Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 4, No. 2, (October 2012), pp. 365–375
ISSN 2093–9310
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

On soft int-groups

Kenan Kaygisiz

Received 29 November 2011; Revised 20 January 2012; Accepted 14 February 2012

Abstract. In this paper, we present soft intersection groups (soft int-groups) on a soft set and obtain some other properties of soft int-groups. We also investigate some relations on α-inclusion, soft product and soft int-groups.

2010 AMS Classification: 03G25, 20N25, 08A72, 06D72

Keywords: Soft set, α-inclusion, Soft product, Soft int-group.

Corresponding Author: Kenan Kaygisiz (kenan.kaygisiz@gop.edu.tr)

1. Introduction

There are many problems in economy, engineering, environmental and social sciences that may not be successfully modeled by the classical mathematics because of various types of uncertainties. Zadeh [24] introduced the notion of a fuzzy set in 1965 to deal with such kinds of problems.

In 1971, Azriel Rosenfeld [22] defined the fuzzy subgroup of a group. Rosenfeld's paper made important contributions to the development of fuzzy abstract algebra. Since then, various researchers have studied on fuzzy group theory analogues of results derived from classical group theory. These include [1, 3, 6, 7, 8, 11, 12, 17, 21]. Mordeson et al. [20] combined all the above papers and many others in their book titled Fuzzy Group Theory.

There is another theory, called soft sets, defined by Molodtsov [19] in 1999 in order to deal with uncertainties. Since then Maji et al.[18] studied the operations of soft sets, Çağman and Enginoğlu [10] modified definition and operations of soft sets and Ali et al. [5] presented some new algebraic operations for soft sets. Sezgin and Atagün [23] analyzed operations of soft sets. Using these definitions, researches have been very active on the soft sets and many important results have been achieved in theoretical and practical aspects.

An algebraic structure of soft sets was first studied by Aktaş and Çağman [4]. They introduced the notion of the soft group and derived some basic properties.

Since then, many papers have been prepared on soft algebraic structures, such as [2, 13, 14, 15, 16].

Çağman et al. [9] studied on soft int-groups, which are different from the definition of soft groups in [4]. This new approach is based on the inclusion relation and intersection of sets. It brings the soft set theory, set theory and the group theory together. In this paper, we give some supplementary properties of soft sets and soft int-groups, analogues to classical group theory and fuzzy group theory. We, finally, present some important relations on α-inclusion, soft product and soft int-groups, to construct notions of soft group theory.

2. Preliminaries

2.1. **Soft sets.** In this section, we present basic definitions of soft set theory according to [10]. For more detailed explanations, we refer to the earlier studies [18, 19].

Throughout this paper, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U. $\subset$ and $\supset$ stands for proper subset and superset, respectively.

Definition 2.1 ([19]). For any subset A of E, a soft set f_A over U is a set defined by a function f_A representing a mapping

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \varnothing \text{ if } x \notin A.$$

A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E,\ f_A(x) \in P(U)\}.$$

Note that the set of all soft sets over U will be denoted by $S(U)$. From here on "soft set" will be used without over U.

Definition 2.2 ([10]). Let f_A be a soft set. If $f_A(x) = \varnothing$ for all $x \in E$, then f_A is called an empty soft set and denoted by Φ_A.

If $f_A(x) = U$ for all $x \in A$, then f_A is called A-universal soft set and denoted by $f_{\widetilde{A}}$.

If $f_A(x) = U$, for all $x \in E$, then f_A is called a universal soft set and denoted by $f_{\widetilde{E}}$.

If $f_A \in S(U)$, then the image(value class) of f_A is defined by

$$\operatorname{Im}(f_A) = \{f_A(x) : x \in A\}$$

and if $A = E$, then $\operatorname{Im}(f_E)$ is called image of E under f_A.

Definition 2.3. Let f_A be a soft set and $A \subseteq E$. Then, the set f_A^* defined by $f_A^* = \{x \in A : f_A(x) \neq \varnothing\}$ is called the support of f_A.

Definition 2.4. Let $f_A : E \longrightarrow P(U)$ be a soft set and $K \subseteq E$. Then, the image of a set K under f_A is defined by

$$f_A(K) = \cup \{f_A(x_i) : x_i \in K\}.$$

Definition 2.5 ([10]). Let f_A, f_B be two soft sets. Then, f_A is a soft subset of f_B, denoted by $f_A \widetilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

f_A is called a soft proper subset of f_B, denoted by $f_A \widetilde{\subset} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$ and $f_A(x) \neq f_B(x)$ for at least one $x \in E$.

f_A and f_B are called soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$.

