Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 4, No. 2, (October 2012), pp. 343–348
ISSN 2093–9310
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The lattice structure of weakly induced principal L-topologies

Raji George, T. P. Johnson

Received 8 August 2011; Accepted 24 February 2012

Abstract. We study the lattice structure of the set $W_P(X)$ of all weakly induced principal L-topologies on a given set X. It is proved that this lattice is complete, not atomic, not complemented and not dually atomic. Some other properties of the lattice $W_{P\tau}$, the set of all weakly induced principal L-topologies defined by families of (completely) scott continuous functions on X are also discussed.

2010 AMS Classification: 54A40

Keywords: Weakly induced principal L-topology, Induced principal L topology, Complete lattice, Atom, Dual atom, Join complement, Meet complement, Semi complement, Complement.

Corresponding Author: Raji George (bobyraji@gmail.com)

1. Introduction

The concept of induced fuzzy topological space was introduced by Weiss [14]. Lowen called these spaces a topologically generated spaces. Martin [10] introduced a generalized concept, weakly induced spaces, which was called semi induced space by Mashhour et al. [11]. The notion of lower semi continuous functions plays an important tool in defining the above concepts. In [1, 2], Aygun et al. introduced a new class of functions from a topological space (X, τ) to a fuzzy lattice L with its scott topology called (completely) scott continuous functions, as a generalization of (completely) lower semi continuous functions from (X, τ) to $[0, 1]$.

It is known that [6] lattice of L-topologies is complete, atomic and not complemented. In [7], Jose and Johnson generalised weakly induced spaces introduced by Martin [10] using the tool (completely) scott continuous functions and studied the lattice structure of the set $W(X)$ of all weakly induced L-topologies on a given set X. A related problem is to find subfamilies in $W(X)$ having certain properties. The collection of all weakly induced principal L topologies $W_P(X)$ form a lattice with the natural order of set inclusion. The concept of principal topologies in the crisp case is studied by Steiner [12]. The lattice of principal topologies is both atomic and dually atomic. Analogously we study the lattice structure of the set of all weakly induced principal L-topologies on a given set X. Here we study properties of the lattice W_{P_τ} of weakly induced principal L topologies defined by families of (completed) scott continuous functions with reference to τ on X. From the lattice W_{P_τ} we deduce that lattice $W_P(X)$ of weakly induced principal L-topologies on X. It is not complemented but join complemented.

2. Preliminaries

Let X be a nonempty ordinary set and $L = (\leq, \vee, \wedge, ')$ be a complete completely distributive lattice with smallest element 0 and largest element 1, $0 \neq 1$, and with an order reversing involution $a \rightarrow a'$ $(a \in L)$. We identify the constant function from X to L with value α by $\underline{\alpha}$. The fundamental definition of L-fuzzy set theory and L-topology are assumed to be familiar to the reader in the sense of Chang [3].

A topological space is called principal if it is discrete or if it can be written as the meet of principal ultra topologies. Steiner [12] proved that this is equivalent to requiring that the arbitrary intersection of open sets is open. Analogously we define principal L-topology as

Definition 2.1. An L-topology is called principal L-topology if arbitrary intersection of open L subsets is an open L subset.

Definition 2.2 ([8])**.** An element of a lattice L is called an atom if it is the minimal element of $L \backslash \{0\}$.

Definition 2.3 ([8])**.** An element of a lattice L is called a dual atom if it is the maximal element of $L \backslash \{1\}$.

Definition 2.4 ([4])**.** A lattice is said to be bounded if it possess smallest element 0 and largest element 1.

Definition 2.5 ([8])**.** A bounded lattice L is said to be join complemented if for all x in L, there exists y in L such that $x \vee y = 1$.

Definition 2.6 ([8])**.** A bounded lattice L is said to be meet complemented if for all x in L, there exist y in L such that $x \wedge y = 0$.

Definition 2.7 ([8])**.** A bounded lattice is said to be complemented if it is both join complemented and meet complemented.

