline{B} = (B_1, B_2) \in \eta^*$, then $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2)) \in \eta$ consequently, $f^{-1}(B_1) \in \pi_1$. Thus $f : (X, \pi_1) \rightarrow (Y, \pi_1^*)$ is F-continuous.
The proof for the case i=2 can be done in a similar way. □

Note: The following example shows that the converse of the above theorem may not be true in general.

Example 3.3. Let (X, τ) be any FTS. Then the map $I_X : (X, \tau) \rightarrow (X, \tau)$ is F-continuous. But $I_X : (X, \eta_\Delta) \rightarrow (X, \tau \hat{\times} \tau)$ is not an IVF-continuous map where, $\eta_\Delta = \{(A, A) : A \in \tau\}$.

Theorem 3.4. *Let (X, η) be the G-IVFTS and (Y, η^*) be any IVFTS. Then $f : (X, \tau) \rightarrow (Y, \tau^*)$ is an IVF-continuous map iff the maps $f : (X, \pi_i) \rightarrow (Y, \pi_i^*), i = 1, 2$ are F-continuous.*

Proof. " ⇒ " follows from Theorem(3.2). Conversely, let the maps $f : (X, \pi_i) \rightarrow (Y, \pi_i^*), i = 1, 2$ are F-continuous and $\underline{A} = (A_1, A_2) \in \eta^*$. Then $A_1 \in \pi_1^*, A_2 \in \pi_2^*$ and so $f^{-1}(A_1) \in \pi_1, f^{-1}(A_2) \in \pi_2$. Since $f^{-1}(\underline{A}) = (f^{-1}(A_1), f^{-1}(A_2)) \in \eta$ and hence $f : (X, \tau) \rightarrow (Y, \tau^*)$ is IVF-continuous. □

Theorem 3.5. *Let $(X, \eta), (Y, \eta^*)$ be any two IVFTSs. If the map $f : (X, \eta) \rightarrow (Y, \eta^*)$ is IVF-continuous, then the maps $f : (X, \pi_i) \rightarrow (Y, \pi_i^*), i = 3, 4$ are F-continuous, where $(\pi_3, \pi_4^*), i = 3, 4$ are defined as in Theorem (2.6).*

Proof. The proof for the case $i = 3$. Let $f : (X, \eta) \rightarrow (Y, \eta^*)$ be an IVF-continuous map and $B_1 \in \pi_3^*$, then $\underline{B} = (B_1, Y) \in \eta^*$. Since $f : (X, \eta) \rightarrow (Y, \eta^*)$ is IVF-continuous, then $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(Y)) = (f^{-1}(B_1), X) \in \eta$ and so, $f^{-1}(B_1) \in \pi_3$. Hence $f : (X, \pi_3) \rightarrow (Y, \pi_3^*)$ is F-continuous. The proof of the case $i = 4$ can be done in a similar way. □

Note: The Example (3.3) shows that the converse of the above theorem may not be true in general.

Theorem 3.6. *Let (X, η) be the G-IVFTS and $(Y, \eta_i^*), i = 3, 4$ are IVFTS defined in Theorem(2.8). Then the maps $f : (X, \eta) \rightarrow (Y, \eta_i^*)$ are IVF-continuous iff $f : (X, \pi_i) \rightarrow (Y, \pi_i^*), i = 3, 4$ are F-continuous maps, $i = 3, 4$.*

Proof. The proof of the case $i = 3$. " ⇒ " follows from the above theorem.
Conversely, let the maps $f : (X, \pi_3) \rightarrow (Y, \pi_3^*)$ is F-continuous and $\underline{B} = (B_1, Y) \in \eta_3^*$, then $B_1 \in \pi_3^*$. Since $f : (X, \pi_3) \rightarrow (Y, \pi_3^*)$ is F-continuous, then $f^{-1}(B_1) \in \pi_3$ and so $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(Y)) = (f^{-1}(B_1, X) \in \eta$. Hence $f : (X, \eta) \rightarrow (Y, \eta_3^*)$ is IVF-continuous. The proof of the rest case can be done in a similar way. □

Theorem 3.7. *Let $f : (X, \eta) \rightarrow (Y, \eta^*)$ be an IVF-open(IVF-closed)map. Then $f : (X, \pi_i) \rightarrow (Y, \pi_i^*), i = 1, 2$ are fuzzy open(closed)maps.*

