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Some results on T-fuzzy ideals of Γ-near-rings

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Abstract. In this paper the notion of fuzzy ideals of a Γ-near-ring with respect to a t-norm is introduced and investigated some related properties. This concept of T-fuzzy ideals of a Γ-near-ring is a generalization of the concept of T-fuzzy ideals in near-rings. Also the notions of T-fuzzy ideals of a Γ-near-ring, quotient Γ-near-ring with respect to a t-norm and the sum of T-fuzzy ideals of a Γ-near-ring are introduced. Further, it is shown that an onto homomorphic image of a T-fuzzy ideal with Sup property is a T-fuzzy ideal and an epimorphic pre-image of a T-fuzzy ideal of a Γ-near-ring is a T-fuzzy ideal.

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1. Introduction

The notion of a fuzzy set was introduced by L. A. Zadeh [14] in 1965. In 1971, A. Rosenfeld [9] used the notion of a fuzzy subset of a set to introduce the concept of a fuzzy subgroup of a group. Rosenfeld's paper inspired the development of fuzzy abstract algebra. W. J. Liu [7] studied fuzzy ideals in rings and Bh. Satyanarayana [10] introduced Γ-near-rings. In [6] W. A. Dudek and Y. B. Jun introduced fuzzy subgroups over a t-norm. In [12] M. Shabir and M. Hussan characterized the sum of fuzzy ideals. In [13] Srinivas, Nagaiah and Narasimha Swamy studied anti fuzzy ideals of Γ-near-rings. P. Deena, G. Mohanraj [5] and M. Akram [1] have studied several properties of T-fuzzy ideals of rings and T-fuzzy ideals of near-rings. We extended the results of Akram [1] to Γ-near-rings. For more other study on the fuzzy theory in Γ-near-rings we refer [3, 8, 12].

In this paper, by using the t-norm T, we define T-fuzzy ideals of a Γ-near-ring and prove that every fuzzy ideal of a Γ-near-ring is a T-fuzzy ideal. Also we prove that an onto homomorphic image of a T-fuzzy ideal with Sup property is a T-fuzzy ideal and an epimorphic pre-image of a T-fuzzy ideal of a Γ-near-ring is a T-fuzzy ideal. Further we introduce the notion of direct product of T-fuzzy ideals and prove that the sum of T-fuzzy left ideals of a Γ-near-ring is a T-fuzzy left ideal. The proofs are almost similar to that of T-fuzzy ideals in near-rings [1] and fuzzy algebras on K(G)-algebras [4].

2. Preliminaries

In this section we summarize the preliminary definitions that will be required in this paper. Most of the contents of this section are contained in [1, 2, 4, 13].

Definition 2.1. A non-empty set N with two binary operations " + " (addition) and " $\cdot$ " (multiplication) is called a near-ring if it satisfies the following axioms:
(i) $(N, +)$ is a group,
(ii) $(N, \cdot)$ is a semigroup,
(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$,
for all $x, y, z \in N$.

Precisely speaking it is a right near-ring, because it satisfies the right distributive law. We will use the word "near-ring" to mean "right near-ring". We denote xy instead of $x \cdot y$. Moreover, a near-ring N is said to be a zero-symmetric if $r \cdot 0 = 0$ for all $r \in N$, where 0 is the additive identity in N.

Definition 2.2. Let $(R, +)$ be a group and Γ be a non empty set. Then R is said to be a Γ-near-ring if there exists a mapping $R \times \Gamma \times R \to R$ (The image of (x, α, y) is denoted by $x\alpha y$) satisfying the following conditions:
(i) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$
for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

Definition 2.3. Let R be a Γ-near-ring. A normal subgroup $(I, +)$ of $(R, +)$ is called
(i) a left ideal if $x\alpha(y + i) - x\alpha y \in I$ for all $x, y \in R, \alpha \in \Gamma, i \in I$,
(ii) a right ideal if $i\alpha x \in I$ for all $x \in R, \alpha \in \Gamma, i \in I$,
(iii) an ideal if it is both a left ideal and a right ideal of R.
A Γ-near-ring R is said to be a zero-symmetric if $a\alpha 0 = 0$ for all $a \in R$ and $\alpha \in \Gamma$, where 0 is the additive identity in R.