Definition 2.6 ([10]). Let f_A, f_B be two soft sets. Then, union $f_A \widetilde{\cup} f_B$ and intersection $f_A \widetilde{\cap} f_B$ of f_A and f_B are defined by

$$f_{A\widetilde{\cup}B}(x) = f_A(x) \cup f_B(x),\ f_{A\widetilde{\cap}B}(x) = f_A(x) \cap f_B(x),$$

respectively.

2.2. **Definitions and basic properties of soft int-groups.** In this section, we review soft int-groups and their basic properties according to paper by Çağman et al. [9].

Definition 2.7 ([9]). Let G be a group and f_G be a soft set. Then, f_G is called a soft intersection groupoid over U if $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in G$ and is called a soft intersection group over U if it satisfies $f_G(x^{-1}) = f_G(x)$ for all $x \in G$ as well.

Throughout this paper, G denotes an arbitrary group with identity element e and the set of all soft int-groups with parameter set G over U will be denoted by $S_G(U)$, unless otherwise stated. For short, instead of "f_G is a soft int-group with the parameter set G over U" we say "f_G is a soft int-group".

Theorem 2.8 ([9]). *Let f_G be a soft int-group. Then*

(1) $f_G(e) \supseteq f_G(x)$ *for all* $x \in G$,
(2) $f_G(xy) \supseteq f_G(y)$ *for all* $y \in G$ *if and only if* $f_G(x) = f_G(e)$.

Theorem 2.9 ([9]). *A soft set f_G is a soft int-group if and only if $f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in G$.*

Definition 2.10 ([9]). Let f_G be a soft set. Then, e-set of f_G, denoted by e_{f_G}, is defined as

$$e_{f_G} = \{x \in G : f_G(x) = f_G(e)\}.$$

If f_G is a soft int-group, then the largest set in $\mathrm{Im}(f_G)$ is called the tip of f_G, which is equal to $f_G(e)$.

Theorem 2.11 ([9]). *If f_G is a soft int-group, then e_{f_G} is a subgroup of G.*

3. Some new results on soft sets and soft int-groups

In this section, we first give some new results on soft sets and soft int-groups. Then, we define soft singleton and soft product, and give some additional properties of soft int-groups.

Theorem 3.1. *If f_G is a soft int-group, then $f_G(x^n) \supseteq f_G(x)$ for all $x \in G$ where $n \in \mathbb{N}$.*

Proof. Proof is direct from definition of soft int-group by induction. □

Theorem 3.2. *Let f_G be a soft int-group and $x, y \in G$. If $f_G\left(xy^{-1}\right) = f_G(e)$, then $f_G(x) = f_G(y)$.*

Proof. For any $x, y \in G$,

$$f_G(x) = f_G\left(\left(xy^{-1}\right)y\right) \supseteq f_G\left(xy^{-1}\right) \cap f_G(y) = f_G(e) \cap f_G(y) = f_G(y)$$

and

$$\begin{aligned} f_G(y) &= f_G\left(y^{-1}\right) \\ &= f_G\left(x^{-1}\left(xy^{-1}\right)\right) \\ &\supseteq f_G\left(x^{-1}\right) \cap f_G\left(xy^{-1}\right) \\ &= f_G\left(x^{-1}\right) \cap f_G(e) \\ &= f_G(x). \end{aligned}$$

Hence $f_G(x) = f_G(y)$. □

Theorem 3.3. *If f_G is a soft int-group and $H \leq G$, then the restriction $f_G|_H$ is a soft int-group with the parameter set H.*

Proof. Since $H \leq G$, $f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in H$. Let's define $f_H(x) = f_G(x)$ for all $x \in H$. Since H is a group, $xy^{-1} \in H$ for all $x, y \in H$. Then for all $x, y \in H$

$$f_H(xy^{-1}) = f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y) = f_H(x) \cap f_H(y),$$

so f_H is a soft int-group with the parameter set H. □

Theorem 3.4. *Let $A_i \leq G$ for all $i \in I$ and $\{f_{A_i} : i \in I\}$ be a family of soft int-groups. Then, $\widetilde{\bigcap}_{i\in I} f_{A_i}$ is a soft int-group.*