Proof. Let $f : (X, \eta) \rightarrow (Y, \eta^*)$ be an IVF-open map and $A_1 \in \pi_1$. Then there is $A_2 \in \pi_2$ such that $\underline{A} = (A_1, A_2) \in \eta$. Since $f : (X, \eta) \rightarrow (Y, \eta^*)$, then $f(\underline{A}) = (f(A_1), f(A_2)) \in \eta^*$ and so $f(A_1) \in \pi_1^*$, $f(A_2) \in \pi_2^*$. Thus $f : (X, \pi_1) \rightarrow (Y, \pi_1^*)$ is a fuzzy open map. The proof of the rest case can be done in analogous way. □

Note: The following example shows that the converse of the above theorem may not be true in general.

Example 3.8. Let $X = \{x, y, z\}, \eta = \{\underline{X}, \underline{\phi}, (\phi, X), (\phi, (x_1, y_0, z_1)), (\phi, (x_0, y_1, z_0)), ((x_1, y_0, z_0), (x_1, y_0, z_1)), ((x_0, y_1, z_1), X), ((x_1, y_0, z_0), X)\}$ and $Y = \{a, b, c\}$, $\eta^* = \{\underline{Y}, \underline{\phi}, (\phi, Y), ((a_1, b_0, c_0), Y), ((a_0, b_1, c_1), Y), (\phi, (a_0, b_1, c_0)), (\phi, (a_1, b_0, c_1))\}$. If $f : X \to Y$ a map given by $f((x_1, y_0, z_0)) = (a_1, b_0, c_0), f((x_0, y_1, z_0)) = (a_0, b_1, c_0)$ and $f((x_0, y_0, z_1)) = (a_0, b_0, c_1)$, then $f : (X, \pi_i) \to (Y, \pi_i^*), i = 1, 2$ are fuzzy open maps. But $f : (X, \eta) \to (Y, \eta^*)$ is not an IVF-open map. In fact,

$$f((x_1, y_0, z_0), (x_1, y_0, z_1)) = ((a_1, b_0, c_0), (a_1, b_0, c_1)) \notin \eta^*.$$

Theorem 3.9. *Let (Y, η^*) be the G-IVFTS and (X, η) be any IVFTS. Then the map $f : (X, \eta) \to (Y, \eta^*)$ is IVF-open(IVF-closed) iff the maps $f : (X, \pi_i) \to (Y, \pi_i^*), i = 1, 2$ are fuzzy open(closed).*

Proof. " $\Rightarrow$ " Follows from Theorem(3.7).
Conversely, let $f : (X, \pi_i) \to (Y, \pi_i^*), i = 1, 2$ be fuzzy open(closed) maps and $\underline{A} = (A_1, A_2) \in \eta$. Then $A_1 \in \pi_1, A_2 \in \pi_2$ and so $f(A_1) \in \pi_1^*, f(A_2) \in \pi_2^*$. Consequently, $f(\underline{A}) = (f(A_1), f(A_2)) \in \eta^*$. Hence the result follows. □

Theorem 3.10. *Let $(X, \eta), (Y, \eta^*)$ be any two IVFTSs. If the map $f : (X, \eta) \to (Y, \eta^*)$ is an IVF-open(IVF-closed)map then $f : (X, \pi_i) \to (Y, \pi_i^*), i = 3, 4$ are fuzzy open(closed)maps, where $(\pi_i, \pi_i^*, i = 3, 4)$ are defined as in Theorem(2.6).*

Proof. The proof can be done is a similar to that of Theorem(3.7). □

Note: The Example(3.8) shows that the converse of the above theorem may not be true in general.