Definition 2.4. A sub set M of a Γ-near-ring R is said to a sub Γ-near-ring if there exists a mapping $M \times \Gamma \times M \to M$ such that
(i) $(M, +)$ be a subgroup of $(R, +)$,
(ii) $(x + y)\gamma z = x\gamma z + y\gamma z$ for every $x, y, z \in M$ and $\gamma \in \Gamma$,
(ii) $(x\gamma y)\omega z = x\gamma(y\omega z)$ for every $x, y, z \in M$ and $\gamma, \omega \in \Gamma$.

Definition 2.5. Let R be a Γ-near-ring. A fuzzy set of R is a function $A : R \to [0, 1]$. Let A be a fuzzy set of R. For $\alpha \in [0, 1]$, the set $U(A; \alpha) = \{x \in R : A(x) \geq \alpha\}$ is called a level set of A.

Definition 2.6. A fuzzy subset A of a Γ-near-ring R is said to be a fuzzy sub Γ-near-ring of R if it satisfies the following conditions:
(P1) $A(x - y) \geq \min\{A(x), A(y)\}$ for all $x, y \in R$,

(P2) $A(x\alpha y) \geq \min\{A(x), A(y)\}$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Definition 2.7. A fuzzy sub Γ-near-ring A of R is called a fuzzy ideal if it satisfies the following conditions:
(P3) $A(y + x - y) \geq A(x)$ for all $x, y, z \in R$,
(P4) $A(x\alpha y) \geq A(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$,
(P5) $A(x\alpha(z + y) - x\alpha y) \geq A(z)$ for all $x, y, z \in R$ and $\alpha \in \Gamma$.

Note that if A is a fuzzy ideal of a Γ-near-ring R then $A(0) \geq A(x)$ for all $x \in R$.

Definition 2.8 ([1]). A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions:
(N1) $T(x, 1) = x$,
(N2) $T(x, y) = T(y, x)$,
(N3) $T(x, T(y, z)) = T(T(x, y), z)$,
(N4) $T(x, y) \leq T(x, z)$ whenever $y \leq z$,
for all $x, y, z \in [0, 1]$.

For a t-norm T, let Δ_T denote the set of elements $\alpha \in [0, 1]$ such that $T(\alpha, \alpha) = \alpha$, that is $\Delta_T = \{\alpha \in [0, 1] : T(\alpha, \alpha) = \alpha\}$. Note that every t-norm T has a useful property $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.9. Let A and B be the fuzzy subsets of a non-empty set X. A fuzzy subset $A \wedge B$ is defined by

$$(A \wedge B)(x) = T(A(x), B(x))$$

for all $x \in X$.

Definition 2.10. Let R_1 and R_2 be two Γ-near-rings. A mapping $f : R_1 \to R_2$ is called a Γ-near-ring homomorphism if $f(x+y) = f(x)+f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma$. If f is one-to-one and onto, we say that f is a Γ-near-ring isomorphism.

Definition 2.11 ([9]). A fuzzy set μ of a Γ-near-ring R has the Sup property if for any subset N of R, there exists $a_0 \in N$ such that

$$\mu(a_o) = \sup_{a \in N} \mu(a).$$

Definition 2.12. Let A and B be two fuzzy ideals of a Γ-near-ring R. Then the sum $A + B$ is a fuzzy set of R defined by

$$(A + B)(x) = \begin{cases} sup(min(A(y), B(z))) & \text{if } x = y + z \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y, z \in R$.

Definition 2.13. A fuzzy ideal A of a Γ-near-ring R is said to be normal if there exists an element $a \in R$ such that $A(a) = 1$.

We note that A is normal of a Γ-near-ring R if and only if $A(1) = 1$.

Definition 2.14. Let M and N be any two sets and let $f : M \to N$ be any function. A fuzzy subset μ of M is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in M$.

3. T-fuzzy ideals of Γ-near-rings

In this section we introduce the concepts of T-fuzzy sub Γ-near-ring and T-fuzzy ideal of a Γ-near-ring.