Proof. By Çağman et al. ([9, Theorem 7]), we have the proof for two soft int-groups. To prove the general form, let $x, y \in G$, then

$$\begin{aligned} \bigcap_{i\in I} f_{A_i}\left(xy^{-1}\right) &= \bigcap\left\{f_{A_i}\left(xy^{-1}\right) : i \in I\right\} \\ &\supseteq \bigcap\left\{f_{A_i}(x) \cap f_{A_i}(y) : i \in I\right\} \\ &= \left(\bigcap\left\{f_{A_i}(x) : i \in I\right\}\right) \cap \left(\bigcap\left\{f_{A_i}(y) : i \in I\right\}\right) \\ &= \left(\bigcap_{i\in I} f_{A_i}(x)\right) \cap \left(\bigcap_{i\in I} f_{A_i}(y)\right). \end{aligned}$$

So the proof is complete by Theorem 2.9. □

Lemma 3.5. *Let f_G be a soft int-group such that either $f_G(x) \subseteq f_G(y)$ or $f_G(x) \supseteq f_G(y)$ for any $x, y \in G$. If $f_G(x) \neq f_G(y)$, then $f_G(xy) = f_G(x) \cap f_G(y)$ for any $x, y \in G$.*

Proof. If $f_G(x) \neq f_G(y)$, then either $f_G(x) \supset f_G(y)$ or $f_G(x) \subset f_G(y)$. Suppose

$$f_G(x) \subset f_G(y) \tag{3.1}$$

then

$$f_G(x) = f_G(xyy^{-1}) \supseteq f_G(xy) \cap f_G(y^{-1}) = f_G(xy) \cap f_G(y) \tag{3.2}$$

Thus, from (3.1) and (3.2) we have

$$f_G(x) \supseteq f_G(xy) \cap f_G(y) \supseteq f_G(xy) \supseteq f_G(x) \cap f_G(y) = f_G(x). \tag{3.3}$$

So, all expressions are equal in (3.3). Hence, $f_G(xy) = f_G(x) \cap f_G(y)$.

For the other case, proof is similar. □

Corollary 3.6. *Let f_G be a soft int-group as in Lemma 3.5. If $f_G(x) \subset f_G(y)$ then $f_G(x) = f_G(xy) = f_G(yx)$ for all $x, y \in G$.*

Proof. Obvious from the Lemma 3.5 and (3.3). □

Remark 3.7. Corollary 3.6 is not true if we replace, in the hypothesis, the strict inclusion $f_A(x) \subset f_A(y)$ with the inclusion $f_A(x) \subseteq f_A(y)$.

We show this fact by the following example.

Example 3.8. Let G be the Dihedral group D_3, where $D_3 = \{e, u, u^2, v, vu, vu^2\}$, and $u^3 = v^2 = e$, $uv = vu^2$. Define a mapping $f_G : G \longrightarrow P(U)$ such that

$$f_G(x) = \begin{cases} U & \text{for } x = e \\ \alpha & \text{for } x = v \\ \beta & \text{otherwise} \end{cases}$$

where $\phi \subset \beta \subset \alpha \subset U$. It is easy to verify that f_G is a soft group. In the notation of Corollary 3.6, let $x = u$ and $y = vu$, then although $\beta = f_G(u) \subseteq f_G(vu) = \beta$, $f_G((u)(vu)) = f_G(v) = \alpha \neq \beta = f_G(u)$. So $f_G(x) = f_G(xy)$ is not true.

Remark 3.9. The converse of Corollary 3.6 is not true. Let us show it by a counter example, using the Dihedral group given in Example 3.8. Let $x = u^2$ and $y = vu^2$. Although $f_G(x) = f_G(u^2) = \beta$ and $f_G(yx) = f_G((vu^2)(u^2)) = f_G(vu) = \beta$, the inclusion $\beta = f_G(u^2) = f_G(x) \subset f_G(y) = f_G(vu^2) = \beta$ is not true.

Theorem 3.10. *Let G be a cyclic group of prime order and $A \subseteq G$. Then, the soft set f_A, defined by*

$$f_A(x) = \begin{cases} \alpha & \text{for } x = e \\ \beta & \text{otherwise} \end{cases}$$

where $\alpha \supset \beta$ and $\alpha, \beta \in P(U)$, is a soft int-group.