Theorem 3.11. *Let $(X, \eta_i), i = 3, 4$ are the IVFTSs defined in Theorem(2.8) and (Y, η^*) be the G-IVFTS. Then the map $f : (X, \eta_i) \to (Y, \eta^*)$ is IVF-open(IVF-closed) iff the maps $f : (X, \pi_i) \to (Y, \pi_i^*), i = 3, 4$ are fuzzy open(closed).*

Proof. The proof of the case $i = 3$. "$\Rightarrow$" follows from the above theorem.
Conversely, let the maps $f : (X, \pi_3) \to (Y, \pi_3^*)$ are fuzzy open(closed) and $\underline{A} = (A_1, X) \in \eta_3$, then $A_1 \in \pi_3$. Since $f : (X, \pi_3) \to (Y, \pi_3^*)$ is a fuzzy open(closed) map, then $f(A_1) \in \pi_3^*$ and so $f(\underline{A}) = (f(A_1), f(X)) = (f(A_1), Y) \in \eta^*$. Hence the result obtains. The proof of the case $i = 4$ can be done in a similar way. □

Theorem 3.12. *Let $(X, \eta), (Y, \eta^*)$ are two G-IVFTSs. Then $f : (X, \eta) \to (Y, \eta^*)$ is IVF-homeomorphism iff $f : (X, \pi_i) \to (Y, \pi_i^*), i = 1, 2$ are F-homeomorphisms.*

Proof. The proof follows from Theorem(3.4)and Theorem(3.9). □

Theorem 3.13. *Let $(X, \eta_i), (Y, \eta_i^*), i = 3, 4$ are G-IVFTSs. Then the maps $f : (X, \eta_i) \to (Y, \eta_i^*)$ are IVF-homeomorphisms iff $f : (X, \pi_i) \to (Y, \pi_i^*), i = 3, 4$ are F-homeomorphisms.*

Proof. The proof follows from Theorem(3.6)and Theorem(3.11). □

4. The category of IVFTSs and its subcategories

In this section, we study the relationship between the category of interval valued fuzzy topological spaces (IVFTSs, briefly) with some of its subcategories and the category of fuzzy topological spaces (FTSs, briefly).
Let Ψ be the category of all interval valued fuzzy topological spaces with IVF-continuous maps and Φ be the category of all fuzzy topological spaces with F-continuous maps. First, let us define some functors between Ψ and Φ.

Theorem 4.1. *For the categories Ψ and Φ. Define:*
1) $F_1 : \Psi \to \Phi$ by $F_1(X, \eta) = (X, F_1(\eta))$ and $F_1(f) = f$ where,
$F_1(\eta) = \{A_1 : (A_1, A_2) \in \eta\}$.
2) $F_2 : \Psi \to \Phi$ by $F_2(X, \eta) = (X, F_2(\eta))$ and $F_2(f) = f$ where,
$F_2(\eta) = \{A_2 : (A_1, A_2) \in \eta\}$.
3) $P_1 : \Phi \to \Psi$ by $P_1(X, \tau) = (X, P_1(\tau))$ and $P_1(f) = f$ where,
$P_1(\tau) = \{(A_1, A_2) : A_1 \in \tau, A_1 \subseteq A_2\}$.
4) $P_2 : \Phi \to \Psi$ by $P_2(X, \tau) = (X, P_2(\tau))$ and $P_2(f) = f$ where,
$P_2(\tau) = \{(A_1, A_2) : A_2 \in \tau, A_1 \subseteq A_2\}$. Then F_1, F_2, P_1, P_2 are functors.

Proof. The proofs for 1) and 2) follows from Theorem(2.6), Theorem(3.2) and the proofs for 3) and 4) follows from Theorem(2.8), Theorem(3.4). □

Note: $F_1(\eta) = \pi_1, F_2(\eta) = \pi_2, P_1(\tau) = \eta_1$ and $P_2(\tau) = \eta_2$.

Theorem 4.2. *For the categories Ψ and Φ. Define:*
i) $F_3 : \Psi \to \Phi$ by $F_3(X, \eta) = (X, F_3(\eta))$ and $F_3(f) = f$ where,
$F_3(\eta) = \{A_1 : (A_1, X) \in \eta\} \cup \{\phi\}$.
ii) $F_4 : \Psi \to \Phi$ by $F_4(X, \eta) = (X, F_4(\eta))$ and $F_4(f) = f$ where,
$F_4(\eta) = \{A_2 : (\phi, A_2) \in \eta\} \cup \{X\}$.
iii) $P_3 : \Phi \to \Psi$ by $P_3(X, \tau) = (X, P_3(\tau))$ and $P_3(f) = f$ where,
$P_3(\tau) = \{(A_1, X) : A_1 \in \tau\} \cup \{\underline{\phi}\}$.
iv) $P_4 : \Phi \to \Psi$ by $P_4(X, \tau) = (X, P_4(\tau))$ and $P_4(f) = f$ where,
$P_4(\tau) = \{(\phi, A_2) : A_2 \in \tau\} \cup \{\underline{X}\}$. Then F_3, F_4, P_3, P_4 are functors.

