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Inverse system of fuzzy soft modules

Sadi Bayramov, Ciğdem Gunduz (Aras), Murat Ibrahim Yazar

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Abstract. Molodtsov initiated the concept of soft sets in [19]. Maji et al. defined some operations on soft sets in [16]. Aktaş et al. generalized soft sets by defining the concept of soft groups in [2]. After then, Qiu-Mei Sun et al. gave soft modules in [23]. Gunduz and Bayramov [8] introduce fuzzy soft module and investigate some of fuzzy soft module basic properties. In this paper, we introduce the concept of inverse system in the category of fuzzy soft modules and give some operations on fuzzy soft modules. Finally, we investigate whether or not the limit of inverse system of fuzzy soft modules.

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Corresponding Author: Murat Ibrahim Yazar (miy248@yahoo.com)

1. Introduction

Many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. Probability theory, fuzzy sets, rough sets, and other mathematical tools have their inherent difficulties ([14, 20, 21]). The reason for these difficulties may be due to the inadequacy of the theories of parameterization tools.

Molodtsov [19] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Later, work on the soft set theory is progressing rapidly. Maji et al. [16, 22] have published a detailed theoretical study on soft sets. After Molodtsov's work, some different applications of soft sets were studied in [16]. Aktaş and Çağman [2] introduced the basic properties of soft sets, and compared soft sets with the related concepts of fuzzy sets and rough sets. At the same time, they gave a definition of soft groups, soft rings and derived their basic properties ([1], [6], [10], [23]). Qiu- Mei Sun et al. [23] defined soft modules and investigated their basic properties. Gunduz and Bayramov [8] introduce fuzzy soft module and investigate some of fuzzy soft module basic properties. As above mentioned, the algebraic structure of set theories dealing with uncertainties has also been studied by some authors.

It is known that the inverse limit is not only an important concept in the category theory, but also plays an important role in topology, algebra, homology theory etc. ([4, 5, 17]). To the date, inverse system and its limit was defined in the different categories. Furthermore, some of its pr operties was investigated.

Sheng-Gang Li [11, 12] defined inverse (direct) system of fuzzy topological spaces and their limits and obtained their properties for the case of category $L-Top$. M. Ghadiri and B.Davvaz [7] introduced direct system and direct limit of $H_\nu-$modules. V. Leoreanu [13] introduced direct limits and inverse limits of SHR semigroups. Gunduz and Bayramov [9] defined inverse (direct) system of fuzzy modules and their limits and obtained their properties.

This paper begins with the basic concepts of fuzzy soft module. Later, we introduce inverse system in the category of fuzzy soft modules and prove that its limit exists in this category. Generally, limit of inverse system of exact sequences of fuzzy soft modules is not exact ([18]). Then we define the notion $\underleftarrow{lim}^{(1)}$ which is first derived functor of the inverse limit functor. Finally, using methods of homology algebra, we prove that the inverse system limit of exact sequence of fuzzy soft modules is exact.

2. Preminilaries

In this section, we recall necessary information commonly used in fuzzy soft module.

Definition 2.1 ([16]). Let X be an initial universe set and E be a set of parameters. A pair (F,E) is called a soft set over X if only if F is a mapping from E into the set of all subsets of the set X, i.e., $F:E\rightarrow P(X)$, where $P(X)$ is the power set of X.

In other words, the soft set is a parameterized family of subsets of the set X. Every set $F(e)$, for every $e\in E$, may be considered as the set of $e-$elements of the soft set (F,E), or as the set of $e-$approximate elements of the soft set.

According to this manner, a soft set (F,E) is given as consisting of collection of approximations:

$$(F,E)=\{F(e):e\in E\}.$$

Definition 2.2 ([15]). Let I^X denote the set of all fuzzy sets on X and $A\subset E$. A pair (f,A) is called a fuzzy soft set over X, where f is a mapping from A into I^X. That is, for each $a\in A$, $f(a)=f_a:X\rightarrow I$, is a fuzzy set on X.

Definition 2.3 ([15]). Union of two fuzzy soft sets (f,A) and (g,B) over a common universe X is the fuzzy soft set (h,C), where $C=A\cup B$ and

$$h(c)=\begin{cases} f_c\,, & \text{if } c\in A-B \\ g_c\,, & \text{if c} \in \text{B - A} \\ f_c\vee g_c, & \text{if c} \in \text{A} \cap \text{B} \end{cases}\quad,\quad \forall c\in C.$$

It is denoted as $(f, A) \cup (g, B) = (h, C)$.

Definition 2.4 ([15])**.** Intersection of two fuzzy soft sets (f, A) and (g, B) over a common universe X is the fuzzy soft set (h, C), where $C = A \cap B$ and $h_c = f_c \wedge g_c$, $\forall c \in C$.

It is written as $(f, A) \cap (g, B) = (h, C)$.

Definition 2.5 ([15])**.** If (f, A) and (g, B) are two soft sets, then (f, A) and (g, B) is denoted as $(f, A) \wedge (g, B)$. $(f, A) \wedge (g, B)$ is defined as $(h, A \times B)$ where $h\left(a, b\right) = h_{a,b} = f_a \wedge g_b$, $\forall \left(a, b\right) \in A \times B$.

Now, let M be a left $R-$module, A be any nonempty set. $F : A \to P\left(M\right)$ refer to a set-valued function and the pair (F, A) is a soft set over M.

Definition 2.6 ([23])**.** Let (F, A) be a soft set over M. (F, A) is said to be a soft module over M if and only if $F\left(x\right) < M$ for all $x \in A$.

Definition 2.7 ([23])**.** Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B) = (H, A \times B)$ is defined as $H\left(x, y\right) = F\left(x\right) \times G\left(y\right)$ for all $(x, y) \in A \times B$.

Proposition 2.8 ([23])**.** *Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B)$ is soft module over $M \times N$.*

Definition 2.9 ([23])**.** Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \to N$, $g : A \to B$ be two functions. Then we say that (f, g) is a soft homomorphism if the following conditions are satisfied:
(1) *f is a homomorphism from M onto N,*
(2) *g is a mapping from A onto B, and*
(3) *$f\left(F\left(x\right)\right) = G\left(g\left(x\right)\right)$ for all $x \in A$.*

Definition 2.10 ([8])**.** Let (F, A) be a fuzzy soft set over M. Then (F, A) is said to be a fuzzy soft module over M iff for each $a \in A$, $F\left(a\right)$ is a fuzzy submodule of M and denoted as F_a.

