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Generalized intuitionistic fuzzy interior ideals in a semigroup

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Abstract. In this paper, using t-norm $\triangle$ and s-norm $\triangledown$, we introduce the notion of generalized intuitionistic fuzzy interior ideal of a semigroup and investigate some properties of these ideals.

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1. Introduction

In [18], Zadeh introduced the notion of fuzzy set and fuzzy set operations. Since then, fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathematics. Kuroki initiated the study of fuzzy semigroups [9]. Hong et. al. in [8] studied various properties of fuzzy interior ideals of semigroups (also see [21, 22, 15, 16]).

In 1986, Atanassov [3] introduced the concept of intuitionistic fuzzy set as a generalization of fuzzy set (also see [4, 5]. Intuitionistic fuzzy set theory has been applied in different fields, for example logic programming, decision making problems, etc. De et. al. in [7] applied intuitionistic fuzzy set theory in medical diagnosis. Kim and Jun in [10, 11] studied various properties of intuitionistic fuzzy ideals of semigroups and intuitionistic fuzzy interior ideals in semigroups. S. Abdullah et al. introduced direct product of intuitionistic fuzzy ideals and bi-ideals in [1, 2].

Kim in [12] considered the fuzzification of R-subgroups of nearring with respect to an s-norm. In [13], Kim and Lee studied the concept of intuitionistic (T, S) normed fuzzy ideals of Γ-rings. Zhan in [19] introduced the concept of the fuzzy left h-ideals in hemirings with t-norms. Ma and Zhan (S, T)-fuzzy M-subsemigroups of an M-semigroup in [14]. Zhan and Dudek in [20] defined interval valued intuitionistic

(S,T)-fuzzy H_v -submodules. Akram and Dar in [6] introduced the idea of fuzzy left h-ideal in hemirings with respect to an s-norm.

In this paper, we introduce the concept of a generalized intuitionistic fuzzy interior ideal in a semigroup and investigate some properties of these ideals.

2. Preliminaries

Throughout this paper S will denote a semigroup. By a subsemigroup of S we mean a non-empty subset A of S such that $A\ A \subseteq A$. By a left (right) ideal of S we mean a non-empty subset A of S such that $SA \subseteq A\,(AS \subseteq A)$. By a two sided ideal or simply an ideal, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$. A semigroup S is said to be regular if, for each $x \in S$ there exists $y \in S$ such that $x = xyx$.

A fuzzy subset λ of a universe X is a function from X into the unit closed interval of real numbers $[0, 1]$. Two fuzzy subsets λ, μ of X are said to be equal if $\lambda\,(x) = \mu\,(x)$ for all $x \in X$. Let λ, μ be two fuzzy subsets of X. Then $\lambda \leq \mu$ if and only if $\lambda\,(x) \leq \mu\,(x)$ for all $x \in X$. The union and intersection of two fuzzy subsets λ, μ of X is defined as

$(\lambda \vee \mu)\,(x) = \lambda\,(x) \vee \mu\,(x)$

$(\lambda \wedge \mu)\,(x) = \lambda\,(x) \wedge \mu\,(x)$

for all $x \in X$, respectively.

An intuitionistic fuzzy set A in S is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in S\}$$

where the function $\mu_A \colon S \to [0, 1]$ and $\gamma_A \colon S \to [0, 1]$ denote the degree of membership and the degree of nonmembership of each element $x \in S$ to A, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in S\}$.

Definition 2.1 ([3])**.** Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets in a set S. Then

(1) $A \subseteq B$ if and only if $\mu_A \leq \mu_B$ and $\gamma_A \geq \gamma_B$.
(2) $A^c = (\gamma_A, \mu_A)$.
(3) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
(4) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.
(5) $\Box A = (\mu_A, \bar{\mu}_A)$ where $\bar{\mu}_A = 1 - \mu_A$.
(6) $\Diamond A = (\bar{\gamma}_A, \gamma_A)$ where $\bar{\gamma}_A = 1 - \gamma_A$.

Definition 2.2 ([10])**.** An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy subsemigroup of S if

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$$

for all x, $y \in S$.

Definition 2.3 ([10]). An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy left (right) ideal of S if

$$\mu_A(xy) \geq \mu_A(y)\ (\mu_A(xy) \geq \mu_A(x))$$

and

$$\gamma_A(xy) \leq \gamma_A(y)\ (\gamma_A(xy) \leq \gamma_A(x))$$

for all $x,\ y \in S$.

An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy ideal of S if it is both an intuitionistic fuzzy left ideal and intuitionistic fuzzy right ideal of S.

Definition 2.4 ([11]). An intuitionistic fuzzy subsemigroup $A = (\mu_{A,}\gamma_A)$ of a semigroup S is called an intuitionistic fuzzy interior ideal of S if

$$\begin{aligned} \mu_A(xyz) &\geq \mu_A(y) \\ \gamma_A(xyz) &\leq \gamma_A(y) \end{aligned}$$

for all $x, y, z \in S$.