Definition 3.1. A fuzzy set A of a Γ-near-ring R is called a fuzzy sub Γ-near-ring with respect to a t-norm (shortly, T-fuzzy sub Γ-near-ring) of R if
(NP1) $A(x-y) \geq T(A(x), A(y))$ for all $x, y \in R$,
(NP2) $A(x\alpha y) \geq T(A(x), A(y))$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Definition 3.2. Let R be a $\Gamma-$near-ring. Then T-fuzzy sub Γ-near-ring A of R is called a T-fuzzy ideal of R if it satisfies the following conditions:
(NP3) $A(y+x-y) \geq A(x)$ for all $x, y \in R$,
(NP4) $A(x\alpha y) \geq A(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$,
(NP5) $A(x\alpha(z+y) - x\alpha y) \geq A(z)$ for all $x, y, z \in R$ and $\alpha \in \Gamma$.

Note that A is a T-fuzzy left ideal of R if it satisfies (NP1), (NP2), (NP3) and (NP4) and A is a T-fuzzy right ideal of R if it satisfies (NP1), (NP2), (NP3) and (NP5). A is called a T-fuzzy ideal of R if A is both left and right T-fuzzy ideal of R.

Example 3.3. Let $R = \{0, a, b, c\}$ and $\Gamma = \{0_\Gamma, 1\}$. Define a binary operation "+" on R and a mapping $R \times \Gamma \times R \to R$ by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

0_Γ	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

1	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Clearly, $(R, +)$ is a group and
(i) $(x+y)\gamma z = x\gamma z + y\gamma z$ for every $x, y, z \in R$, $\gamma \in \Gamma$,
(ii) $(x\gamma y)\omega z = x\gamma(y\omega z)$ for every $x, y, z \in R$ and $\gamma, \omega \in \Gamma$.
Thus R is a Γ-near-ring. Let T be a t-norm defined by $T(\alpha, \beta) = \max\ (\alpha+\beta-1, 0)$ for all $\alpha, \beta \in [0, 1]$. Define a fuzzy set $A : R \to [0, 1]$ by $A(0) \geq A(a) = A(b) = A(c)$. Then it can be easily verified that A is a T-fuzzy ideal of R.

Example 3.4. If $(G, +)$ is a non-abelian group and X is a non-empty set then

$$R = \{f \mid f \text{ is a mapping from X to G}\}$$

is a non-abelian group under the point wise addition. Let

$$\Gamma = \{g \mid g \text{ is a mapping from G to X}\}.$$

Let $f_1, f_2 \in R, g \in \Gamma$ then $f_1 g f_2 \in R$. Then the map $R \times \Gamma \times R \to R$ satisfies the following:
(i) $(f_1 g_1 f_2) f_3 = f_1 g_1 (f_2 g_2 f_3)$ and
(ii) $(f_1 + f_2) g_1 f_3 = f_1 g_1 f_3 + f_2 g_1 f_3$
for all $f_1, f_2, f_3 \in R$ and for all $g_1, g_2 \in \Gamma$. Thus R is a Γ-near-ring. Let T be a t-norm defined by

$$T(\alpha, \beta) = \max(\alpha+\beta-1,\ 0) \text{ for all } \alpha, \beta \in [0,\ 1].$$

Define a fuzzy set $A : R \to [0,\ 1]$ by

$$A(0_R) = 0.7 \text{ and } A(f) = 0.3 \text{ where } f \text{ is any element of } R \text{ with } f \neq 0_R.$$

Then it can be easily verified that A is a T-fuzzy ideal of a Γ-near-ring R.

Example 3.5. Let $R = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. Define a binary operation " $+$ " on R and a mapping $R \times \Gamma \times R \to R$ by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

α	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

Clearly, $(R, +)$ is a group and

(i) $(x + y)\gamma z = x\gamma z + y\gamma z$ for every $x, y, z \in R$, $\gamma \in \Gamma$,

(ii)$(x\gamma y)\omega z = x\gamma(y\omega z)$ for every $x, y, z \in R$ and $\gamma, \omega \in \Gamma$.

Then R is a Γ-near-ring. Let T be a t-norm defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0) \text{ for all } \alpha, \beta \in [0, 1].$$

Define a fuzzy subset $A : R \to [0, 1]$ by

$$A(0) = 0.9 \text{ and } A(x) = 0.4 \text{ for all } x \neq 0.$$

The routine calculation shows that A is a T-fuzzy ideal of a Γ-near-ring R.