Proof. For any $x, y \in G$, there are four conditions;

(1) $xy \neq e$ and neither x nor y equals to e. Then,

$$f_A(xy) = \beta \supseteq \beta \cap \beta = f_A(x) \cap f_A(y)$$

and since $x \neq e$, then $x^{-1} \neq e$, so $f_A(x) = f_A(x^{-1}) = \beta$.

(2) $xy \neq e$ and only one of x or y equals to e. Firstly, let $x = e$. Then,

$$f_A(xy) = f_A(ey) = \beta \supseteq \alpha \cap \beta = f_A(x) \cap f_A(y).$$

For the second condition of soft int-group, if $x = e$, then

$$f_A(x) = f_A(e) = \alpha = f_A(e^{-1}) = f_A(x^{-1}),$$

and since $y \neq e$, then $y^{-1} \neq e$ so, $f_A(y) = \beta = f_A(y^{-1})$.

(3) $xy = e$ and neither x nor y equals to e. Then,

$$f_A(xy) = \alpha \supseteq \beta \cap \beta = f_A(x) \cap f_A(y)$$

and $f_A(x) = f_A(x^{-1}) = \beta$, since $x \neq e$ implies $x^{-1} \neq e$.

(4) The last condition is $x = y = e$, which satisfies all conditions as well. □

Definition 3.11 ([9]). Let f_A be a soft set and $\alpha \in P(U)$. Then, α-inclusion of the soft set f_A, denoted by f_A^{α}, is defined as

$$f_A^{\alpha} = \{x \in A : f_A(x) \supseteq \alpha\}.$$

We define the set $f_A^{\alpha'} = \{x \in A : f_A(x) \supset \alpha\}$, which is called strong α-inclusion.

Corollary 3.12. *For any soft sets f_A and f_B,*

(1) $f_A \widetilde{\subseteq} f_B,\ \alpha \in P(U) \Rightarrow f_A^{\alpha} \subseteq f_B^{\alpha}$,
(2) $\alpha \subseteq \beta,\ \alpha, \beta \in P(U) \Rightarrow f_A^{\beta} \subseteq f_A^{\alpha}$,
(3) $f_A = f_B \Leftrightarrow f_A^{\alpha} = f_B^{\alpha}$, *for all* $\alpha \in P(U)$.

Theorem 3.13 ([9]). *Let I be an index set and $\{f_{A_i} : i \in I\}$ be a family of soft sets. Then, for any $\alpha \in P(U)$,*

(1) $\bigcup_{i\in I} (f_{A_i}^{\alpha}) \subseteq \left(\widetilde{\bigcup}_{i\in I} f_{A_i}\right)^{\alpha}$,
(2) $\bigcap_{i\in I} (f_{A_i}^{\alpha}) = \left(\widetilde{\bigcap}_{i\in I} f_{A_i}\right)^{\alpha}$.

Theorem 3.14. *Let f_A be a soft set and $\{\alpha_i : i \in I\}$ be a non-empty subset of $P(U)$ for each $i \in I$. Let $\beta = \bigcap_{i\in I} \alpha_i$ and $\gamma = \bigcup_{i\in I} \alpha_i$. Then, the following assertions hold:*

(1) $\bigcup_{i\in I} f_A^{\alpha_i} \subseteq f_A^{\beta}$,
(2) $\bigcap_{i\in I} f_A^{\alpha_i} = f_A^{\gamma}$.

Proof. Let $x \in A$. Then,

$$\begin{aligned} x &\in \bigcup_{i\in I} f_A^{\alpha_i} \Rightarrow \exists i \in I \text{ such that } x \in f_A^{\alpha_i} \\ &\Rightarrow \exists i \in I \text{ such that } f_A(x) \supseteq \alpha_i \\ &\Rightarrow \exists i \in I \text{ such that } f_A(x) \supseteq \alpha_i \supseteq \bigcap_{i\in I} \alpha_i = \beta \\ &\Rightarrow x \in f_A^{\beta}. \end{aligned}$$

So, the result follows. Second part is similar. □

Definition 3.15. Let f_A be a soft set and $\alpha \in P(U)$. Then, the soft set $f_{A\alpha}$, defined by, $f_{A\alpha}(x) = \alpha$, for all $x \in A$, is called $A - \alpha$ soft set. If A is a singleton, say $\{w\}$, then $f_{w\alpha}$ is called a soft singleton (or soft point). If $\alpha = U$, then $f_{\widetilde{A}}$ is the characteristic function of A.