Definition 2.11 ([8])**.** Let (F, A) and (H, B) be two fuzzy soft modules over M and N respectively, and let $f : M \to N$ be a homomorphism of modules, and let $g : A \to B$ be a mapping of sets. Then we say that $(f, g) : (F, A) \to (H, B)$ is a fuzzy soft homomorphism of fuzzy soft modules, if the following condition is satisfied:

$$f\left(F_a\right) = H\left(g\left(a\right)\right) = H_{g(a)}.$$

Proposition 2.12 ([8])**.** *Let (F, A) be a fuzzy soft module over M and N be a module, $f : M \to N$ be a homomorphism of modules. Then $\left(f\left(F\right), A\right)$ is a fuzzy soft module over N, where $f\left(F\right) : A \to PF\left(N\right)$ is defined by $f\left(F\right)_a\left(y\right) = \underset{f(x)=y}{\vee} F_a\left(x\right)$. $(f, 1_A) : (F, A) \to \left(f\left(F\right), A\right)$ is a fuzzy soft homomorphism of fuzzy soft modules.*

Proposition 2.13 ([8])**.** *Let (G, B) be a fuzzy soft module over N and $f : M \to N$ be a homomorphism of modules. Then $\left(f^{-1}\left(G\right), B\right)$ is a fuzzy soft module over M, where $f^{-1}\left(G\right) : B \to PF\left(M\right)$ is defined by $f^{-1}\left(G\right)_b\left(x\right) = G_b\left(f\left(x\right)\right)$. $(f, 1_B) : \left(f^{-1}\left(G\right), B\right) \to (G, B)$ is a fuzzy soft homomorphism of fuzzy soft modules.*

Proposition 2.14 ([8]). *If (F, A) is a fuzzy soft module over M and $N \subset M$ is a submodule of M and $i : N \to M$ is an embedding homomorphism, then $\left(i^{-1}(F), A\right)$ is a fuzzy soft module over N.*

Proposition 2.15 ([8]). *If (F, A) is a fuzzy soft module over M and $M/\sim$ is a quotient module and $p : M \to M/\sim$ is a canonical homomorphism, then $(p(F), A)$ is a fuzzy soft module over $M/\sim$.*

Theorem 2.16 ([8]). *If $\{(F_i, A_i)\}_{i\in I}$ is a family of fuzzy soft modules over $\{M_i\}_{i\in I}$, then $\prod\limits_{i\in I}(F_i, A_i)$ is a fuzzy soft module over $\prod\limits_{i\in I} M_i$.*

Theorem 2.17 ([8]). *If $\{(F_i, A_i)\}_{i\in I}$ is a family of fuzzy soft modules over $\{M_i\}_{i\in I}$, then $\underset{i\in I}{\oplus}(F_i, A_i)$ is a fuzzy soft module over $\underset{i\in I}{\oplus} M_i$.*

3. Inverse System of Fuzzy Soft Modules

This category of fuzzy soft modules denoted as FSM.

Definition 3.1. Any functor $D : \wedge^{op} \to FSM$, where $\wedge$ is a directed set, is called an inverse system of fuzzy soft modules.

Now we consider the following any inverse system

$$\left(\{(F_\alpha, A_\alpha)\}_{\alpha\in\wedge}, \left\{(p_\alpha^{\alpha'}, q_\alpha^{\alpha'}) : \left(F'_{\alpha'}, A'_{\alpha'}\right) \to (F_\alpha, A_\alpha)\right\}_{\alpha\prec\alpha'}\right) \tag{3.1}$$

It is clear that parameter sets in (3.1) consist of the following inverse system of sets

$$\left(\{A_\alpha\}_{\alpha\in\wedge}, \{q_\alpha^{\alpha'} : A'_\alpha \to A_\alpha\}_{\alpha\prec\alpha'}\right). \tag{3.2}$$

Similarly, $\{M_\alpha\}_{\alpha\in\wedge}$ in (3.1) consist of the following inverse system of modules

$$\left(\{M_\alpha\}_{\alpha\in\wedge}, \left\{p_\alpha^{\alpha'} : M'_\alpha \to M_\alpha\right\}_{\alpha\prec\alpha'}\right). \tag{3.3}$$

Let $A = \underset{\overleftarrow{\alpha}}{\lim} A_\alpha$ be inverse limit of (3.2) and $M = \underset{\overleftarrow{\alpha}}{\lim} M_\alpha$ be inverse limit of (3.3). Since $p_\alpha^{\alpha'}(a_{\alpha'}) = a_\alpha$ for all $a = \{a_\alpha\} \in A$,

$$\left(\{(M_\alpha, (F_\alpha)_{a_\alpha})\}_{\alpha\in\wedge}, \left\{p_\alpha^{\alpha'} : \left(M_{\alpha'}, (F_{\alpha'})_{a_{\alpha'}}\right) \to \left(M_\alpha, (F_\alpha)_{a_\alpha}\right)\right\}_{\alpha\prec\alpha'}\right) \tag{3.4}$$

is an inverse system of fuzzy modules [9].

We denote inverse limit of (3.4) as (M, F_a). We define $F : A \to PF(M)$ as $F(a) = F_a$. Then (F, A) is a fuzzy soft module over M.

If $\pi_\alpha : \underset{\leftarrow}{\lim} M_\alpha \to M_\alpha$ and $q_\alpha : \underset{\leftarrow}{\lim} A_\alpha \to A_\alpha$ are projection mappings, then $(\pi_\alpha, q_\alpha) : (F, A) \to (F_\alpha, A_\alpha)$ is a homomorphism of fuzzy soft modules, and, for $\alpha \prec \alpha'$, the following diagram is commutative:

$$\begin{array}{ccc} (F, A) & \overset{(\pi_\alpha, q_\alpha)}{\longrightarrow} & (F_\alpha, A_\alpha) \\ {\scriptstyle (\pi_{\acute{\alpha}}, q_{\acute{\alpha}})}\downarrow & \nearrow{\scriptstyle p_\alpha^{\alpha'}} & \\ (F'_\alpha, A'_\alpha) & & \end{array}$$

Theorem 3.2. *Every inverse system of fuzzy soft modules has limit. This limit is unique and this limit is equal to* (F, A).

Proof. We get inverse system (3.1). Let (G, B) be a fuzzy soft module over N. For $\{(h_\alpha, \varphi_\alpha) : (G, B) \to (F_\alpha, A_\alpha)\}_{\alpha \in \wedge}$ be a family of fuzzy soft homomorphisms of fuzzy soft modules. Then for this family and, for all $\alpha \prec \alpha'$, let $(p_\alpha^{\alpha'}, q_\alpha^{\alpha'})\,(h_{\alpha'}, \varphi_{\alpha'}) = (h_\alpha, \varphi_\alpha)$. Now we define fuzzy soft homomorphism $(\psi, \gamma) : (G, B) \to (F, A)$, where $\gamma : B \to A = \varprojlim_\alpha A_\alpha$, $\gamma(b) = \{\varphi_\alpha(b)\}$ and $\psi : N \to M = \varprojlim_\alpha M_\alpha$, $\psi(x) = \{h_\alpha(x)\}$. Then $(\psi, \gamma) : (G, B) \to (F, A)$ is a fuzzy soft homomorphism of fuzzy soft modules. It is clear that for all $\alpha \in \wedge$, the following diagram is commutative:

$$\begin{tikzcd}
(G,B) \arrow[r, "(h_\alpha,\varphi_\alpha)"] \arrow[d, "(\psi,\gamma)"'] & (F_\alpha, A_\alpha) \\
(F,A) \arrow[ur, "(\pi_\alpha,q_\alpha)'"'] &
\end{tikzcd}$$