Definition 3.6 ([5]). Let A and B be T-fuzzy ideals of a Γ-near-ring R. Then the direct product of T-fuzzy ideals is defined by $(A \times B)(x, y) = T(A(x), B(y))$ for all $x, y \in R$.

Definition 3.7 ([2]). Let R be a Γ-near-ring. Let μ be a fuzzy set of a T-fuzzy ideal of R and f be a function defined on R, then the fuzzy set μ^f in $f(R)$ is defined by

$$\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(R)$ and is called the image of μ under f. Similarly, if ν is a fuzzy set in $f(R)$, then $\mu = \nu \circ f$ in R is defined as $\mu(x) = \nu(f(x))$ for all $x \in R$ and is called the pre-image of ν under f.

4. Main results

The following theorems can be proved similar to that of fuzzy subquasigroups over a t-norm [6] and in T-fuzzy ideals in rings [5].

Theorem 4.1. *Let A be a T-fuzzy ideal of a Γ-near-ring R and $\alpha \in [0, 1]$.*

(i) *If $\alpha = 1$, then $U(A; \alpha)$ is either empty or an ideal of R.*

(ii) *If $T = \min$, then $U(A; \alpha)$ is either empty or an ideal of R.*

(iii) *$A(0) \geq A(x)$ for all $x \in R$.*

Theorem 4.2. *Every fuzzy ideal of a Γ-near-ring R is a T-fuzzy ideal of R.*

Theorem 4.3. *If $A_i, i \in I$, is a T-fuzzy ideal of a Γ-near-ring R then $\bigwedge_{i \in I} A_i$ is also a T-fuzzy ideal of R where $\bigwedge_{i \in I} A_i$ is defined by $(\bigwedge_{i \in I} A_i)(x) = \inf\{A_i(x) : i \in I\}$ for all $x \in R$.*

For the sake of understanding the above theorem first we prove the following result and then we give an example.

Theorem 4.4. *If A and B are T-fuzzy ideals of a* Γ*-near-ring R then* $A \wedge B$ *is a T-fuzzy ideal of R.*

Proof. Let A and B be T-fuzzy ideals of a Γ-near-ring R. Let $x, y, z \in R$ and $\alpha \in \Gamma$. Then

$$(A \wedge B)(x-y) = T(A(x-y), B(x-y)) \geq T(T(A(x), A(y)), T(B(x), B(y)))$$
$$= T(T(T(A(x), A(y)), B(x)), B(y)) = T(T(T(A(x), B(x)), A(y)), B(y))$$
$$= T(T(A(x), B(x)), T(A(y), B(y))) = T((A \wedge B)(x), (A \wedge B)(y)).$$

Since A and B are T-fuzzy ideals of R, we have $A(x\alpha y) \geq T(A(x), A(y))$ and $B(x\alpha y) \geq T(B(x), B(y))$. Also we have

$$(A \wedge B)(x\alpha y) = T(A(x\alpha y), B(x\alpha y)) \geq T(T(A(x), A(y)), T(B(x), B(y)))$$
$$= T(T(T(A(x), A(y)), B(x)), B(y)) = T(T(T(A(x), B(x)), A(y)), B(y))$$
$$= T(T(A(x), B(x)), T(A(y), B(y))) = T((A \wedge B)(x), (A \wedge B)(y)).$$

Since $A(x\alpha y) \geq A(y)$ and $B(x\alpha y) \geq B(y)$, we have

$$\begin{aligned} (A \wedge B)(x\alpha y) &= T(A(x\alpha y), B(x\alpha y)) \\ &\geq T(A(y), B(y)) \\ &= (A \wedge B)(y). \end{aligned}$$

Since $A(y + x - y) \geq A(x)$ and $B(y + x - y) \geq B(x)$, we have

$$\begin{aligned} (A \wedge B)(y + x - y) &= T(A(y + x - y), B(y + x - y)) \\ &\geq T(A(x), B(x)) \\ &= (A \wedge B)(x). \end{aligned}$$