Proposition 3.16. *Let $f_{A\alpha}$ be an $A - \alpha$ soft set. Then,*

(1) $f_{A\alpha} \widetilde{\cap} f_{B\alpha} = f_{(A\cap B)\alpha}$ *and* $f_{A\alpha} \widetilde{\cup} f_{B\alpha} = f_{(A\cup B)\alpha}$,
(2) $f_{A\alpha} \widetilde{\cap} f_{\widetilde{A}} = f_{A\alpha}$ *and* $f_{A\alpha} \widetilde{\cup} f_{\widetilde{A}} = f_{\widetilde{A}}$.

Lemma 3.17. *Let f_A be a soft set and $f_{(f_A^\alpha)\alpha}$ be an $(f_A^\alpha)-\alpha$ soft set. Then,*

$$f_A = \widetilde{\bigcup_{\alpha\in P(U)}} f_{(f_A^\alpha)\alpha} = \widetilde{\bigcup_{\alpha\in Im(f_A)}} f_{(f_A^\alpha)\alpha}.$$

It is clear that, instead of $A-\alpha$ soft set of all α subset of U, it is enough to consider $A-\alpha$ soft set taken α from $\mathrm{Im}(f_A)$.

Proof. For any $x \in A$,

$$\widetilde{\bigcup_{\alpha\in P(U)}} f_{(f_A^\alpha)\alpha}(x) = \bigcup \{\alpha \in P(U) : \alpha \subseteq f_A(x)\} = f_A(x).$$

So, $f_A = \widetilde{\bigcup}_{\alpha\in P(U)} f_{(f_A^\alpha)\alpha}$.
Similarly,

$$\widetilde{\bigcup_{\alpha\in Im(f_A)}} f_{(f_A^\alpha)\alpha}(x) = \bigcup \{\alpha \in Im(f_A) : \alpha \subseteq f_A(x)\} = f_A(x)$$

and thus $f_A = \widetilde{\bigcup}_{\alpha\in Im(f_A)} f_{(f_A^\alpha)\alpha}$. □

Theorem 3.18. *Let G be a group and $\alpha \in P(U)$. Then, f_G is a soft int-group if and only if f_G^α is a subgroup of G, whenever it is nonempty.*

Proof. The necessary condition is proven in ([9, Theorem 11]). To prove sufficient condition, suppose $f_G^\alpha \leq G$ for any nonempty f_G^α.

Let $x, y \in G$, $f_G(x) = \beta$ and $f_G(y) = \delta$, and let $\gamma = \beta \cap \delta$. Then, $x, y \in f_G^\gamma$ and f_G^γ is a subgroup of G by hypothesis. So $xy^{-1} \in f_G^\gamma$. Hence,

$$f_G(xy^{-1}) \supseteq \gamma = \beta \cap \delta = f_G(x) \cap f_G(y).$$

Thus, f_G is a soft int-group. □

Definition 3.19. Let f_G be a soft int-group. Then, the subgroups f_G^α are called level subgroups of G for any $\alpha \in P(U)$.

Definition 3.20. Let G be a group and $A, B \subseteq G$. Then, soft product of soft sets f_A and f_B is defined as

$$(f_A * f_B)(x) = \bigcup \{f_A(u) \cap f_B(v) : uv = x,\ u, v \in G\}$$

Inverse of f_A is defined as

$$f_A^{-1}(x) = f_A(x^{-1})$$

for all $x \in G$.

The following theorem reduces the soft product to the product of singletons.

Theorem 3.21. *Let G be a group and f_A, f_B, $f_{x\alpha}$, $f_{y\beta}$ be soft sets with the parameter set G. Then, for any $x, y \in G$ and $\phi \subset \alpha, \beta \subseteq U$,*

(1) $f_{x\alpha} * f_{y\beta} = f_{(xy)(\alpha\cap\beta)}$,

(2) $f_A * f_B = \widetilde{\bigcup}_{\substack{f_{x\alpha}\in f_A \\ f_{y\beta}\in f_B}} (f_{x\alpha} * f_{y\beta}) = \widetilde{\bigcup} \{f_{x\alpha} * f_{y\beta} : f_{x\alpha} \in f_A,\ f_{y\beta} \in f_B\}$.