The proof is completed. $\square$

Now we consider the following inverse system of fuzzy soft modules over $\{N_\beta\}_{\beta \in \wedge'}$

$$(3.5) \quad (\underline{G}, \underline{B}) = \left(\{(G_\beta, B_\beta)\}_{\beta \in \wedge'}, \left\{ \left(r_\beta^{\beta'}, \chi_\beta^{\beta'}\right) : \left(G_{\beta'}, B_{\beta'}\right) \to \left(G_\beta, B_\beta\right) \right\}_{\beta \prec \beta'} \right).$$

Let $\varphi : \wedge' \to \wedge$ be an isotone mapping and the following mapping

$$(f_\beta, g_\beta) : \left(F_{\varphi(\beta)}, A_{\varphi(\beta)}\right) \to (G_\beta, B_\beta)$$

be a fuzzy soft homomorphism of fuzzy soft modules, for all $\beta \in \wedge'$.

Definition 3.3. If for all $\beta \prec \beta'$, the condition

$$\left(r_\beta^{\beta'}, \chi_\beta^{\beta'}\right) \circ \left(f'_\beta, g'_\beta\right) = (f_\beta, g_\beta) \circ \left(p_{\varphi(\beta)}^{\varphi(\beta')}, q_{\varphi(\beta)}^{\varphi(\beta')}\right)$$

is satisfild, then the family $\left(\varphi, \{(f_\beta, g_\beta)\}_{\beta \in \wedge'}\right)$ is said to by morphism of inverse systems.

It is clear that inverse systems of fuzzy soft modules and morphisms of their consist of a category. This category is denoted as $Inv\,(FSM)$.

Let $\left(\varphi, \{(f_\beta, g_\beta)\}_{\beta \in \wedge'}\right) : (\underline{F}, \underline{A}) \to (\underline{G}, \underline{B})$ be a morphism of inverse systems of fuzzy soft modules. Here $\underline{B} = \left(\{B_\beta\}_{\beta \in \wedge'}, \left\{\chi_\beta^{\beta'}\right\}_{\beta \prec \beta'}\right)$ is an inverse system of sets and $\left(\varphi, \{g_\beta\}_{\beta \in \wedge'}\right) : \underline{A} \to \underline{B}$ is a morphism of inverse systems of sets. Then the mapping $g = \varprojlim \left(\varphi, \{g_\beta\}_{\beta \in \wedge'}\right) : \varprojlim_\alpha A_\alpha = A \to \varprojlim_\beta B_\beta = B$ is a mapping of limit sets of this inverse systems. Similarly,

$$\left(\varphi, \{f_\beta\}_{\beta \in \wedge'}\right) : \{M_\alpha\}_{\alpha \in \wedge} \to \{N_\beta\}_{\beta \in \wedge'}$$

is a morphism of inverse systems of modules.

Proposition 3.4. *Let* $\lim\limits_{\leftarrow}\left(\varphi, \{f_\beta\}_{\beta\in\wedge'}\right) = f$. *Then*

$$(f,g) : \lim_{\overleftarrow{\alpha}} (F_\alpha, A_\alpha) \to \lim_{\overleftarrow{\beta}} (G_\beta, B_\beta)$$

is a morphism of limits of inverse systems of fuzzy soft modules.

Proof. Since the product operation of fuzzy soft modules is a functor, the following diagram is commutative:

$$\begin{array}{ccc} \prod\limits_\beta A_{\varphi(\beta)} & \xrightarrow{\quad\wedge F_\beta\quad} & \prod\limits_\beta M_{\varphi(\beta)} \\ \Pi g_\beta \downarrow & & \downarrow \Pi f_\beta \\ \prod\limits_\beta B_\beta & \xrightarrow{\quad\wedge G_\beta\quad} & \prod\limits_\beta N_\beta \end{array}$$

For all $\left\{a_{\varphi(\beta)}\right\} \in \prod\limits_\beta A_{\varphi(\beta)}$,

$$\left(\varphi, \{f_\beta\}_{\beta\in\wedge'}\right) : \left\{\left(M_{\varphi(\beta)}, F_{a_{\varphi(\beta)}}\right)\right\} \to \left\{\left(N_\beta, G_{g_\beta\left(a_{\varphi(\beta)}\right)}\right)\right\}_{\beta\in\wedge'}$$

is a morphism of inverse systems of fuzzy modules. Then

$$\lim_{\leftarrow}\left(\varphi, \{f_\beta\}_{\beta\in\wedge'}\right) : \lim_{\leftarrow}\left\{\left(M_{\varphi(\beta)}, F_{a_{\varphi(\beta)}}\right)\right\} \to \lim_{\leftarrow}\left\{\left(N_\beta, G_{g_\beta\left(a_{\varphi(\beta)}\right)}\right)\right\}_{\beta\in\wedge'}$$

is a fuzzy homomorphism of fuzzy modules and the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\qquad F \qquad} & \lim\limits_{\overleftarrow{\beta}} M_{\varphi(\beta)} \\ g \downarrow & & \downarrow f \\ B & \xrightarrow{\qquad G \qquad} & \lim\limits_{\overleftarrow{\beta}} N_\beta \end{array}$$

Thus the diagram is proved. □

Theorem 3.5. *The corresponding*

$$\{(F_\alpha, A_\alpha)\}_{\alpha\in\wedge} \mapsto \lim_{\overleftarrow{\alpha}} (F_\alpha, A_\alpha)$$

is a covariant functor from the category $Inv\,(FSM)$ *to the category of* FSM.

4. Operation on fuzzy soft modules

Let (F_1, A_1), (F_2, A_2), (G_1, B_1) and (G_2, B_2) be the four fuzzy soft modules over M, $(f,g) : (F_1, A_1) \to (F_2, A_2)$ and $(h,p) : (G_1, B_1) \to (G_2, B_2)$ be two fuzzy soft homomorphism of fuzzy soft modules and $A_1 \cap B_1 = A_2 \cap B_2 = \varnothing$. Then we get the fuzzy soft modules $(F_1, A_1)\,\widetilde{\cup}\,(G_1, B_1) = (H_1, C_1)$ and $(F_2, A_2)\,\widetilde{\cup}\,(G_2, B_2) = (H_2, C_2)$. Here $C_1 = A_1 \cup B_1$, $C_2 = A_2 \cup B_2$. Hence we define the mapping $\varphi : C_1 \to C_2$ such that

$$\varphi(c) = \begin{cases} g(c), & c \in A_1 \\ p(c), & c \in B_1 \end{cases}.$$

Also, we give the homomorphism $\psi : M \to N$ such that

$$\psi(c) = \begin{cases} f(c), & c \in A_1 \\ h(c), & c \in B_1 \end{cases}.$$

Then $(\psi,\varphi):(H_1,C_1)\to(H_2,C_2)$ is a fuzzy soft homomorphism of fuzzy soft modules.