Definition 2.5 ([13]). Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in S and $t \in [0, 1]$. Then the sets

$$U(\mu_A\colon t) = \{x \in S\colon \mu_A(x) \geq t\}$$

and

$$L(\gamma_A\colon t) = \{x \in S\colon \gamma_A(x) \leq t\}$$

are called μ-level t-cut and γ-level t-cut of A, respectively.

Definition 2.6 ([17]). By a t-norm $\triangle$, we mean a function $\triangle\colon\ [0, 1]\times[0, 1] \to [0, 1]$ satisfying the following conditions

($t1$) $x \triangle 1 = x$
($t2$) $x \triangle y = y \triangle x$
($t3$) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$
($t4$) if $w \leq x$ and $y \leq z$ then $w \triangle y \leq x \triangle z$, for all $x, y,\ z,\ w \in [0, 1]$.

Remark 2.7 ([17]). Every t-norm $\triangle$ has a useful property

$$(x \triangle y) \leq \min(x, y)$$

for all $x, y \in [0, 1]$.

Definition 2.8 ([17]). By an s -norm $\bigtriangledown$, we mean a function $\bigtriangledown\colon\ [0, 1]\times[0, 1] \to [0, 1]$ satisfying the following conditions

($s1$) $x \bigtriangledown 0 = x$
($s2$) $x \bigtriangledown y = y \bigtriangledown x$
($s3$) $x \bigtriangledown (y \bigtriangledown z) = (x \bigtriangledown y) \bigtriangledown z$
($s4$) if $w \leq x$ and $y \leq z$ then $w \bigtriangledown y \leq x \bigtriangledown z$
for all $x, y, z,\ w \in [0, 1]$.

Remark 2.9 ([17]). Every s-norm has a useful property

$$\max(x, y) \leq x \bigtriangledown y$$

for all $x, y \in [0, 1]$.

Definition 2.10 ([17]). A mapping $\eta\colon [0,1] \to [0,1]$ is called a negation if it satisfies

($\eta 1$) $\eta(0) = 1,\ \eta(1) = 0$

($\eta 2$) η is non increasing.

($\eta 3$) $\eta(\eta(x)) = x$

The most frequently used negation is $x \to 1 - x$.

Remark 2.11 ([17]). The t-norm and s-norm are said to be dual with respect to the negation $\eta(x) = 1 - x$, if

$$x \bigtriangledown y = \eta(\eta(x) \bigtriangleup \eta(y)).$$

This holds with respect to η if and only if $x \bigtriangleup y = \eta(\eta(x) \bigtriangledown \eta(y))$.

3. Generalized intuitionistic fuzzy interior ideals in a semigroup

In this paper we denote by $\bigtriangleup$ and $\bigtriangledown$, the t-norm and s-norm which are dual with respect to the negation $\eta(x) = 1 - x$.

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy subsemigroup of S if

$$\mu_A(xy) \geq \mu_A(x) \bigtriangleup \mu_A(y)$$

and

$$\gamma_A(xy) \leq \gamma_A(x) \bigtriangledown \gamma_A(y)$$

for all $x,\ y \in S$.

Example 3.2. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in a semigroup S. Where $\mu_A\colon S \to [0,1]$ is defined by $\mu_A(a) = 0.\ 6,\ \mu_A(b) = 0.\ 5,\ \mu_A(c) = 0.\ 7 = \mu_A(d)$ and $\gamma_A\colon S \to [0,1]$ is defined by $\gamma_A(a) = 0.\ 3,\ \gamma_A(b) = 0.\ 4 = \gamma_A(c),\ \gamma_A(d) = 0.3$.

Let $\bigtriangleup$ be the t-norm defined by

$$x \bigtriangleup y = \max(x + y - 1, 0)$$

and $\bigtriangledown$ be the s-norm defined by

$$x \bigtriangledown y = \min(x + y, 1)$$

for all $x, y \in [0,1]$. By routine calculations we can check that the intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy subsemigroup of S.

Definition 3.3. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy left (right) ideal of S if

$$\mu_A(xy) \geq \mu_A(y) \qquad (\mu_A(xy) \geq \mu_A(x)\ )$$

and

$$\gamma_A(xy) \leq \gamma_A(y) \qquad (\gamma_A(xy) \leq \gamma_A(x))$$

for all $x,\ y \in S$.

Definition 3.4. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy two sided ideal (or generalized intuitionistic fuzzy ideal) of S if it is both a generalized intuitionistic fuzzy left ideal and a generalized intuitionistic fuzzy right ideal of S.