Proof. The proofs for i), ii) follow from Theorem(2.6), Theorem(3.5) and the proofs for iii), iv) follow from Theorem(2.8), Theorem(3.6). □

Note: $F_3(\eta) = \pi_3, F_4(\eta) = \pi_4, P_3(\tau) = \eta_3$ and $P_4(\tau) = \eta_4$.

Theorem 4.3. *The functor $P_1 : \Phi \to \Psi$ is a left adjoint of the functor $F_1 : \Psi \to \Phi$.*

Proof. By using the universal property. Let $(X, \tau) \in \Phi$ and the map $I_X : (X, \tau) \to F_1(P_1(X, \tau)) = (X, \tau)$ be F-continuous. We only need to prove that the map I_X is universal. Let $(Y, \xi) \in \Psi$ and the map $f : (X, \tau) \to F_1(Y, \xi)$ be F-continuous, then it is sufficient to prove that the map $f^* : P_1(X, \tau) = (X, P_1(\tau)) \to F_1(Y, \xi)$ is IVF-continuous. So let $\underline{B} = (B_1, B_2) \in \xi$, then $B_1 \in F_1(\xi)$. Since $f : (X, \tau) \to F_1(Y, \xi)$ is IVF-continuous, then $f^{-1}(B_1) \in \tau$ and hence $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2)) \in P_1(\tau)$. Thus $f^* : P_1(X, \tau) \to F_1(Y, \xi)$ is an IVF-continuous map. Therefore I_X is an F_1-universal map for (X, τ) in Φ. Hence the result follows. □

Theorem 4.4. *The functor $P_2 : \Phi \to \Psi$ is a left adjoint of the functor $F_2 : \Psi \to \Phi$.*

Proof. The proof can be done in a similar way of that of the above theorem. □

Definition 4.5. On the category Ψ, we define five subcategories as follows:
1) Ψ_{η_3} : The full subcategory of Ψ whose objects are all IVFTSs from the type η_3 defined in Theorem(2.8) with an IVF-continuous maps.
2) Ψ_{η_4} : The full subcategory of Ψ whose objects are all IVFTSs from the type η_4 defined in Theorem(2.8) with an IVF-continuous maps.
3) Ψ_{η_1} : The full subcategory of Ψ whose objects are all IVFTSs from the type η_1 defined in Theorem(2.8) with an IVF-continuous maps.
4) Ψ_{η_2} : The full subcategory of Ψ whose objects are all IVFTSs from the type η_2 defined in Theorem(2.8) with an IVF-continuous maps.
5) Ψ_{η_Δ} : The full subcategory of Ψ whose objects are all IVFTSs from the type η_Δ defined in Theorem(2.8) with an IVF-continuous maps.

Note: It is clear that the category $\Psi_{\eta_3}(\Psi_{\eta_4})$ is a fully subcategory of the category $\Psi_{\eta_1}(\Psi_{\eta_2})$.

Theorem 4.6. *Let $F_3^* : \Psi_{\eta_3} \to \Phi$ and $F_4^* : \Psi_{\eta_4} \to \Phi$ be the restrictions of the functors F_3, F_4 , respectively. Then:*
i) The functor $P_3 : \Phi \to \Psi$ is a left adjoint of the functor $F_3 : \Psi \to \Phi$.
ii) The functor $P_4 : \Phi \to \Psi$ is a left adjoint of the functor $F_4 : \Psi \to \Phi$.

Proof. The proof for the case (i) follows from fact that, $F_3 \circ P_3 = I_\Phi$ and $P_3 \circ F_3^* = I_{\Psi_{\eta_3}}$, where $F_3^* = F_3|I_{\Psi_{\eta_3}}$ is the restriction functor of the functor F_3. Indeed, $F_3 \circ P_3(X,\tau) = (X,\tau)\forall(X,\tau) \in \Phi$ and $P_3 \circ F_3^*(X,\xi) = (X,\xi)\forall(X,\xi) \in \Psi_{\eta_3}$.
The proof for the rest case is analogous. □