Since $A(x\alpha(z + y) - x\alpha y) \geq A(z)$ and $B(x\alpha(z + y) - x\alpha y) \geq B(z)$, we have

$$(A \wedge B)(x\alpha(z + y) - x\alpha y) = T(A(x\alpha(z + y) - x\alpha y), B(x\alpha(z + y) - x\alpha y))$$
$$\geq T(A(z), B(z)) = (A \wedge B)(z).$$

Hence $A \wedge B$ is a T-fuzzy ideal of R. This completes the proof. □

Example 4.5. Let $R = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be a Γ-near-ring as in example 3.5. We have $(A \wedge B)(x) = T(A(x), B(x))$ for all $x \in R$ (by definition 2.9). Define a t-norm T by $T(p, q) = \max(p + q - 1,\ 0)$ for all $p, q \in [0,\ 1]$. Define a fuzzy subset $A : R \to [0, 1]$ by $A(0) = 0.9$ and $A(a) = A(b) = A(c) = 0.4$, where $0, a, b, c \in R$. Then $A = \{(0, 0.9), (a, 0.4), (b, 0.4), (c, 0.4)\}$. Again define a fuzzy subset $B : R \to [0, 1]$ by $B(0) = 0.7$ and $B(a) = 0.6, B(b) = 0.5, B(c) = 0.4$, where $0, a, b, c \in R$. Then $B = \{(0, 0.7), (a, 0.6), (b, 0.5), (c, 0.4)\}$. Let $0, a, b, c \in R$. Then

$$(A \wedge B)(0) = T(A(0), B(0)) = T(0.9, 0.7) = \max(0.9 + 0.7 - 1, 0)$$
$$= \max(1.6 - 1, 0) = \max(0.6, 0) = 0.6.$$
$$(A \wedge B)(a) = T(A(a), B(a)) = T(0.4, 0.6) = \max(0.4 + 0.6 - 1, 0) = 0.$$
$$(A \wedge B)(b) = T(A(b), B(b)) = T(0.4, 0.5) = \max(0.4 + 0.5 - 1, 0)$$
$$= \max(0.9 - 1, 0) = \max(-0.1, 0) = 0.$$
$$(A \wedge B)(c) = T(A(c), B(c)) = T(0.4, 0.4) = \max(0.4 + 0.4 - 1, 0)$$
$$= \max(-0.2, 0) = 0.$$

So $A \wedge B = \{(0, 0.6), (a, 0), (b, 0), (c, 0)\}$ is a fuzzy subset on R, that is, $A \wedge B : R \to$

$[0,1]$ defined by $(A \wedge B)(0_R) = 0.6, (A \cap B)(x) = 0$ for all $x \neq 0_R$. Then it can be easily verified that $A \wedge B$ is a T-fuzzy ideal of a Γ-near-ring R.

Now we state the following theorem which can be proved similar to that of T-fuzzy ideals of near-rings [1].

Theorem 4.6. *If $A_i, i \in I$, is a T-fuzzy ideal of a Γ-near-ring R, then $\bigvee_{i\in I} A_i$ is also a T-fuzzy ideal of R where $\bigvee_{i\in I} A_i$ is defined by $(\bigvee_{i\in I} A_i)(x) = \sup\{A_i(x) : i \in I\}$ for all $x \in R$.*

Theorem 4.7. *Let A be a T-fuzzy ideal of a Γ-near-ring R and $A^\star$ be a fuzzy set in R defined by $A^\star(x) = \frac{A(x)}{A(1)}$ for all $x \in R$. Then $A^\star$ is normal T-fuzzy ideal of R contains A.*

Proof. Let A be a T-fuzzy ideal of a Γ-near-ring R. For any $x, y, z \in R$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} A^\star(x-y) &= \tfrac{A(x-y)}{A(1)} \geq \tfrac{1}{A(1)}(T(A(x), T(A(y))) \\ &= T(\tfrac{1}{A(1)}A(x), \tfrac{1}{A(1)}A(y)) = T(A^\star(x), A^\star(y)), \end{aligned}$$

and

$$\begin{aligned} A^\star(x\alpha y) &= \tfrac{A(x\alpha y)}{A(1)} \geq \tfrac{1}{A(1)}(T(A(x), T(A(y))) \\ &= T(\tfrac{1}{A(1)}A(x), \tfrac{1}{A(1)}A(y)) = T(A^\star(x), A^\star(y)). \end{aligned}$$