Let A be a non-empty subset of a semigroup S. Then the intuitionistic characteristic function of A is denoted by $\ddot{A} = (\Phi_A, \Psi_A)$ and is defined as

$$\begin{aligned} \Phi_A(x) &= \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}, \\ \Psi_A(x) &= \begin{cases} 0 & if\ x \in A \\ 1 & if\ x \notin A \end{cases} \end{aligned}$$

for all $x \in S$.

Theorem 3.5. *Let A be a non-empty subset in a semigroup of S. Then A is a subsemigroup of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy subsemigroup of S.*

Proof. Assume that A is a subsemigroup of S. We show that $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy subsemigroup of S. Suppose that there exist $x, y \in S$, such that

$$\Phi_A(xy) < \Phi_A(x) \vartriangle \Phi_A(y).$$

Take $t = \frac{1}{2}(\Phi_A(xy) + \Phi_A(x) \vartriangle \Phi_A(y))$, then $t \in (0, 1]$, such that

$$\Phi_A(xy) < t < \Phi_A(x) \vartriangle \Phi_A(y) \leq \min(\Phi_A(x), \Phi_A(y)).$$

Thus $\Phi_A(x) > t$ and $\Phi_A(y) > t$. This implies $x, y \in A$. Since A is a subsemigroup of S, $xy \in A$. This implies $\Phi_A(xy) = 1 > t$. This is a contradiction. Hence

$$\Phi_A(xy) \geq \Phi_A(x) \vartriangle \Phi_A(y).$$

Similarly, if there exist $x, y \in S$, such that

$$\Psi_A(xy) > \Psi_A(x) \triangledown \Psi_A(y).$$

Take $t' = \frac{1}{2}(\Psi_A(xy) + \Psi_A(x) \triangledown \Psi_A(y))$. Then $t' \in (0, 1]$, such that

$$\Psi_A(xy) > t' > \Psi_A(x) \triangledown \Psi_A(y) \geq \max(\Psi_A(x), \Psi_A(y)).$$

This implies $\Psi_A(x) < t'$ and $\Psi_A(y) < t'$ that is $x, y \in A$. So $xy \in A$. This implies $\Psi_A(xy) = 0 < t'$ which is a contradiction. Hence

$$\Psi_A(xy) \leq \Psi_A(x) \triangledown \Psi_A(y)$$

for all $x, y \in S$. This shows that $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy subsemigroup of S.

Conversely, let $\ddot{A} = (\Phi_A, \Psi_A)$ be a generalized intuitionistic fuzzy subsemigroup of S. We show that A is a subsemigroup of S. Let $x, y \in A$, $\Phi_A(x) = 1$, $\Phi_A(y) = 1$ and $\Psi_A(x) = 0$, $\Psi_A(y) = 0$. Since $\Phi_A(xy) \geq \Phi_A(x) \vartriangle \Phi_A(y) = 1 \vartriangle 1 = 1$ and $\Psi_A(xy) \leq \Psi_A(x) \triangledown \Psi_A(y) = 0 \triangledown 0 = 0$, we have $\Phi_A(xy) = 1$ and $\Psi_A(xy) = 0$. This implies $xy \in A$. Hence A is a subsemigroup of S. $\square$

Definition 3.6. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ be two intuitionistic fuzzy subsets of a semigroup S. The product $A \circledcirc B = (\mu_A \circledcirc \mu_B,\ \gamma_A \circledcirc \gamma_B)$ is defined by

$$\mu_A \circledcirc \mu_B = \begin{cases} \underset{x=yz}{\vee} \mu_A(y) \bigtriangleup \mu_B(z) \text{ if } \exists\ y, z \in S, \text{ such that } x = yz \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$$

$$\gamma_A \circledcirc \gamma_B = \begin{cases} \underset{x=yz}{\wedge} \gamma_A(y) \bigtriangledown \gamma_B(z) \text{ if } \exists\ y, z \in S, \text{ such that } x = yz \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$$

The operation $\circledcirc$ is associative.

We considered the semigroup S as an intuitionistic fuzzy subset of itself. It is written as $S = (\mathcal{S}, \acute{\mathcal{S}})$ and defined as $\mathcal{S}(x) = 1$ and $\acute{\mathcal{S}}(x) = 0$ for all $x \in S$.

Definition 3.7. A generalized intuitionistic fuzzy subsemigroup $A = (\mu_{A,}\gamma_A)$ of a semigroup S is called a generalized intuitionistic fuzzy interior ideal of S if

$$\mu_A\left(xyz\right) \geq \mu_A\left(y\right)$$

and

$$\gamma_A\left(xyz\right) \leq \gamma_A\left(y\right)$$

for all $x, y, z \in S$.