Proposition 4.1. *The operation* $\widetilde{\cup}: FSM\times FSM\to FSM$ *is a covariant functor. Now, let* $(F,A)=\left(\{(F_\alpha,A_\alpha)\}_{\alpha\in\wedge},\{(p_\alpha^{\alpha'},q_\alpha^{\alpha'})\}_{\alpha\prec\alpha'}\right)$ *and*
$(G,B)=\left(\{(G_\alpha,B_\alpha)\}_{\alpha\in\wedge},\{(r_\alpha^{\alpha'},t_\alpha^{\alpha'})\}_{\alpha\prec\alpha'}\right)$ *be two inverse systems of fuzzy soft modules over* $\{M_\alpha\}_{\alpha\in\wedge}$ *and* $\{(f_\alpha,g_\alpha)\}:(F,A)\to(G,B)$ *be a morphism of the systems and* $A_\alpha\cap B_\alpha=\emptyset$*, for all* $\alpha\in\wedge$*. From the proposition 4.1,*

$$\left(\left\{(F_\alpha,A_\alpha)\,\widetilde{\cup}\,(G_\alpha,B_\alpha)\right\}_{\alpha\in\wedge},\left\{(p_\alpha^{\alpha'},q_\alpha^{\alpha'})\,\widetilde{\cup}\,(r_\alpha^{\alpha'},t_\alpha^{\alpha'})\right\}_{\alpha\prec\alpha'}\right) \tag{4.1}$$

is an inverse system of fuzzy soft modules.

Theorem 4.2. $\varprojlim_\alpha\left[(F_\alpha,A_\alpha)\,\widetilde{\cup}\,(G_\alpha,B_\alpha)\right]=\left[\varprojlim_\alpha(F_\alpha,A_\alpha)\right]\widetilde{\cup}\left[\varprojlim_\alpha(G_\alpha,B_\alpha)\right]$.

Proof. The parameter set of the fuzzy soft module $\varprojlim_\alpha\left[(F_\alpha,A_\alpha)\,\widetilde{\cup}\,(G_\alpha,B_\alpha)\right]$ is $\varprojlim_\alpha(A_\alpha\cup B_\alpha)$. Since $A_\alpha\cap B_\alpha=\emptyset$, $\varprojlim_\alpha(A_\alpha\cup B_\alpha)=\varprojlim_\alpha A_\alpha\cup\varprojlim_\alpha B_\alpha$. Also since $\left(\prod_\alpha A_\alpha\right)\cap\left(\prod_\alpha B_\alpha\right)=\emptyset$, $\varprojlim_\alpha A_\alpha\cap\varprojlim_\alpha B_\alpha=\emptyset$. Then for all $c=\{c_\alpha\}\in\left(\varprojlim_\alpha A_\alpha\right)\cup\left(\varprojlim_\alpha B_\alpha\right)$, $c\in\varprojlim_\alpha A_\alpha$ or $c\in\varprojlim_\alpha B_\alpha$. Suppose that $c=\{c_\alpha\}\in\varprojlim_\alpha A_\alpha$. Then from (4.1), the following inverse system of fuzzy modules

$$\left(\left\{\left(M_\alpha,(F_\alpha)_{c_\alpha}\right)\right\},\{p_\alpha^{\alpha'}\}_{\alpha\prec\alpha'}\right) \tag{4.2}$$

is obtained. Thus,

$$\left(\varprojlim_\alpha\left[(F_\alpha,A_\alpha)\,\widetilde{\cup}\,(G_\alpha,B_\alpha)\right]\right)(\{c_\alpha\})=\varprojlim_\alpha\left(M_\alpha,(F_\alpha)_{c_\alpha}\right) \tag{4.3}$$

Also, the parameter set of the fuzzy soft module $\left[\varprojlim_\alpha(F_\alpha,A_\alpha)\right]\widetilde{\cup}\left[\varprojlim_\alpha(G_\alpha,B_\alpha)\right]$ is $\varprojlim_\alpha A_\alpha\cup\varprojlim_\alpha B_\alpha$. For all $c=\{c_\alpha\}\in\varprojlim_\alpha A_\alpha$,

$$\left[\varprojlim_\alpha(F_\alpha,A_\alpha)\right]\widetilde{\cup}\left[\varprojlim_\alpha(G_\alpha,B_\alpha)\right](\{c_\alpha\})=\varprojlim_\alpha\left(M_\alpha,(F_\alpha)_{c_\alpha}\right) \tag{4.4}$$

From (4.3) and (4.4) the proof is completed. □

Theorem 4.3. *If* $\{(F,A)\}_{j\in J}$ *is a family of inverse systems of fuzzy soft modules, then*

$$\varprojlim\prod_j(F,A)_j=\prod_j\varprojlim(F,A)_j\,.$$

Proof. The proof of the theorem is straightforward. □

5. Derivative functor of $\varprojlim$ functor

Let us review the problem of exact limit for inverse system of exact sequence of fuzzy soft modules.

Example 5.1. Let $M_n = \mathbb{Z}$, $M'_n = \mathbb{Z}$, $M''_n = \mathbb{Z}_2$ be modules an $\mathbb{Z}$ ring $\forall n \in \mathbb{N}$. Then,

$$\underline{M} = \left(\{M_n\}_{n\in\mathbb{N}}, \left\{p_n^{n+1}(m) = 3m\right\}\right)$$
$$\underline{M}' = \left(\{M'_n\}_{n\in\mathbb{N}}, \left\{q_n^{n+1}(m) = 3m\right\}\right)$$
$$\underline{M}'' = \left(\{M''_n\}_{n\in\mathbb{N}}, \left\{r_n^{n+1}([m]) = [m]\right\}\right)$$

are inverse systems of modules and

$$f = \{f_n : M'_n \to M_n \; f_n(m) = 2m\}$$
$$g = \{g_n : M_n \to M''_n \; g_n(m) = [m]\}$$

are morphisms of inverse systems. The following sequence

$$0 \longrightarrow \underline{M'} \xrightarrow{f} \underline{M} \xrightarrow{g} \underline{M^n} \longrightarrow 0$$

is short exact sequence of inverse systems of $\mathbb{Z}$-modules.