This shows that $A^\star$ is a T-fuzzy sub Γ-near-ring of R.

$$A^\star(y+x-y) = \tfrac{A(y+x-y)}{A(1)} \geq \tfrac{1}{A(1)}A(x) = A^\star(x),$$

$$A^\star(x\alpha y) = \tfrac{A(x\alpha y)}{A(1)} \geq \tfrac{1}{A(1)}A(y) = A^\star(y),$$

and

$$A^\star(x\alpha(z+y) - x\alpha y) = \tfrac{A(x\alpha(z+y)-x\alpha y)}{A(1)} \geq \tfrac{1}{A(1)}A(z) = A^\star(z).$$

Hence $A^\star$ is a T-fuzzy ideal of R. Clearly $A^\star(1) = \frac{1}{A(1)}A(1) = 1$ and $A \subset A^\star$. This completes the proof. □

Theorem 4.8. *Let A be a T-fuzzy ideal of a Γ-near-ring R and let A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(1)$ for all $x \in R$. Then A^+ is normal T-fuzzy ideal of R containing A.*

Proof. Let A be a T-fuzzy ideal of a Γ-near-ring R. For any $x, y, z \in R$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} A^+(x-y) &= A(x-y) + 1 - A(1) \geq T(A(x), A(y)) + 1 - A(1) \\ &= T(A(x) + 1 - A(1), A(y) + 1 - A(1)) = T(A^+(x), A^+(y)), \\ A^+(x\alpha y) &= A(x\alpha y) + 1 - A(1) \geq T(A(x), A(y)) + 1 - A(1) \\ &= T(A(x) + 1 - A(1), A(y) + 1 - A(1)) = T(A^+(x), A^+(y)). \\ A^+(y+x-y) &= A(y+x-y) + 1 - A(1) \geq A(x) + 1 - A(1) = A^+(x), \\ A^+(x\alpha y) &= A(x\alpha y) + 1 - A(1) \geq A(y) + 1 - A(1) = A^+(y), \end{aligned}$$

and

$$\begin{aligned} A^+(x\alpha(z+y) - x\alpha y) &= A(x\alpha(z+y) - x\alpha y) + 1 - A(1) \\ &\geq A(z) + 1 - A(1) = A^+(z). \end{aligned}$$

Hence A^+ is a T-fuzzy ideal of a Γ-near-ring R. Clearly $A^+(1) = 1$ and $A \subset A^+$. This completes the proof. □

Theorem 4.9. *An onto homomorphic image of a T-fuzzy ideal with Sup property is a T-fuzzy ideal.*

Proof. Let R and S be Γ-near-rings. Let $f : R \to S$ be a epimorphism and A be a T-fuzzy ideal of R with sup property. Let $x^{'}, y^{'} \in S$, $x_0 \in f^{-1}(x^{'}), y_0 \in f^{-1}(y^{'})$ and $z_0 \in f^{-1}(z^{'})$ be such that $A(x_o) = \sup\limits_{n\in f^{-1}(x')} A(n), A(y_o) = \sup\limits_{n\in f^{-1}(y')} A(n), A(z_o) = \sup\limits_{n\in f^{-1}(z')} A(n)$ respectively. Then for any $\alpha \in \Gamma$, we have