Proof. Let $f_{x\alpha}, f_{y\beta}, f_A, f_B \in S(U)$.
(1) From the Definition 3.20, we have for any $w \in G$

$$\left(f_{x\alpha} * f_{y\beta}\right)(w) = \bigcup\{f_{x\alpha}(u) \cap f_{y\beta}(v) : uv = w,\ u, v \in G\}.$$

If $u = x$ and $v = y$, then $f_{x\alpha}(u) \cap f_{y\beta}(v) \neq \phi$, otherwise $f_{x\alpha}(u) \cap f_{y\beta}(v) = \phi$. Hence,

$$\left(f_{x\alpha} * f_{y\beta}\right)(w) = \bigcup\{\phi, f_{x\alpha}(x) \cap f_{y\beta}(y)\} = \bigcup\{\phi, \alpha \cap \beta\} = \alpha \cap \beta$$

for $w = xy$.

(2) For any point $w \in G$, we may assume that, there exists $u, v \in G$ such that $uv = w$ and $f_A(u) \neq \phi$, $f_B(v) \neq \phi$ without loss of generality. Then,

$$\begin{aligned}\left(f_A * f_B\right)(w) &= \bigcup\{f_A(u) \cap f_B(v) : uv = w \text{ and } u, v \in G\} \\ &= \bigcup\{f_{x\alpha}(x) \cap f_{y\beta}(y) : xy = w,\ f_{x\alpha} \in f_A,\ f_{y\beta} \in f_B\} \\ &= \widetilde{\bigcup_{\substack{f_{x\alpha} \in f_A \\ f_{y\beta} \in f_B}}} \left(f_{x\alpha} * f_{y\beta}\right)(w).\end{aligned}$$

□

Corollary 3.22. *Let $A \subseteq G$, $f_A \in S(U)$ and $f_{x\alpha}, f_{y\beta}, f_{z\gamma}$ be singletons in f_A. Then,*

(1) $\left(f_{x\alpha} * f_{y\beta}\right) * f_{z\gamma} = f_{x\alpha} * \left(f_{y\beta} * f_{z\gamma}\right)$,
(2) $f_{x\alpha} * f_{y\beta} = f_{y\beta} * f_{x\alpha}$, *if G is commutative,*
(3) $f_{x\alpha} * f_{e(f_A(e))} = f_{e(f_A(e))} * f_{x\alpha} = f_{x\alpha}$, *if $f_A \in S_G(U)$.*

Proof. Let $f_A \in S(U)$.
(1) For any $f_{x\alpha}, f_{y\beta}, f_{z\gamma} \in f_A$,

$$\begin{aligned}\left(f_{x\alpha} * f_{y\beta}\right) * f_{z\gamma} &= f_{(xy)(\alpha\cap\beta)} * f_{z\gamma} \\ &= f_{((xy)z)((\alpha\cap\beta)\cap\gamma)} \\ &= f_{(x(yz))(\alpha\cap(\beta\cap\gamma))} \\ &= f_{x\alpha} * f_{(yz)(\beta\cap\gamma)} \\ &= f_{x\alpha} * \left(f_{y\beta} * f_{z\gamma}\right).\end{aligned}$$

(2) Clear from Theorem 3.21.
(3) For any $f_{x\alpha} \in f_A$,

$$f_{x\alpha} * f_{e(f_A(e))} = f_{(xe)(\alpha\cap(f_A(e)))} = f_{x\alpha} = f_{(ex)((f_A(e))\cap\alpha)} = f_{e(f_A(e))} * f_{x\alpha}.$$

□

Theorem 3.23. *Let $A, B, C \subseteq G$ and f_A, f_B, f_C be soft sets. Then,*

(1) $\left(f_A * f_B\right) * f_C = f_A * \left(f_B * f_C\right)$,
(2) $f_A * f_B = f_B * f_A$, *if G is commutative,*
(3) *If $f_A \in S_G(U)$, then the identity element of operation "$*$" is $f_{e(f_A(e))}$.*

Proof. Proofs are direct from definition of soft product, Theorem 3.21 and Conclusion 3.22. □

Theorem 3.24. *Let $A, B \subseteq G$ and f_A, f_B be soft sets. Then,*

$$(f_A * f_B)(x) = \bigcup_{v \in G}\left(f_A(v) \cap f_B(v^{-1}x)\right) = \bigcup_{v \in G}\left(f_A(xv^{-1}) \cap f_B(v)\right)$$

for any $x \in G$.