Let A be a parameter set

$$F'_n : A \to FSM(M'_n), \; F_n : A \to FSM(M_n), \; F''_n : A \to FSM(M''_n)$$

fuzzy soft modules defined by the formula

$$\forall a \in A, \; F'_{na} = (\chi(0))_{M'_n}, \; F_{na} = (\chi(0))_{M_n}, \; F''_{na} = (\chi(0))_{M''_n}$$

The sequence

$$0 \longrightarrow (M'_n, F'_{na}) \xrightarrow{\overline{f_n}} (M_n, F_{na}) \xrightarrow{\overline{g_n}} (M''_n, F''_{na}) \longrightarrow 0$$

is also short exact sequence of fuzzy modules for each $a \in A$. Then the sequence

$$0 \to (F', A) \to (F, A) \to (F'', A) \to 0$$

is short exact sequence of inverse systems of fuzzy soft modules. Taking the limits of this sequence is not exact.

Example 5.2. As it seen the limit of inverse system of exact sequence of fuzzy soft modules is not exact. So it is necessary to define derivative functor of inverse limit functor in category of fuzzy soft modules.

We get inverse system in (3.1). We define the following homomorphism of modules

$$d : \prod_\alpha M_\alpha \to \prod_\alpha M_\alpha$$

by the formula:

$$d(\{x_\alpha\}) = \{x_\alpha - p_\alpha^{\alpha'}(x'_\alpha)\}_{\alpha \prec \alpha'}.$$

We demonstrate that $\forall a \in A \quad d$ is a homomorphism of fuzzy modules. Indeed,

$$F_{\mathrm{A}a}(d(\{x_\alpha\})) = F_{\mathrm{A}a}(\{x_\alpha - p_\alpha^{\alpha'}(x_{\alpha'})\}) = \bigwedge_\alpha F_{\alpha a}(x_\alpha - p_\alpha^{\alpha'}(x_{\alpha'}))$$
$$\geq \bigwedge_\alpha \min\{F_{\alpha a}(x_\alpha), F_{\alpha a}(p_\alpha^{\alpha'}(x_{\alpha'}))\}.$$

Since $F_{\alpha a}\left(p_{\alpha}^{\alpha'}\left(x_{\alpha'}\right)\right) \geq F_{\alpha' a}\left(x_{\alpha'}\right)$,

$$\begin{aligned} F_{\mathrm{A}a}\left(d\left(\{x_{\alpha}\}\right)\right) &\geq \bigwedge_{\alpha} \min\left\{F_{\alpha a}\left(x_{\alpha}\right), F_{\alpha' a}\left(x_{\alpha'}\right)\right\} \\ &= \bigwedge_{\alpha}\left(F_{\alpha a}\left(x_{\alpha}\right) \wedge F_{\alpha'' a}\left(x_{\alpha'}\right)\right) \\ &= \bigwedge_{\alpha} F_{\alpha a}\left(x_{\alpha}\right) = F_{\mathrm{A}a}\left(\{x_{\alpha}\}\right). \end{aligned}$$

Then $\overline{d}$ is a homomorphism of fuzzy modules. Therefore $\left(\ker d, F_{\mathrm{A}a}\left|\ker d\right.\right)$ and $\left(co\ker d, \left(F_{\mathrm{A}a}\right)_p\right)$ are defined.

For inverse system of modules $\left(\{M_{\alpha}\}_{\alpha\in\wedge}, \{p_{\alpha}^{\alpha'}\}_{\alpha\prec\alpha'}\right)$, $\underset{\leftarrow}{\lim}^{(1)} M_{\alpha} = \prod_{\alpha} M_{\alpha} \Big/ Imd$ is derivative functor.

If $\pi = \prod_{\alpha} M_{\alpha} \to \underset{\leftarrow}{\lim}^{(1)} M_{\alpha}$ is the canonical homomorphism, we can define fuzzy modules by $\left(\underset{\leftarrow}{\lim}^{(1)} M_{\alpha}, \left(F_A\right)_a^{\pi}\right)$. Then $\left(F_A\right)^{\pi} : A \to \prod_{\alpha} M_{\alpha}$ is fuzzy soft module.

Definition 5.3. $\left(\left(F_A\right)^{\pi}, A\right)$ is called "first derived functor" of the inverse system of fuzzy soft modules given (3.1).

Proposition 5.4. $\underset{\leftarrow}{\lim}^{(1)}$ *is a functor.*

Proof. For this reason, it suffices to show that for each the morphism
$\underline{\overline{f}} = \left(\rho : B \to A, \left\{\left(\overline{f}_{\beta}, g_{\beta}\right) : \left(F_{\rho(\beta)}, A_{\rho(\beta)}\right) \to \left(G_{\beta}, B_{\beta}\right)\right\}_{\beta\in B}\right)$,

$\underset{\leftarrow}{\lim}^{(1)} \underline{\overline{f}} : \left(\left(F_A\right)^{\pi}, A\right) \to \left(\left(G_A\right)^{\pi}, B\right)$ is the homomorphism of fuzzy soft modules.

Since

$$\begin{aligned} \left(F_{\mathrm{A}}\right)^{\pi}\left(x + imd\right) &= \sup_{z\in Imd} {}_{\mathrm{A}}\left(x+z\right) \leq \sup_{z\in Imd} G_B\left(f\left(x+z\right)\right) \\ &= \sup_{z\in Imd} G_B\left(f\left(x\right) + f\left(z\right)\right) \\ &= \sup_{y=f(z)} G_B\left(f\left(x\right)+y\right) \leq \sup_{y\in Imd} G_B\left(f\left(x\right)+y\right) \\ &= \left(G_B\right)^{\pi}\left(\underset{\leftarrow}{\lim}^{(1)} \underline{f}\left(x + Imd\right)\right), \end{aligned}$$

$\underset{\leftarrow}{\lim}^{(1)}$ is a functor. □

We investigate another properties of $\underset{\leftarrow}{\lim}^{(1)}$ functor, let us introduce the category of chain complexes of fuzzy soft modules ([3]).

Let $\{(F_n, A)\}_{n\in Z}$ be fuzzy soft modules over $\{M_n\}_{n\in Z}$ and let for $\forall n \in Z$,

$$\left(\partial_n, 1_A\right) : \left(F_n, A\right) \to \left(F_{n-1}, A\right)$$

be homomorphism of fuzzy soft modules.

Definition 5.5. If for all $a \in A \quad \{(M_n, F_{na}), \partial_n : (M_n, F_{na}) \to (M_{n-1}, F_{n-1a})\}$ is chain complex of fuzzy modules, then the following sequence is said to be a chain complex of fuzzy soft modules

$$(F, A) = \{(F_n, A), (\partial_n, 1_A) : (F_n, A) \to (F_{n-1}, A)\}.$$

Let $(F, A) = \{(F_n, A), (\partial_n, 1_A)\}$ be a chain complex of fuzzy soft modules. Then for each we obtain the fuzzy homology module

$$H_n(F, \alpha) = \ker \partial_n \setminus \mathrm{Im}\partial_{n+1}$$

for the fuzzy chain complex

$$\{(M_n, F_{na}), \partial_n : (M_n, F_{na}) \longrightarrow (M_{n-1}, F_{n-1a})\}.$$

Thus, for all $a \in A$ the fuzzy module $H_n(F, a)$ is a quotient module in (M_n, F_{na}). If there exist an one to one and covered connection with every fuzzy submodule of fuzzy quotient module of (M_n, F_{na}) and fuzzy submodule of M_n we can think the fuzzy module $H_n(F, a)$ as a fuzzy submodule of (M_n, F_{na}). Thus,

$$H_n(F, -) : A \to FSM(M_n)$$

is a fuzzy soft module.