$$\begin{aligned}
A^f(x^{'} - y^{'}) &= \sup_{z\in f^{-1}(x'-y')} A(z) \geq A(x_0 - y_0) \geq \min(A(x_0), A(y_0)) \\
&\geq T(A(x_0), A(y_0)) = T(\sup_{n\in f^{-1}(x')} A(n), \sup_{n\in f^{-1}(y')} A(n)) \\
&= T(A^f(x^{'}), A^f(y^{'})), \\
A^f(x^{'}\alpha y^{'}) &= \sup_{z\in f^{-1}(x'\alpha y')} A(z) \geq A(x_0\alpha y_0) \geq \min(A(x_0), A(y_0)) \\
&\geq T(A(x_0), A(y_0)) = T(\sup_{n\in f^{-1}(x')} A(n), \sup_{n\in f^{-1}(y')} A(n)) \\
&= T(A^f(x^{'}), A^f(y^{'})), \\
A^f(y^{'} + x^{'} - y^{'}) &= \sup_{z\in f^{-1}(y'+x'-y')} A(z) \geq A(y_0 + x_0 - y_0) \\
&\geq A(x_0) = \sup_{n\in f^{-1}(x')} A(n) = A^f(x^{'}), \\
A^f(x^{'}\alpha y^{'}) &= \sup_{z\in f^{-1}(x'\alpha y')} A(z) \geq A(x_0\alpha y_0) \geq A(y_0) = \sup_{n\in f^{-1}(y')} A(n) = A^f(y^{'}),
\end{aligned}$$

and

$$\begin{aligned}
A^f(x^{'}\alpha(z^{'} + y^{'}) - x^{'}\alpha y^{'}) &= \sup_{z\in f^{-1}(x'\alpha(z'+y')-x'\alpha y')} A(z) \\
&\geq A(x_0\alpha(z_0 + y_0) - x_o\alpha y_0)) \geq A(z_0) = \sup_{n\in f^{-1}(z')} A(n) = A^f(z^{'}).
\end{aligned}$$

This completes the proof. □

Theorem 4.10. *An epimorphic pre-image of a T-fuzzy ideal of a Γ-near-ring is a T-fuzzy ideal.*

Proof. Let R and S be Γ-near-rings. Let $f : R \to S$ be an epimorphism. Let ν be a T-fuzzy ideal of S and μ be the pre-image of ν under f. Then for any $x, y, z \in R$ and $\alpha \in \Gamma$, we have

$$\begin{aligned}
\mu(x - y) &= (\nu \circ f)(x - y) = \nu(f(x - y)) = \nu(f(x) - f(y)) \\
&\geq T(\nu(f(x)), \nu(f(y))) = T((\nu \circ f)(x), (\nu \circ f)(y)) = T(\mu(x), \mu(y)), \\
\mu(x\alpha y) &= (\nu \circ f)(x\alpha y) = \nu(f(x\alpha y)) = \nu(f(x)\alpha f(y)) \\
&\geq T(\nu(f(x)), \nu(f(y))) = T((\nu \circ f)(x), (\nu \circ f)(y)) = T(\mu(x), \mu(y)), \\
\mu(y + x - y) &= (\nu \circ f)(y + x - y) = \nu(f(y + x - y)) = \nu(f(y) + f(x) - f(y)) \\
&\geq \nu(f(x)) = (\nu \circ f)(x) = \mu(x), \\
\mu(x\alpha y) &= (\nu \circ f)(x\alpha y) = \nu(f(x\alpha y)) = \nu(f(x)\alpha f(y)) \\
&\geq \mu(f(y)) = (\nu \circ f)(y) = \mu(y),
\end{aligned}$$

and

$$\begin{aligned}
\mu(x\alpha(z + y) - x\alpha y) &= (\nu \circ f)(x\alpha(z + y) - x\alpha y) \\
&= \nu(f(x\alpha(z + y) - x\alpha y)) = \nu(f(x)\alpha f(z))
\end{aligned}$$

$\geq \nu(f(z)) = (\nu \circ f)(z) = \mu(z).$

Hence μ is a T-fuzzy ideal of a Γ-near-ring R. This completes the proof. □

Lemma 4.11 ([13]). . *Let R and S be Γ-near-rings and $f : R \to S$ be a homomorphism. Let A be f-invariant fuzzy ideal of R. If $x = f(a)$, then $f(A)(x) = A(a)$ for all $a \in R$.*

Theorem 4.12. *Let $f : R \to S$ be an epimorphism of Γ-near-rings R and S. If A is f-invariant T-fuzzy ideal of R, then $f(A)$ is a T-fuzzy ideal of S.*