Proof. For all $v \in G$, $v(v^{-1}x) = x$ or $(xv^{-1})v = x$ includes all compositions of x in the definition of soft product, so equality holds. □

Corollary 3.25. *Let $A \subseteq G$ and $f_A, f_{u\alpha}$ be soft sets where $\alpha = f_A(A)$. Then, for any $x, u \in G$,*

$$(f_{u\alpha} * f_A)(x) = f_A(u^{-1}x) \text{ and } (f_A * f_{u\alpha})(x) = f_A(xu^{-1}).$$

Proof. For any $x, u \in G$, we have

$$\begin{aligned}(f_{u\alpha} * f_A)(x) &= \bigcup_{v\in G} (f_{u\alpha}(v) \cap f_A(v^{-1}x)) \\ &= \alpha \cap f_A(u^{-1}x) \\ &= f_A(u^{-1}x)\end{aligned}$$

by Theorem 3.24. Second part is similar. □

Theorem 3.26. *Let $A_i \subseteq G$ and f_{A_i} $(i \in I)$ be soft int-groups. Then,*

$$\left(\widetilde{\bigcap_{i\in I}} f_{A_i}\right)^{-1} = \widetilde{\bigcap_{i\in I}} (f_{A_i})^{-1}.$$

Proof. For all $x \in G$,

$$\left(\widetilde{\bigcap_{i\in I}} f_{A_i}\right)^{-1}(x) = \left(\widetilde{\bigcap_{i\in I}} f_{A_i}\right)(x^{-1}) = \bigcap_{i\in I} f_{A_i}\left(x^{-1}\right) = \bigcap_{i\in I} f_{A_i}^{-1}(x)$$

so, equality holds. □

Theorem 3.27. *Let $A, B \subseteq G$ and f_A, f_B be soft int-groups. Then,*

$$(f_A * f_B)^{-1} = f_B^{-1} * f_A^{-1}.$$

Proof. For all $x \in G$,

$$\begin{aligned}&(f_A * f_B)^{-1}(x) = (f_A * f_B)\left(x^{-1}\right) \\ &= \bigcup\{f_A(u) \cap f_B(v) : uv = x^{-1},\ u, v \in G\} \\ &= \bigcup\{f_B\left(v^{-1}\right)^{-1} \cap f_A\left(u^{-1}\right)^{-1} : v^{-1}u^{-1} = x,\ u^{-1}, v^{-1} \in G\} \\ &= \bigcup\{f_B^{-1}\left(v^{-1}\right) \cap f_A^{-1}\left(u^{-1}\right) : v^{-1}u^{-1} = x,\ u^{-1}, v^{-1} \in G\} \\ &= \left(f_B^{-1} * f_A^{-1}\right)(x).\end{aligned}$$

□

Theorem 3.28. *Let f_G be a soft set. Then, f_G is a soft int-group if and only if f_G satisfies the following conditions:*

(1) $(f_G * f_G) \widetilde{\subseteq} f_G$,

(2) $f_G^{-1} = f_G$ (*or* $f_G \widetilde{\subseteq} f_G^{-1}$ *or* $f_G^{-1} \widetilde{\subseteq} f_G$).

Proof. Assume that $f_G \in S_G(U)$. Firstly, for all $x \in G$,

$$\begin{aligned}(f_G * f_G)(x) &= \bigcup\{f_G(u) \cap f_G(v) :\ uv = x,\ u, v \in G\} \\ &\subseteq \bigcup\{f_G(uv) :\ uv = x,\ u, v \in G\} \\ &= f_G(x)\end{aligned}$$

since $f_G \in S_G(U)$. So $(f_G * f_G) \widetilde{\subseteq} f_G$.

Second part is obvious by the definition of soft int-group since

$$f_G^{-1}(x) = f_G\left(x^{-1}\right) = f_G(x)$$

for all $x \in G$.