Definition 5.6. Fuzzy soft module $(H_n(F, -), A)$is said to be $n-$dimensional fuzzy soft homology module of chain complex of fuzzy soft modules

$$(F, A) = \{(F_n, A), (\partial_n, 1_A)\}$$

Definition 5.7. Let $\{(F_n, A), (\partial_n, 1_A)\}$ and $\{(G_n, B), (\partial'_n, 1_B)\}$ be chain complexes of fuzzy soft modules over $\{M_n\}_{n\in Z}$ and $\{N_n\}_{n\in Z}$, respectively and let $\{f_n : M_n \to N_n\}$ is homomorphism of modules, $g : A \to B$ is a mapping of sets. If for all $a \in A$, $f_n : (M_n, F_{na}) \to (N_n, G_{ng(a)})$ is a fuzzy homomorphism of fuzzy modules and the condition $\partial'_n \circ f_n = f'_{n-1} \circ \partial_n$ is satisfied, then

$$(\{f_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial\prime_n\}$$

is said to be morphism of chain complexes of fuzzy soft modules.

Definition 5.8. Let $(\{\varphi_n\}, g), (\{\psi_n\}, g) : \{(F_n, A), \partial_n\} \to \{(G_n, B), \partial\prime_n\}$ be morphisms of chain complexes of fuzzy soft modules and let

$$D = (\{D_n\}, g) : \{(F_n, A), \partial_n\} \to \left\{(G_{n+1}, B), \partial'{}_{n+1}\right\}$$

be a family of homomorphisms of fuzzy soft modules. If the equation $\varphi_n - \psi_n = D_{n-1}\partial_n + \partial\prime_{n+1}D_n$ is satisfied then the family of homomorphisms of fuzzy soft modules $D = (\{D_n\}, g)$ is said to be chain homotopy morphism $(\{\varphi_n\}, g), (\{\psi_n\}, g)$is said to be chain homotopy mappings and denoted by $(\{\varphi_n\}, g) \sim (\{\psi_n\}, g)$.

The following theorem can be easily proved.

Theorem 5.9. *The chain homotopy relation is an equivalence relation and homology (cohomology) modules are invariant with respect to this relation. Let*

$$\left(\{(F_\alpha, A)\}_{\alpha\in\wedge}, \left\{\left(p_\alpha^{\alpha'}, 1_A\right) : (F_\alpha\prime, A) \to (F_\alpha, A)\right\}_{\alpha\prec\alpha\prime}\right)$$

be a chain complex of fuzzy soft modules.

Let us consider the following cochain complex of fuzzy soft modules

$$\overline{0} \to \left(\prod F_\alpha, A\right) \xrightarrow{\overline{d}} \left(\prod F_\alpha, A\right) \to \overline{0}.$$

Cohomology modules of this complex are $\ker \overline{d}$*and* $co\ker \overline{d}$*.*

Lemma 5.10. $\lim\limits_{\leftarrow} (F_\alpha, A) = \ker \overline{d}$ *and* $\lim\limits_{\leftarrow}^{(1)} (F_\alpha, A) = co\ker \overline{d}$*.*

Proof. The proof of lemma is trivial. □

We accept natural numbers set which is index set of inverse system.

Theorem 5.11. *Let the sequence*

$$(F_1, A) \xleftarrow{p_1^2} (F_2, A) \xleftarrow{p_2^3} \cdots$$

be inverse sequence of fuzzy soft modules. For each infinite subsequence of this sequence, $\lim\limits_{\leftarrow}^{(1)}$ *dose not change.*

Proof. Let $S = \{i, j, k, ...\}$be infinite subsequence of natural numbers $\mathbb{N}$. From Lemma 1, $\lim\limits_{\leftarrow}^{(1)}$ is defined by the following homomorphism of fuzzy soft modules as appropriate subsequence S

$$\overline{d}' : \left(\prod_{s\in S} F_s, A\right) \to \left(\prod_{s\in S} F_s, A\right).$$

We may define

$$f_0, f_1 : \prod_{s\in S} M_s \to \prod_{n\in \mathbb{N}} M_n$$

homomorphisms of modules with this formula:

$$f_0 (x_i, x_j, x_k, ...) = \left(p_1^i (x_i), p_2^i (x_i), ..., p_{i-1}^i (x_i), x_i, p_{i+1}^j (x_j), ..., p_{j-1}^j (x_j), x_j, ...\right)$$

$$f_1 (x_i, x_j, x_k, ...) = (0, 0, ..., x_i, 0, ..., x_j, 0, ..., x_k, 0, ...).$$

Also, for each $a \in A$

$$\begin{array}{l} \left(\underset{n\in\mathbb{N}}{\wedge} F_{na}\right) \left(p_1^i (x_i), ..., p_{i-1}^i (x_i), x_i, p_{i+1}^j (x_j), ..., p_{j-1}^j (x_j), x_j, ...\right) \\ = F_{1a} \left(p_1^i (x_i)\right) \wedge ... \wedge F_{i-1a} \left(p_{i-1}^i (x_i)\right) \wedge F_{ia} (x_i) \wedge \\ F_{i+1a} \left(p_{i+1}^j (x_j)\right) \wedge ... \wedge \mu_j (x_j) \wedge ... \\ \geq [F_{ia} (x_i) \wedge ... \wedge F_{ia} (x_i) \wedge F_{ia} (x_i)] \wedge [F_{ja} (x_j) \wedge ... \wedge F_{ja} (x_j)] \wedge ... \\ = F_{ia} (x_i) \wedge F_{ja} (x_j) \wedge ... = \underset{s\in S}{\wedge} F_{sa} (x_s), \end{array},$$

and

$$\begin{array}{l} \left(\underset{n\in\mathbb{N}}{\wedge} F_{na}\right) (0, 0, ..., x_i, 0, ..., x_j, 0, ...) \\ = F_{1a} (0) \wedge ... \wedge F_{ia} (x_i) \wedge F_{i+1a} (0) \wedge ... \wedge F_{ja} (x_j) \wedge ... \\ = F_{ia} (x_i) \wedge F_{ja} (x_j) \wedge ... = \underset{s\in S}{\wedge} F_{sa} (x_s), \end{array}$$