Theorem 3.8. *An intuitionistic fuzzy set $A = (\mu_{A,}\gamma_A)$ in a semigroup S is a generalized intuitionistic fuzzy interior ideal of S if and only if $A \circledcirc A \subseteq A$ and $A \circledcirc S \circledcirc A \subseteq A$ that is $\mu_A \circledcirc \mu_A \leq \mu_A$, $\gamma_A \circledcirc \gamma_A \geq \gamma_A$ and $S \circledcirc \mu_A \circledcirc S \leq \mu_A$, $S^{'} \circledcirc \gamma_A \circledcirc S^{'} \geq \gamma_A$.*

Proof. Let $A = (\mu_{A,}\gamma_A)$ be a generalized intuitionistic fuzzy interior ideal of S and $a \in S$. If

$$(\mu_A \circledcirc \mu_A)(a) = 0 \text{ and } (\gamma_A \circledcirc \gamma_A)(a) = 1,$$

then

$$\mu_A \circledcirc \mu_A \leq \mu_A \text{ and } \gamma_A \circledcirc \gamma_A \geq \gamma_A.$$

Otherwise, there exist elements $x, y \in S$ such that $a = xy$. Then,

$$\begin{aligned} (\mu_A \circledcirc \mu_A)(a) &= \underset{a=xy}{\vee} \mu_A(x) \bigtriangleup \mu_A(y) \\ &\leq \underset{a=xy}{\vee} \mu_A(xy) \\ &= \mu_A(a) \end{aligned}$$

and

$$\begin{aligned} (\gamma_A \circledcirc \gamma_A)(a) &= \underset{a=xy}{\wedge} \gamma_A(x) \bigtriangledown \gamma_A(y) \\ &\geq \underset{a=xy}{\vee} \gamma_A(xy) \\ &= \gamma_A(a). \end{aligned}$$

Hence $\mu_A \circledcirc \mu_A \leq \mu_A$ *and* $\gamma_A \circledcirc \gamma_A \geq \gamma_A$.

Now, let x be any element of S. If $x = yz$ for some $y, z \in S$ and $y = uv$ for some $u, v \in S$, then

$$\mu_A\left(uvz\right) \geq \mu_A\left(v\right) \text{ and } \gamma_A\left(uvz\right) \leq \gamma_A\left(v\right)$$

Now

$$
\begin{aligned}
(S \circledcirc \mu_A \circledcirc S)x &= \bigvee_{x=yz} \{(S \circledcirc \mu_A)(y) \triangle S(z)\} \\
&= \bigvee_{x=yz} \left\{ \bigvee_{y=uv} (S(u) \triangle \mu_A(v)) \triangle S(z) \right\} \\
&= \bigvee_{x=yz} \left\{ \bigvee_{y=uv} (1 \triangle \mu_A(v)) \triangle 1 \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=uv} \mu_A(v) \\
&\leq \mu_A(uvz) = \mu_A(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(S' \circledcirc \gamma_A \circledcirc S')(x) &= \bigwedge_{x=yz} \left\{ \left(S' \circledcirc \gamma_A\right)(y) \bigtriangledown S'(z) \right\} \\
&= \bigwedge_{x=yz} \left\{ \bigwedge_{y=uv} \left(S'(u) \bigtriangledown \gamma_A(v)\right) \bigtriangledown S'(z) \right\} \\
&= \bigwedge_{x=yz} \left\{ \bigwedge_{y=uv} (0 \bigtriangledown \gamma_A(v)) \bigtriangledown 0 \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=uv} \gamma_A(v) \\
&\geq \gamma_A(uvz) = \gamma_A(x).
\end{aligned}
$$

In the other case, we have

$$
\begin{aligned}
(S \circledcirc \mu_A \circledcirc S)(x) &= 0 \leq \mu_A(x) \\
(S' \circledcirc \gamma_A \circledcirc S')(x) &= 1 \geq \gamma_A(x).
\end{aligned}
$$

Therefore, $S \circledcirc \mu_A \circledcirc S \leq \mu_A$ and $S' \circledcirc \gamma_A \circledcirc S' \geq \gamma_A$.

Conversely, assume that $\mu_A \circledcirc \mu_A \leq \mu_A$, $\gamma_A \circledcirc \gamma_A \geq \gamma_A$ and $S \circledcirc \mu_A \circledcirc S \leq \mu_A$, $S' \circledcirc \gamma_A \circledcirc S' \geq \gamma_A$ holds. Let x, a and y be any elements of S. Then

$$
\begin{aligned}
\mu_A(xy) &\geq (\mu_A \circledcirc \mu_A)(xy) \\
&= \bigvee_{xy=bc} \mu_A(b) \triangle \mu_A(c) \\
&\geq \mu_A(x) \triangle \mu_A(y),
\end{aligned}
$$

$$
\begin{aligned}
\gamma_A(xy) &\leq (\gamma_A \circledcirc \gamma_A)(xy) \\
&= \bigwedge_{xy=bc} \gamma_A(b) \bigtriangledown \gamma_A(c) \\
&\leq \gamma_A(x) \bigtriangledown \gamma_A(y),
\end{aligned}
$$