Definition 2.8 ([8])**.** A bounded lattice L is said to be semi-complemented if it is either join complemented or meet complemented.

Definition 2.9 ([5])**.** An element p of L is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by Pr (L).

Definition 2.10 ([13])**.** The scott topology on L is the topology S, generated by the sets of the form $\{t \in L : t \nleq p\}$ where $p \in \Pr(L)$. Let (X, τ) be a topological space and $f : (X, \tau) \rightarrow L$ be a function, where L has its scott topology. We say that f is scott continuous if for every $p \in \Pr(L)$, $f^{-1}(\{t \in L : t \nleq p\}) \in \tau$.

Remark 2.11. When $L = [0, 1]$, the scott topology coincides with the topology of topologically generated spaces of Lowen [9]. The set

$$\omega_L(\tau) = \{f \in L^X; f : (X, \tau) \to L \text{ is scott continuous }\}$$

is an L-topology. It is the largest element in W_τ. If τ is a principal topology $\omega_L(\tau)$ is a principal L-topology, we can denote it by $\omega_{PL}(\tau)$. An L-topology F on X is called an induced principal L-topology if there exist a principal topology τ on X such that $F = \omega_{PL}(\tau)$.

Definition 2.12 ([2, 1]). Let (X, τ) be a topological space and $a \in X$. A function $f : (X, \tau) \to L$, where L has its scott topology, is said to be completely scott continuous at $a \in X$ if for every $p \in \Pr(L)$ with $f(a) \not\leq p$, there is a regular open neighbourhood U of a in (X, τ) such that $f(x) \not\leq p$ for every $x \in U$. That is $U \subset f^{-1}(\{t \in L : t \not\leq p\})$ and f is called completely scott continuous on X, if f is completely scott continuous at every point of X.

Note. Let F be a principal L-topology on the set X, let F_c denote the 0–1 valued members of F, that is, F_c is the set of all characteristic mappings in F. Then F_c is a principal L-topology on X. Define $F_c^* = \{A \subset X : \mu_A \in F_c$ where μ_A is the characteristic function of $A\}$. The principal L-topological space (X, F_c) is same as the principal topological space (X, F_c^*).

Definition 2.13. A principal L-topological space (X, F) is said to be a weakly induced principal L topological space, if for each $f \in F$, f is a scott continuous function from (X, F_c^*) to L.

Definition 2.14. If F is the collection of all scott continuous functions from (X, F_c^*) to L, then F is an induced space and $F = \omega_{PL}(F_c^*)$.

3. Lattice of weakly induced principal L-topologies

For a given principal topology τ on X, the family $W_{P\tau}$ of all weakly induced principal L-topologies defined by families of scott continuous functions from (X, τ) to L forms a lattice under the natural order of set inclusion. The least upper bound of a collection of weakly induced principal L-topologies belonging to $W_{P\tau}$ is the weakly induced principal L-topology which is generated by their union and the greatest lower bound is their intersection. The smallest element is the indiscrete L-topology, and denoted by 0 and the largest element is denoted by $1 = \omega_{PL}(\tau)$.

Also for a principal topology τ on X, the family $CW_{P\tau}$ of all weakly induced principal L topologies defined by families of completely scott continuous function from (X, τ) to L forms a lattice under the natural order of set inclusion. Since every completely scott continuous function is scott continuous, it follows that $CW_{P\tau}$ is a sublattice if $W_{P\tau}$. We note that $W_{P\tau}$ and $CW_{P\tau}$ coincide when each openset in τ is regular open.

When $\tau = D$, the discrete topology on X, these lattices coincide with lattice of weakly induced principal L-topologies on X.