Then $\overline{f}_0, \overline{f}_1 : \left(\prod_{s\in S} F_s, A\right) \to \left(\prod_{n\in\mathbb{N}} F_n, A\right)$ are homomorphisms of fuzzy soft modules. It is clear that the following diagram is commutative:

$$\begin{array}{ccc}
\left(\prod_{s\in S} F_s, A\right) & \xrightarrow{\qquad\qquad} & \left(\prod_{n\in\mathbb{N}} F_n, A\right) \\
\overline{d'} \downarrow & & \downarrow_{\overline{d}} \\
\left(\prod_{s\in S} F_s, A\right) & \xrightarrow{\qquad\qquad} & \left(\prod_{n\in\mathbb{N}} F_n, A\right)
\end{array}$$

i.e., $\{\overline{f}_0, \overline{f}_1\}$ are morphisms of cochain complexes. Now, let us define

$$g_0, g_1 : \prod_{n\in\mathbb{N}} M_n \to \prod_{s\in S} M_s$$

homomorphisms with this formula:

$$g_0(x_1, x_2, x_3, ...) = (x_i, x_j, x_k, ...)$$

$$g_1(x_1, x_2, x_3, ...) = \left(\begin{array}{c} x_i + p_i^{i+1}(x_{i+1}) + ... + p_i^{j-1}(x_{j-1}), x_j \\ +p_j^{j+1}(x_{j+1}) + ... + p_j^{k-1}(x_{k-1}), ... \end{array} \right).$$

For

$$\left(\underset{s\in S}{\wedge} F_{sa}\right)(x_i, x_j, x_k, ...) = F_{ia}(x_i) \wedge F_{ja}(x_j) \wedge ... \geq \underset{n\in\mathbb{N}}{\wedge} F_{na}(x_n)$$

and

$$\begin{aligned}
&\left(\underset{s\in S}{\wedge} F_{sa}\right)\left(x_i + p_i^{i+1}(x_{i+1}) + ... + p_i^{j-1}(x_{j-1}), x_j + ... + p_j^{k-1}(x_{k-1}), ...\right) \\
&= F_{ia}\left(x_i + p_i^{i+1}(x_{i+1}) + ... + p_i^{j-1}(x_{j-1})\right) \wedge F_{ja}\left(x_j + ... + p_j^{k-1}(x_{k-1})\right) \wedge ... \\
&\geq \min\left\{F_{ia}(x_i), F_{ia}\left(p_i^{i+1}(x_{i+1})\right), ..., F_{ia}\left(p_i^{j-1}(x_{j-1})\right)\right\} \wedge \\
&\min\left\{F_{ja}(x_j), ..., F_{ja}\left(p_j^{k-1}(x_{k-1})\right)\right\} \wedge ... \\
&\geq \min\left\{F_{ia}(x_i), F_{i+1a}(x_{i+1}), ..., F_{j-1a}(x_{j-1})\right\} \wedge \\
&\min\left\{F_{ja}(x_j), F_{j+1a}(x_{j+1}), ..., F_{k-1a}\mu_{k-1}(x_{k-1})\right\} \wedge ... \\
&= \underset{m\in S}{\wedge} F_{ma}(x_m) \geq \underset{n\in\mathbb{N}}{\wedge} F_{na}(x_n),
\end{aligned}$$

thus, $\overline{g}_0, \overline{g}_1 : \left(\prod_{n\in\mathbb{N}} F_n, A\right) \to \left(\prod_{s\in S} F_s, A\right)$ are homomorphisms of fuzzy soft modules and $\overline{d}' \circ \overline{g_o} = \overline{g_1} \circ \overline{d}$ are satisfied. i.e., $\{\overline{g}_0, \overline{g}_1\}$ are homomorphisms of cochain complexes. It is clear that

$$\overline{g_o} \circ \overline{f_o} = \overline{g_1} \circ \overline{f_1} = \overline{1}_{\left(\prod_{s\in S} F_s, A\right)}.$$

Hence, we give

$$D : \prod_{n\in\mathbb{N}} M_n \to \prod_{n\in\mathbb{N}} M_n$$

homomorphism of modules with this formula:

$$\begin{aligned}
&D(x_1, x_2, x_3, ...) = (x_1 + p_1^2(x_2) + ... + p_1^{i-1}(x_{i-1}), x_2 + p_2^3(x_3) + ... + p_2^{i-1}(x_{i-1}) \\
&, ..., x_{i-1}, 0, x_{i+1} + p_{i+1}^{i+2}(x_{i+2}) + ... + p_{i+1}^{j-1}(x_{j-1}), x_{i+2} + ... + p_{i+2}^{j-1}(x_{j-1}), 0, ...).
\end{aligned}$$

For,

$$\begin{aligned}
&\left(\underset{n\in\mathbb{N}}{\wedge} F_{na}\right)\begin{pmatrix} x_1+p_1^2(x_2)+...+p_1^{i-1}(x_{i-1}),x_2+p_2^3(x_3)+...+p_2^{i-1}(x_{i-1}), \\ ...,x_{i-1},0,... \end{pmatrix}\\
&=F_{1a}\left(x_1+p_1^2(x_2)+...+p_1^{i-1}(x_{i-1})\right)\wedge\\
&F_{2a}\left(x_2+p_2^3(x_3)+...+p_2^{i-1}(x_{i-1})\right)\wedge...\\
&\wedge F_{i-1a}(x_{i-1})\wedge F_{ia}(0)\wedge F_{i+1a}\left(x_{i+1}+p_{i+1}^{i+2}(x_{i+2})+...+p_{i+1}^{j-1}(x_{j-1})\right)\wedge...\\
&\geq\min\left\{F_{1a}(x_1),F_{1a}\left(p_1^2(x_2)\right),...,F_{1a}\left(p_1^{i-1}(x_{i-1})\right)\right\}\wedge\\
&\min\left\{F_{2a}(x_2),F_{2a}\left(p_2^3(x_3)\right),...,F_{2a}\left(p_2^{i-1}(x_{i-1})\right)\right\}\wedge F_{i-1a}(x_{i-1})\wedge 1\wedge\\
&\min\left\{F_{i+1a}(x_{i+1}),F_{i+1a}\left(p_{i+1}^{i+2}(x_{i+2})\right),...,F_{i+1a}\left(p_{i+1}^{j-1}(x_{j-1})\right)\right\}\wedge...\\
&\geq\min\left\{F_{1a}(x_1),F_{2a}\mu_2(x_2),...,F_{i-1a}(x_{i-1})\right\}\wedge\\
&\min\left\{F_{2a}(x_2),F_{3a}(x_3),...,F_{i-1a}(x_{i-1})\right\}\wedge F_{i-1a}(x_{i-1})\wedge F_{i+1a}(x_{i+1})\wedge...\\
&=\underset{k=1}{\overset{i-1}{\wedge}}F_{ka}(x_k)\wedge\underset{k=2}{\overset{i-1}{\wedge}}F_{ka}(x_k)\wedge...=\underset{n\in\mathbb{N}}{\wedge}F_{na}(x_n),
\end{aligned}$$