Conversely; suppose $(f_G * f_G) \widetilde{\subseteq} f_G$, then for all $x \in G$, $(f_G * f_G)(x) \subseteq f_G(x)$. So, for all $x \in G$

$$\begin{aligned} f_G(x) &\supseteq (f_G * f_G)(x) \\ &= \bigcup \{f_G(u) \cap f_G(v) : u, v \in G,\ uv = x\}. \end{aligned}$$

Hence, for any $u, v \in G$ such that $uv = x$, we have $f_G(uv) \supseteq f_G(u) \cap f_G(v)$ and by the second part of assumption, $f_G \in S_G(U)$. □

Theorem 3.29. *Let $A, B \subseteq G$ and f_A, f_B be soft int-groups. Then, $f_A * f_B$ is a soft int-group if and only if $f_A * f_B = f_B * f_A$.*

Proof. Assume that $f_A * f_B \in S_G(U)$. Then,

$$f_A * f_B = f_A^{-1} * f_B^{-1} = (f_B * f_A)^{-1} = f_B * f_A.$$

Conversely, suppose $f_A * f_B = f_B * f_A$. Then,

$$\begin{aligned} (f_A * f_B) * (f_A * f_B) &= f_A * (f_B * f_A) * f_B \\ &= f_A * (f_A * f_B) * f_B \\ &= (f_A * f_A) * (f_B * f_B) \\ &\subseteq f_A * f_B \end{aligned}$$

and

$$(f_A * f_B)^{-1} = (f_B * f_A)^{-1} = f_A^{-1} * f_B^{-1} = f_A * f_B.$$

Consequently, by Theorem 3.28, $f_A * f_B$ is a soft int-group. □

4. Conclusions

In this paper, we present soft int-groups on a soft set and give some of their supplementary properties. In addition, we give relations between α-inclusion, soft product and soft int-groups. This study affords us an opportunity to go further on soft group theory, that is, soft normal int-group, quotient group, isomorphism theorems etc.

Acknowledgements. The author is highly grateful to Dr. Naim Çağman for his valuable comments and suggestions.

References

[1] S. Abou-Zaid, On fuzzy subgroups, Fuzzy Sets and Systems 55 (1993) 237–240.
[2] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Comput. Math. Appl. 59 (20100 3458–3463.
[3] M. Akgül, Some properties of fuzzy groups, J. Math. Anal. Appl. 133 (1988) 93–100.
[4] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[5] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.

[6] J. M. Anthony and H. Sherwood, Fuzzy subgroups redefined, J. Math. Anal. Appl. 69 (1979) 124–130.
[7] M. Asaad, Groups and fuzzy subgroups, Fuzzy Sets and Systems 39 (1991) 323–328.
[8] K. R. Bhutani, Fuzzy sets, fuzzy relations and fuzzy groups: Some interrelations, Inform. Sci. 73 (1993) 107–115.
[9] N. Çağman, F. Çıtak and H. Aktaş, Soft int-group and its applications to group theory, Neural Comput. Appl. DOI:10.1007/s00521-011-0752-x.
[10] N. Çağman and S. Enginoğlu, Soft set theory and uni-int decision making, European J. Oper. Res. 207 (2010) 848–855.
[11] P. S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1981) 264–269.
[12] V. N. Dixit, R. Kumar and N. Ajamal, Level subgroups and union of fuzzy subgroups, Fuzzy Sets and Systems 37 (1990) 359–371.
[13] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[14] Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.
[15] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.
[16] Y. B. Jun, K. J. Lee and A. Khan, Soft ordered semigroups, MLQ Math. Log. Q. 56(1) (2010) 42–50.
[17] J. G. Kim, Fuzzy orders relative to fuzzy subgroups, Inform. Sci. 80 (1994) 341–348.
[18] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[19] D. A. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.
[20] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, Fuzzy group theory, Springer, 2005.
[21] N. P. Mujherjee and P. Bhattacharya, Fuzzy groups, some group theoretic analogs, Inform. Sci. 39 (1986) 247–268.
[22] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[23] A. Sezgin and A. O. Atagün , On operations of soft sets, Comput. Math. Appl. 61(5) (2011) 1457–1467.
[24] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Kenan Kaygisiz (kenan.kaygisiz@gop.edu.tr)
Gaziosmanpaşa University, Faculty of Arts and Sciences, Department of Mathematics, Tokat, Turkey.