Theorem 3.1. *The lattice $W_{P\tau}$ is complete.*

Proof. Let S be a subset of $W_{P\tau}$ and let $G = \bigcap_{F\in S} F$. Clearly G is a principal L-topology. Let $g \in G$. Since each $F \in S$ is a weakly induced principal L topology, g is a scott continuous mapping from (X, F_C^*) to L, that is $g^{-1}\{t \in L : t \nleq p$ where $p \in \Pr L\} \in F_c^*$ for each $F \in S$. Therefore $g^{-1}\{t \in L : t \nleq p$ where $p \in \Pr L\} \in \bigcap_{F\in S} F_C^*$. Hence g is a scott continuous function from (X, G_c^*) to L, where $(X, G_c^*) = (X, \bigcap_{F\in S} F_c^*)$. That is $G \in W_{P\tau}$ and G is the greatest lower bound of S. Let K be the set of upper bounds of S. Then K is non empty, since $1 = \omega_{PL}(\tau) \in K$.

Using the above argument K has a greatest lower bound, say H, then this H is a least upper bound of S. Thus every subset S of $W_{P\tau}$ has greatest lower bound and least upper bound. Hence $W_{P\tau}$ is complete. □

Theorem 3.2. *$W_{P\tau}$ is not atomic.*

Proof. Atoms in $W_{P\tau}$ are either of the form $\{\underline{0}, \underline{1}, \underline{\alpha}\}$ or $\{\underline{0}, \underline{1}, \mu_A\}$, where μ_A is the characteristic function of open subsets A of (X, τ) and $\alpha \in (0, 1)$. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and

$$F = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}}, f, g, h, i, j, k\},$$

where $f(a) = 0.6$, $f(b) = 0.5$, $f(c) = 0.4$, $g(a) = 1$, $g(b) = 1$, $g(c) = 0.4$, $h(a) = 1$, $h(b) = 0.5$, $h(c) = 0.4$, $i(a) = 1$, $i(b) = 0.5$, $i(c) = 0$, $j(a) = 0.6$, $j(b) = 0.5$, $j(c) = 0$, $k(a) = 0.6$, $k(b) = 0$, and $k(c) = 0$. $F_c = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}}\}$. $F_c^* = \{\phi, X, \{a\}, \{a, b\}\} = \tau$ and $F \in W_{P\tau}$. But this F cannot be expressed as join of atoms. Hence $W_{p\tau}$ is not atomic. □

Note. A lattice L is modular if and only if it has no sublattice isomorphic to N_5, where N_5 is a standard non modular lattice.

Theorem 3.3. *$W_{P\tau}$ is not distributive.*

Proof. Since every distributive lattice is necessarily modular, we prove that $W_{P\tau}$ is not modular. This can be illustrated with an example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Suppose $F_1 = \{\underline{0}, \underline{1}\}$, $F_2 = \{\underline{0}, \underline{1}, \mu_{\{a\}}\}$, $F_3 = \{\underline{0}, \underline{1}, \mu_{\{b\}}\}$, $F_4 = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}}\}$, $F_5 = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{a,b\}}\}$. Then each element in the collection $S = \{F_1, F_2, F_3, F_4, F_5\}$ belongs to $W_{P\tau}$ and S is a sublattice of $W_{P\tau}$ isomorphic to N_5. Therefore $W_{P\tau}$ is not modular and hence not distributive. □

Proposition 3.4. *If L has no dual atom, then atoms in $W_{P\tau}$ of the form $\{\underline{0}, \underline{1}, \underline{\alpha}\}$ have no complements in $W_{P\tau}$.*

Proof. Let $F = \{\underline{0}, \underline{1}, \underline{\alpha}\}$ be atom in $W_{P\tau}$. We claim that F has no complement. 1 is not a complement of F since $1 \wedge F \neq 0$. Let P be a weakly induced principal L-topology in $W_{P\tau}$ other than $1 = \omega_{PL}(\tau)$. If $F \subset P$, then P cannot be the complement of F, since $F \wedge P \neq 0$. If $F \nsubseteq P$, let $F \vee P = G$ and G has the subbasis $\{f \wedge p | f \in F, p \in P\}$. Then G cannot be equal to $\omega_{PL}(\tau) = 1$. Hence P is not a complement of F. □