$\overline{D}:\left(\prod\limits_{n\in\mathbb{N}}F_n,A\right)\to\left(\prod\limits_{n\in\mathbb{N}}F_n,A\right)$ is a homomorphism of fuzzy soft modules. By using simplicity of calculation, it is shown that $\overline{D}$ is a chain homotopy between $\overline{f_o}\circ\overline{g_o}$ and $\overline{f_1}\circ\overline{g_1}$ homomorphisms. Then the following cohomology modules of cochain complexes

$$\begin{aligned}
0\ \to\ &\left(\prod_{n\in\mathbb{N}}F_n,A\right)\xrightarrow{\overline{d}}\left(\prod_{n\in\mathbb{N}}F_n,A\right)\to 0\\
0\ \to\ &\left(\prod_{s\in S}F_s,A\right)\xrightarrow{\overline{d}}\left(\prod_{s\in S}F_s,A\right)\to 0
\end{aligned}$$

are fuzzy soft isomorfic. Since $\underset{\leftarrow}{\lim}^{(1)}$ is first cohomology module, the theorem is proved. ∎

Since $\underleftarrow{\lim}(F_n,a)=\ker\overline{d}$ and $p_n^{n+1}(x_{n+1})=x_n$ is satisfied for each $\{x_n\}\in\underleftarrow{\lim}M_n$,

$$F_{na}(x_n)=F_{na}(p_n^{n+1}(x_{n+1}))\geq F_{n+1a}(x_{n+1})$$

i.e., for each $\{x_n\}\in\ker\overline{d}$, $\{F_{na}(x_n)\}$ is decreasing sequence.

Theorem 5.12. *For all $\{x_n''\}\in\ker\overline{d}$, if $\underset{n\longrightarrow\infty}{lim}F_{na}''(x_n'')=0$ and the following diagram is short exact sequence of inverse system of fuzzy soft modules*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (F_2',A) & \longrightarrow & (F_2,A) & \longrightarrow & (F_2'',A) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (F_1',A) & \longrightarrow & (F_1,A) & \longrightarrow & (F_1'',A) & \longrightarrow & 0
\end{array}$$

then the sequence

$$\begin{aligned} 0 &\longrightarrow \varprojlim(F'_n, a) \longrightarrow \varprojlim(F_n, a) \longrightarrow \varprojlim(F''_n, a) \longrightarrow \\ \varprojlim{}^{(1)}(F'_n, a) &\longrightarrow \varprojlim{}^{(1)}(F_n, a) \longrightarrow \varprojlim{}^{(1)}(F''_n, a) \longrightarrow 0 \end{aligned}$$

is exact

Proof. For inverse system of fuzzy soft modules $\{(F_n, A)\}_{n\in\mathbb{N}}$,

$$C = 0 \xrightarrow{\overline{0}} \left(\prod_{n\in\mathbb{N}} F_n, A\right) \xrightarrow{\overline{d}} \left(\prod_{n\in\mathbb{N}} F_n, A\right) \xrightarrow{\overline{0}} 0 \xrightarrow{\overline{0}} \cdots$$

is a cochain complexes of fuzzy soft modules.

$$H^0(C) = \varprojlim(F_n, a),\ H^1(C) = \varprojlim{}^{(1)}(F_n, a),\ H^k(C) = 0, k \geq 2 \tag{5.1}$$

are fuzzy soft cohomology modules of this complexes. Similarly, for the inverse system of fuzzy soft modules $\{(F'_n, A)\}$ and $\{(F''_n, A)\}$, we can constitute the following intuitionistic fuzzy cochain complexes

$$\begin{aligned} C' &= 0 \xrightarrow{\overline{0}} \left(\prod_{n\in\mathbb{N}} F'_n, A\right) \xrightarrow{\overline{d}'} \left(\prod_{n\in\mathbb{N}} F'_n, A\right) \xrightarrow{\overline{0}} 0 \xrightarrow{\overline{0}} \cdots \\ C'' &= 0 \xrightarrow{\overline{0}} \left(\prod_{n\in\mathbb{N}} F''_n, A\right) \xrightarrow{\overline{d}''} \left(\prod_{n\in\mathbb{N}} F''_n, A\right) \xrightarrow{\overline{0}} 0 \xrightarrow{\overline{0}} \cdots \quad . \end{aligned}$$

It is clear that fuzzy cohomology modules of this complexes is the form in (5.1). From the condition of this theorem, the following sequence

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

is short exact sequence of cochain complexes of fuzzy soft modules. But generally, the following sequence of cohomology modules of this sequence

$$\begin{aligned} 0 &\longrightarrow H^0(C') \longrightarrow H^0(C) \longrightarrow H^0(C'') \xrightarrow{\overline{\partial}} H^1(C') \\ &\longrightarrow H^1(C) \longrightarrow H^1(C'') \longrightarrow H^2(C') \longrightarrow \cdots \end{aligned}$$

is not exact, because $\overline{\partial}$ is usually not homomorphism of fuzzy soft modules. Since $H^0(C'') = \ker d''$ and $\underset{n\longrightarrow\infty}{lim} F''_{na}(x''_n) = 0$, grade function F'' of fuzzy soft module $\left(H^0(C''), \mu'', \lambda''\right)$ is equal to grade function $\overline{0}$. Thus $\overline{\partial}$ is homomorphism of fuzzy soft modules. Therefore the sequence

$$\begin{aligned} 0 &\longrightarrow H^0(C') \longrightarrow H^0(C) \longrightarrow H^0(C'') \xrightarrow{\overline{\partial}} H^1(C') \\ &\longrightarrow H^1(C) \longrightarrow H^1(C'') \longrightarrow H^2(C') \longrightarrow \cdots \end{aligned}$$

is exact. By using the 5.1, we obtain the following exact sequence of intuitionistic fuzzy modules

$$\begin{aligned} 0 &\longrightarrow \varprojlim(M'_n, \mu'_n, \lambda'_n) \longrightarrow \varprojlim(F_n, a) \longrightarrow \varprojlim(F''_n, a) \\ &\longrightarrow \varprojlim{}^{(1)}(F'_n, a) \longrightarrow \varprojlim{}^{(1)}(F_n, a) \longrightarrow \varprojlim{}^{(1)}(F''_n, a) \longrightarrow 0. \end{aligned}$$

□

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Sadi Bayramov (baysadi@gmail.com)
Department of Mathematics, Kafkas University, Kars, 36100-Turkey

Ciğdem Gunduz (Aras) (carasgunduz@gmail.com)
Department of Mathematics, Kocaeli University, Kocaeli, 41380-Turkey

Murat Ibrahim Yazar (miy248@yahoo.com)
Department of Mathematics, Kafkas University, Kars, 36100-Turkey