$$\begin{aligned}
\mu_A(xay) &\geq (S \circledcirc \mu_A \circledcirc S)(xay) \\
&= \bigvee_{xay=pq} \{(S \circledcirc \mu_A)(p) \triangle S(q)\} \\
&\geq (S \circledcirc \mu_A)(xa) \triangle S(y) \\
&= (S \circledcirc \mu_A)(xa) \triangle 1 \\
&= (S \circledcirc \mu_A)(xa) \\
&= \bigvee_{xa=uv} S(u) \triangle \mu_A(v) \\
&\geq S(x) \triangle \mu_A(a) \\
&= 1 \triangle \mu_A(a) = \mu_A(a)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_A(xay) &\geq \left(S' \circledcirc \gamma_A \circledcirc S'\right)(xay) \\
&= \bigwedge_{xay=pq} \left\{\left(S' \circledcirc \gamma_A\right)(p) \bigtriangledown S'(q)\right\} \\
&\geq \left(S' \circledcirc \gamma_A\right)(xa) \bigtriangledown S'(y) \\
&= \left(S' \circledcirc \gamma_A\right)(xa) \bigtriangledown 0 \\
&= \left(S' \circledcirc \gamma_A\right)(xa) \\
&= \bigwedge_{xa=uv} S'(u) \bigtriangledown \gamma_A(v) \\
&\geq S'(x) \bigtriangledown \gamma_A(a) \\
&= 0 \bigtriangledown \gamma_A(a) = \gamma_A(a).
\end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S. □

Theorem 3.9. *Let A be a non empty subset of a semigroup S. Then A is an interior ideal of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy interior ideal of S.*

Proof. The proof is similar to the proof of the Theorem 3.5. □

Every generalized intuitionistic fuzzy ideal of a semigroup S is a generalized intuitionistic fuzzy interior ideal of S. The following example shows that the converse does not hold.

Example 3.10. Let $S = \{a, b, c, d\}$ be a semigroup under the binary operation $(\cdot)$ defined by

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as

$\mu_A(a) = 0.8$ $\mu_A(b) = 0.4$ $\mu_A(c) = 0.5$ $\mu_A(d) = 0.1$ and
$\gamma_A(a) = 0.15$ $\gamma_A(b) = 0.6$ $\gamma_A(c) = 0.4$ $\gamma_A(d) = 0.5$

Let $(\triangle, \bigtriangledown):\ [0,1] \times [0,1] \rightarrow [0,1]$ be functions defined by

$$\alpha \triangle \beta = \alpha\beta \text{ and } \alpha \bigtriangledown \beta = \alpha + \beta - \alpha\beta$$

for all $\alpha, \beta \in [0,1]$. Then $\triangle$ is a t-norm and $\bigtriangledown$ is an s-norm. The intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S which is not generalized intuitionistic fuzzy ideal of S.

As

$$\mu_A(dc) = \mu_A(b) = 0.4 \ngeq 0.5 = \mu_A(c)$$

and

$$\gamma_A(dc) = \gamma_A(b) = 0.6 \nleq 0.4 = \gamma_A(c).$$

So $A = (\mu_{A,}\gamma_A)$ is not a generalized intuitionistic fuzzy left ideal of S, hence it is not a generalized intuitionistic fuzzy ideal of S.

Lemma 3.11. *Let $A = (\mu_{A,}\gamma_A)$ be an intuitionistic fuzzy subsemigroup of a regular semigroup S. Then the following are equivalent*

(1) $A = (\mu_{A,}\gamma_A)$ *is a generalized intuitionistic fuzzy ideal of S.*

(2) $A = (\mu_{A,}\gamma_A)$ *is a generalized intuitionistic fuzzy interior ideal of S.*

Proof. Assume $A = (\mu_{A,}\gamma_A)$ is a generalized intuitionistic fuzzy ideal of S. So

$$\mu_A \circledcirc S \leq \mu_A\ ,\ \gamma_A \circledcirc S^{'} \geq \gamma_A \text{ and } S \circledcirc \mu_A \leq \mu_A\ ,\ S^{'} \circledcirc \gamma_A \geq \gamma_A.$$

Now

$$S \circledcirc \mu_A \circledcirc S \leq \mu_A \circledcirc S \leq \mu_A \text{ and } S^{'} \circledcirc \gamma_A \circledcirc S^{'} \geq \gamma_A \circledcirc S^{'} \geq \gamma_A.$$

Thus $A = (\mu_{A,}\gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S.