Remark 3.5. The above proposition is not true for an arbitrary lattice L. For example, take $L = \{0, \alpha, 1\}$ ordered by $0 < \alpha < 1$. If (X, τ) is a principal L topological space and $K = \{\underline{0}, \underline{1}, \underline{\alpha}\}$, then clearly K is an atom in W_{P_τ}, when $\underline{\alpha}$ is not a characteristic function. Let $H = \{\underline{0}, \underline{1}\} \cup \{\mu_A : A \in \tau\}$. Then H is an element of W_{P_τ} and $K \wedge H = 0$ and $K \vee H = \omega_{PL}(\tau) = 1$. Hence H is a complement of K.

Theorem 3.6. *W_{P_τ} is not complemented.*

Proof. This follows from the Proposition 3.4. □

Remark 3.7. When $\tau = D$, the discrete topology on X then $W_{PD} = W_P(X)$, the collection of all weakly induced principal L-topologies on X. Let Δ denote the family of all weakly induced principal L-topologies defined by scott continuous functions where each scott continuous function is a characteristic function. Then Δ is a sublattice of $W_P(X)$ and is a lattice isomorphic to the lattice of all principal topologies on X. The elements of Δ are called crisp principal topologies.

Theorem 3.8. *The lattice of weakly induced principal L-topologies $W_P(X)$ is not complemented.*

Proof. This follows from Theorem 3.6. □

Theorem 3.9. *Every atom in $W_P(X)$ of the form $\{\underline{0}, \underline{1}, \mu_A\}$ has complement.*

Proof. Let $F = \{\underline{0}, \underline{1}, \mu_A\}$. Then F is an element of Π, lattice of principal topologies on X. Since Π is complemented there exists τ in Π such that $\tau \vee F$ equal to the discrete principal topology and $\tau \wedge F$ equal to the indiscrete principal topology on X. Then $F \vee \omega_{PL}(\tau) = 1 = \omega_{PL}(D)$ and $F \wedge \omega_{PL}(\tau) = 0$. □

Theorem 3.10. *The lattice $W_P(X)$ of all weakly induced principal L-topologies on any set X is semi complemented.*

Proof. Let $F \in W_P(X)$. Since F is weakly induced there is a topology F_c^* on X such that each element $f \in F$ is a scott continuous function from (X, F_c^*) to L. Since the lattice of principal topologies is complemented, we can find a principal topology τ' such that $F \vee \omega_{PL}(\tau') = 1 = \omega_{PL}(D)$ where (X, D) is a principal topological space and $F \wedge \omega_{PL}(\tau')$ need not be equal to 0, the indiscrete principal L-topology on X. Thus every F in $W_P(X)$ has a join complement. Hence $W_P(X)$ is semi complemented. □

Remark 3.11. Dual atoms in $W_P(X)$ are of the form $\omega_{PL}(\tau)$ where τ is a dual atom in the lattice of principal topologies. Each induced principal L-topology other than the discrete principal L-topology can be expressed as meet of dual atoms. But an arbitrary weakly induced principal L-topology, for example the weakly induced L-topology $F = \{\underline{0}, \underline{1}, \underline{\alpha}\}$ cannot be expressed as a meet of dual atoms. Thus we have

Theorem 3.12. *The lattice $W_P(X)$ of all weakly induced principal L-topologies on any set X is not dually atomic.*

Acknowledgements. The first author wishes to thank the University Grant Commission, India for giving financial support.

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Raji George (bobyraji@gmail.com)
Department of Mathematics, St. Peter's College, Kolenchery - 682 311 Ernakulam Dt. Kerala state, India

T. P. Johnson (tpjohnson@cusat.ac.in)
Applied Sciences and Humanities Division, School of Engineering, Cochin University of Science and Technology, Cochin-22, Kerala state, India.