Theorem 4.7. *The category Ψ_{η_Δ} is isomorphic to the category Φ.*

Proof. We define the functor $P_1^* : \Phi \to \Psi_{\eta_\Delta}$ by $P_1^*(X,\tau) = (X,\eta_\Delta)$, $P_1^*(f) = f$, where, $\eta_\Delta = \{(A,A) : A \in \tau\}$. Consider the restriction $F_1^* : \Psi_{\eta_\Delta} \to \Phi$ of the functor F_1. Then P_1^*, F_1^* are functors. Indeed $F_1^* \circ P_1^*(X,\tau) = F_1^*(X,P_1^*(\tau)) = (X, F_1^*P_1^*(\tau)) = (X,\tau)\ \forall(X,\tau) \in \Phi$. Now we only need to prove that $P_1^* \circ F_1^*(X,\eta) = (X,\eta)\ \forall(X,\eta) \in \Psi_{\eta_\Delta}$. So let $(X,\eta) \in \Psi_{\eta_\Delta}$, $\underline{A} = (A_1,A_2) \in \eta$, then $A_1 = A_2$, $A_1 \in F_1^*(\eta)$ and hence $\underline{A} = (A_1,A_2) = (A_1,A_1) \in (F_1^*(\eta))_\Delta$. Thus $P_1^* \circ F_1^*(X,\eta) = P_1^*(X,F_1^*(\eta)) = (X,(P_1^*(\eta))_\Delta) = (X,\eta)$. Hence the result follows. □

Theorem 4.8. *The category Φ is isomorphic to the categories Ψ_{η_3} and Ψ_{η_4}.*

Proof. The proof for the first case, let us define the restriction functor $P_3^* : \Phi \to \Psi_{\eta_3}$ for the functor P_3 by $P_3^*(X,\tau) = (X,P_3^*(\tau))$, $P_3^*(f) = f$ where, $P_3^*(\tau) = \{(A_1,X) : A_1 \in \tau\} \cup \{\underline{\phi}\}$ and consider the restriction $F_3^* : \Phi\Psi_{\eta_3} \to \Phi$ of the functor F_3. Then P_3^*, F_4^* are functors. It is clear that $F_3^* \circ P_3^*(X,\tau) = F_3^*(X,P_3^*(\tau)) = (X,F_3^*P_3^*(\tau)) = (X,\tau)\ \forall(X,\tau) \in \Phi$. Moreover, $P_3^* \circ F_3^*(X,\eta) = (X,\eta)\ \forall(X,\eta) \in \Psi_{\eta_3}$. Hence the result follows. The proof for the seconde case can be proved in a similar way. □

From the Theorem(4.7)and Theorem(4.8) we obtain the following corollary.

Corollary 4.9. *The categories Φ , Ψ_{η_3} , Ψ_{η_4} and Ψ_{η_Δ} are isomorphic.*

Theorem 4.10. *The category Ψ_{η_Δ} is a bireflective full subcategory of the category Ψ.*

Proof. It is clear that the category Ψ_{η_Δ} is a full subcategory of the category Ψ. Now let $(X, \eta) \in \Psi$ and $\eta^* = \{\underline{A} \in \eta : \underline{A} = (A, A)\}$. Then $(X, \eta^*) \in \Psi_{\eta_\Delta}$ and $I_X : (X, \eta) \to (X, \eta^*)$ is IVF-continuous map. Consider the IVFTS $(Y, \xi) \in \Psi_{\eta_\Delta}$ with an IVF-continuous map $f : (X, \eta) \to (Y, \xi)$. We only need to prove that the map $f : (X, \eta^*) \to (Y, \xi)$ is IVF-continuous. So let $\underline{B} \in \xi$. Since $(Y, \xi) \in \Psi_{\eta_\Delta}$ and $f : (X, \eta) \to (Y, \xi)$ is IVF-continuous, then $\underline{B} = (B, B)$ and $f^{-1}(\underline{B}) \in \eta$. But $f^{-1}(\underline{B}) = (f^{-1}(B), f^{-1}(B)) \in \xi$. Hence the map $f : (X, \eta^*) \to (Y, \xi)$ is IVF-continuous and so the result holds.

□

From the above theorem and Corollary(4.9) we obtain the following result.

Corollary 4.11. *The categories Φ , Ψ_{η_3} and Ψ_{η_4} are a bireflective full subcategories of the category Ψ.*

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