Assume that (2) holds. Let a and b be any elements of S. Since S is regular, there exist elements $x, y \in S$ such that $a = axa$ and $b = byb$. Thus

$$\begin{aligned} \mu_A(ab) &= \mu_A((axa)\,b) = \mu_A((ax)\,ab) \geq \mu_A(a) \\ \gamma_A(ab) &= \gamma_A((axa)\,b) = \gamma_A((ax)\,ab) \leq \gamma_A(a). \end{aligned}$$

and

$$\begin{aligned} \mu_A(ab) &= \mu_A(a(byb)) = \mu_A(ab\,(yb)) \geq \mu_A(b) \\ \gamma_A(ab) &= \gamma_A(a(byb)) = \gamma_A(ab\,(yb)) \leq \gamma_A(b). \end{aligned}$$

Therefore $A = (\mu_{A,}\gamma_A)$ is a generalized intuitionistic fuzzy ideal of S. □

Theorem 3.12. *If an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is a generalized intuitionistic fuzzy interior ideal of S, then so is $\Box A = (\mu_A, \bar{\mu}_A)$.*

Proof. It is sufficient to prove that $\bar{\mu}_A$ satisfies

$$\bar{\mu}_A(xy) \leq \bar{\mu}_A(x) \bigtriangledown \bar{\mu}_A(y) \text{ and } \bar{\mu}_A(xay) \leq \bar{\mu}_A(a)$$

for all $a, x, y \in S$.

For any $a, x, y \in S$, we have

$$\begin{aligned}
\bar{\mu}_A(xy) &= 1 - \mu_A(xy) \leq 1 - \{\mu_A(x) \bigtriangleup \mu_A(y)\} \\
&= \eta \{\mu_A(x) \bigtriangleup \mu_A(y)\} \\
&= \eta [\eta\eta \{\mu_A(x)\} \bigtriangleup \eta\eta \{\mu_A(y)\}] \\
&= \eta [\eta \{\eta(\mu_A(x)\} \bigtriangleup \eta \{\eta(\mu_A(y)\}] \\
&= \eta [\eta \{1 - \mu_A(x)\} \bigtriangleup \eta \{1 - \mu_A(y)\}] \\
&= \{1 - \mu_A(x)\} \bigtriangledown \{1 - \mu_A(y)\} \\
&= \bar{\mu}_A(x) \bigtriangledown \bar{\mu}_A(y)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}_A(xay) &= 1 - \mu_A(xay) \\
&\leq 1 - \mu_A(a) = \bar{\mu}_A(a).
\end{aligned}$$

Therefore $\Box A = (\mu_A, \bar{\mu}_A)$ is a generalized intuitionistic fuzzy interior ideal of S. $\Box$

Theorem 3.13. *If an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is a generalized intuitionistic fuzzy interior ideal of S, then $\Diamond A = (\bar{\gamma}_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S.*

Proof. It is sufficient to prove that $\bar{\gamma}_A$ satisfies

$$\bar{\gamma}_A(xy) \geq \bar{\gamma}_A(x) \bigtriangleup \bar{\gamma}_A(y) \text{ and } \bar{\gamma}_A(xay) \geq \bar{\gamma}_A(a)$$

for all $a, x, y \in S$.

For any $a, x, y \in S$, we have

$$\begin{aligned}
\bar{\gamma}_A(xy) &= 1 - \gamma_A(xy) \geq 1 - \{\gamma_A(x) \bigtriangledown \gamma_A(y)\} \\
&= \eta \{\gamma_A(x) \bigtriangledown \gamma_A(y)\} \\
&= \eta [\eta\eta \{\gamma_A(x)\} \bigtriangledown \eta\eta \{\gamma_A(y)\}] \\
&= \eta [\eta \{\eta(\gamma_A(x)\} \bigtriangledown \eta \{\eta(\gamma_A(y)\}] \\
&= \eta [\eta \{1 - \gamma_A(x)\} \bigtriangledown \eta \{1 - \gamma_A(y)\}] \\
&= \{1 - \gamma_A(x)\} \bigtriangleup \{1 - \gamma_A(y)\} \\
&= \bar{\gamma}_A(x) \bigtriangleup \bar{\gamma}_A(y)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\gamma}_A(xay) &= 1 - \gamma_A(xay) \\
&\leq 1 - \gamma_A(a) = \bar{\gamma}_A(a).
\end{aligned}$$

Therefore $\Diamond A = (\bar{\gamma}_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S. $\Box$

Definition 3.14. A fuzzy set μ in a semigroup S is called a generalized fuzzy interior ideal of S if

$$\mu(xy) \geq \mu(x) \bigtriangleup \mu(y)$$

and

$$\mu(xwy) \geq \mu(w)$$

for all $x, y, w \in S$.

Theorem 3.15. *An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S if and only if the fuzzy sets μ_A and $\bar{\gamma}_A$ are generalized fuzzy interior ideal of S.*

Proof. Let $A = (\mu_{A,}\gamma_A)$ be a generalized intuitionistic fuzzy interior ideal of S. Then clearly μ_A is a generalized fuzzy interior ideal of S. Let $x, w, y \in S$. Then

$$\begin{aligned} \bar{\gamma}_A(xy) &= 1 - \gamma_A(xy) \\ &\geq 1 - (\gamma_A(x) \bigtriangledown \gamma_A(y)) \\ &= \eta(\gamma_A(x) \bigtriangledown \gamma_A(y)) \\ &= \eta\,[\eta(1 - \gamma_A(x)) \bigtriangledown \eta\,(1 - \gamma_A(y))] \\ &= (1 - \gamma_A(x)) \bigtriangleup (1 - \gamma_A(y)) \\ &= \bar{\gamma}_A(x) \bigtriangleup \bar{\gamma}_A(y) \end{aligned}$$

and

$$\begin{aligned} \bar{\gamma}_A(xwy) &= 1 - \gamma_A(xwy) \\ &\geq 1 - \gamma_A(w) = \bar{\gamma}_A\,(w)\,. \end{aligned}$$

Hence $\mu_A, \bar{\gamma}_A$ are generalized fuzzy interior ideals of S.

Conversely, suppose that μ_A and $\bar{\gamma}_A$ are generalized fuzzy interior ideals of S. Let $w, x, y \in S$. Then

$$\begin{aligned} 1 - \gamma_A(xy) &= \bar{\gamma}_A(xy) \geq \bar{\gamma}_A(x) \bigtriangleup \bar{\gamma}_A(y) \\ &\geq (1 - \gamma_A(x)) \bigtriangleup (1 - \gamma_A(y)) \\ &\geq \eta\,[\eta(1 - \gamma_A(x)) \bigtriangledown \eta\,(1 - \gamma_A(y))] \\ &\geq \eta\,(\gamma_A(x) \bigtriangledown \gamma_A(y)) \\ 1 - \gamma_A(xy) &\geq 1 - (\gamma_A(x) \bigtriangledown \gamma_A(y)) \end{aligned}$$

which implies that

$$\gamma_A(xy) \leq \gamma_A(x) \bigtriangledown \gamma_A(y)$$

and

$$\begin{aligned} 1 - \gamma_A(xwy) &= \bar{\gamma}_A(xwy) \geq \bar{\gamma}_A(w) \\ &\geq (1 - \gamma_A(w)) \end{aligned}$$

which implies that

$$\gamma_A(xwy) \leq \gamma_A(w).$$

Therefore $A = (\mu_{A,}\gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S. □

Corollary 3.16. *An intuitionistic fuzzy set $A = (\mu_{A,}\gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S if and only if $\Box A = (\mu_A,\ \bar{\mu}_A)$ and $\Diamond A = (\bar{\gamma}_A,\ \gamma_A)$ are generalized intuitionistic fuzzy interior ideal of S.*

Theorem 3.17. *Let $A = (\mu_{A,}\gamma_A)$ be a generalized intuitionistic fuzzy interior ideal of S. Then the upper level set $U(\mu_A\colon 1)$ and lower level set $L(\gamma_A\colon 0)$ are either empty or an interior ideal of S.*

Proof. (a) Let $x, y \in U(\mu_A\colon 1)$. Then $\mu_A(x) = 1 = \mu_A(y)$. Since $\mu_A(xy) \geq \mu_A(x) \bigtriangleup \mu_A(y) = 1 \bigtriangleup 1 = 1$, we have $xy \in U(\mu_A\colon 1)$. Now let $x, y \in S$ and $w \in U(\mu_A\colon 1)$. Then $\mu_A(xwy) \geq \mu_A(w) = 1$. Hence $xwy \in U(\mu_A\colon 1)$. Hence $U(\mu_A\colon t)$ is an interior ideal of S.

(b) Let $x, y \in L(\gamma_A\colon 0)$. Then $\gamma_A(x) = 0 = \gamma_A(y)$. Since $\gamma_A(xy) \leq \gamma_A(x) \bigtriangledown \gamma_A(y) = 0 \bigtriangledown 0 = 0$, we have $xy \in L(\gamma_A\colon 0)$. Now let $x, y \in S$ and $w \in L(\gamma_A\colon t)$. Then, $\gamma_A(xwy) \leq \gamma_A(w) = 0$, Hence $xwy \in L(\gamma_A\colon 0)$. Hence $L(\gamma_A\colon t)$ is an interior ideal of S. □

Theorem 3.18. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in S such that the non-empty sets $U(\mu_A\colon t)$ and $L(\gamma_A\colon t)$ are interior ideals of S for all $t \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S.*

Proof. Assume that the non-empty sets $U(\mu_A\colon t)$ and $L(\gamma_A\colon t)$ are interior ideals of S for all $t \in [0, 1]$. Suppose $x, y \in S$ such that

$$\mu_A(xy) < \mu_A(x) \bigtriangleup \mu_A(y).$$

Take $t' = \frac{1}{2}(\mu_A(xy) + \mu_A(x) \bigtriangleup \mu_A(y))$. Then $t' \in (0, 1]$ and

$$\mu_A(xy) < t' < \mu_A(x) \bigtriangleup \mu_A(y) \leq \min(\mu_A(x), \mu_A(y)).$$

This implies that $\mu_A(x) > t'$ and $\mu_A(y) > t'$, so $x, y \in U(\mu_A\colon t')$ but $xy \notin U(\mu_A\colon t')$. This shows that $U(\mu_A\colon t')$ is not subsemigroup of S. Which is a contradiction. Hence μ_A satisfies the inequality

$$\mu_A(xy) \geq \mu_A(x) \bigtriangleup \mu_A(y)$$

for all $x, y \in S$. Similarly, for $w, x, y \in S$ it satisfies

$$\mu_A(xwy) \geq \mu_A(w).$$

Now suppose, there exist $x, y \in S$ such that

$$\gamma_A(xy) > \gamma_A(x) \bigtriangledown \gamma_A(y).$$

Take $t'' = \frac{1}{2}(\gamma_A(xy) + \gamma_A(x) \bigtriangledown \gamma_A(y))$. Then $t'' \in (0, 1]$ and

$$\max(\gamma_A(x), \gamma_A(y)) \leq \gamma_A(x) \bigtriangledown \gamma_A(y) < t'' < \gamma_A(xy).$$

This implies that $\gamma_A(x) < t''$ and $\gamma_A(y) < t''$. Thus $x, y \in L(\gamma_A\colon t'')$ but $xy \notin L(\gamma_A\colon t)$. This shows that $L(\gamma_A\colon t'')$ is not a subsemigroup of S. Which is a contradiction. Hence γ_A satisfies the inequality

$$\gamma_A(xy) \leq \gamma_A(x) \bigtriangledown \gamma_A(y)$$

for all $x, y \in S$. Similarly, for $w, x, y \in S$ it satisfies

$$\gamma_A(xwy) \leq \gamma_A(w).$$

Hence $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy interior ideal of S. □

Let f be a map from a set X to a set Y. If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy set in Y, then the pre-image of B under f, denoted by $f^{-1}(B)$, is an intuitionistic fuzzy set in X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)).$$

where $\left(f^{-1}(\mu_B)\,(x)\right) = \mu_B(f(x))$ and $\left(f^{-1}(\gamma_B)\,(x)\right) = \gamma_B(f(x))$, for all $x \in X$.

Theorem 3.19. *Let $f\colon S \to T$ be a homomorphism of semigroups. If $B = (\mu_B, \gamma_B)$ is a generalized intuitionistic fuzzy interior ideal of T, then the pre-image $f^{-1}(B) = (f^{-1}(\mu_B),\ f^{-1}(\gamma_B))$ of B under f is a generalized intuitionistic fuzzy interior ideal of S.*

Proof. Suppose $B = (\mu_B, \gamma_B)$ be a generalized intuitionistic fuzzy interior ideal of T and $x, y \in S$. Then

$$\begin{aligned}\left(f^{-1}(\mu_B\right)(xy)) &= \mu_B(f(xy)) = \mu_B(f(x)f(y)) \\ &\geq \mu_B(f(x)) \triangle \mu_B(f(y)) \\ &\geq \left(f^{-1}(\mu_B\right)(x)) \triangle \left(f^{-1}(\mu_B\right)(y))\end{aligned}$$

and

$$\begin{aligned}\left(f^{-1}(\gamma_B\right)(xy)) &= \gamma_B(f(xy)) = \gamma_B(f(x)f(y)) \\ &\leq \gamma_B(f(x)) \bigtriangledown \gamma_B(f(y)) \\ &\leq \left(f^{-1}(\gamma_B\right)(x)) \bigtriangledown \left(f^{-1}(\gamma_B\right)(y))\end{aligned}$$

Hence $f^{-1}(B) = (f^{-1}(\mu_B),\ f^{-1}(\gamma_B))$ is a generalized intuitionistic fuzzy subsemigroup of S. Now for any $w, x, y \in S$ we have

$$\begin{aligned}\left(f^{-1}(\mu_B\right)(xwy)) &= \mu_B(f(xwy)) \\ &= \mu_B(f(x)f(w)f(y)) \\ &\geq \mu_B(f(w)) = \left(f^{-1}(\mu_B\right)(w)\end{aligned}$$

and

$$\begin{aligned}\left(f^{-1}(\gamma_B\right)(xwy)) &= \gamma_B(f(xwy)) \\ &= \gamma_B(f(x)f(w)f(y)) \\ &\leq \gamma_B(f(w)) = \left(f^{-1}(\gamma_B\right)(w).\end{aligned}$$

Therefore $f^{-1}(B) = (\left(f^{-1}(\mu_B\right)),\ f^{-1}(\gamma_B))$ is a generalized intuitionistic fuzzy interior ideal of S. □

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