Proof. Let $a, b, c \in S$ and $\alpha \in \Gamma$. Then there exists $x, y, z \in R$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. Suppose A is f-invariant T-fuzzy ideal of R, then by Lemma 4.11

$$
\begin{aligned}
f(A)(a-b) &= f(A)(f(x)-f(y)) = f(A)(f(x-y)) = A(x-y)\\
&\geq T(A(x), A(y)) = T(f(A)(a), f(A)(b)),\\
f(A)(a\alpha b) &= f(A)(f(x)\alpha f(y)) = f(A)(f(x\alpha y)) = A(x\alpha y)\\
&\geq T(A(x), A(y)) = T(f(A)(a), f(A)(b)),\\
f(A)(b+a-b) &= f(A)(f(y)+f(x)-f(y)) = f(A)(f(y+x-y))\\
&= A(y+x-y) \geq A(x) = f(A)(a),\\
f(A)(a\alpha b) &= f(A)(f(x)\alpha f(y)) = f(A)(f(x\alpha y)) = A(x\alpha y) \geq A(x) = f(A)(b),
\end{aligned}
$$

and

$$
\begin{aligned}
f(A)[a\alpha(c+b) - a\alpha b] &= f(A)[f(x)\alpha(f(z)+f(y)) - f(x)\alpha f(y)]\\
&= f(A)[f(x\alpha(z+y) - x\alpha y)] = A[x\alpha(z+y) - x\alpha y]\\
&\geq A(z) = f(A)(c).
\end{aligned}
$$

Hence $f(A)$ is a T-fuzzy ideal of S. This completes the proof. □

Theorem 4.13. *Let R_1 and R_2 be Γ-near-rings. If A_1 and A_2 are T-fuzzy ideals of R_1 and R_2 respectively, then $A = A_1 \times A_2$ is a T-fuzzy ideal of the direct product $R_1 \times R_2$.*

Proof. Let A_1 and A_2 be T-fuzzy ideals of a Γ-near-rings R_1 and R_2 respectively. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in R_1 \times R_2$ and $\alpha \in \Gamma$. Then

$$
\begin{aligned}
&A((x_1,x_2)-(y_1,y_2)) = A(x_1-y_1, x_2-y_2) = (A_1\times A_2)(x_1-y_1, x_2-y_2)\\
&\quad = T(A_1(x_1-y_1), A_2(x_2-y_2)) \geq T(T(A_1(x_1), A_1(y_1)), T(A_2(x_2), A_2(y_2)))\\
&\quad = T(T(A_1(x_1), A_2(x_2)), T(A_1(y_1), A_2(y_2)))\\
&\quad = T((A_1\times A_2)(x_1,x_2), (A_1\times A_2)(y_1,y_2))\\
&\quad = T(A(x_1,x_2), A(y_1,y_2)),\\
&A((x_1,x_2)\alpha(y_1,y_2)) = A(x_1\alpha y_1, x_2\alpha y_2) = (A_1\times A_2)(x_1\alpha y_1, x_2\alpha y_2)\\
&\quad = T(A_1(x_1\alpha y_1), A_2(x_2\alpha y_2)) \geq T(T(A_1(x_1), A_1(y_1)), T(A_2(x_2), A_2(y_2)))\\
&\quad = T(T(A_1(x_1), A_2(x_2)), T(A_1(y_1), A_2(y_2)))\\
&\quad = T((A_1\times A_2)(x_1,x_2), (A_1\times A_2)(y_1,y_2)) = T(A(x_1,x_2), A(y_1,y_2)),\\
&A((y_1,y_2)+(x_1,x_2)-(y_1,y_2)) = A(y_1+x_1-y_1, y_2+x_2-y_2)\\
&\quad = (A_1\times A_2)(y_1+x_1-y_1, y_2+x_2-y_2) = T(A_1(y_1+x_1-y_1), A_2(y_2+x_2-y_2))\\
&\quad \geq T(A_1(x_1), A_2(x_2)) = (A_1\times A_2)(x_1,x_2),\\
&A((x_1,x_2)\alpha(y_1,y_2)) = A(x_1\alpha y_1, x_2\alpha y_2) = (A_1\times A_2)(x_1\alpha y_1, x_2\alpha y_2)\\
&\quad = T(A_1(x_1\alpha y_1), A_2(x_2\alpha y_2)) \geq T((A_1(y_1), A_2(y_2)) = (A_1\times A_2)(y_1,y_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&A((x_1,x_2)\alpha((z_1,z_2)+(y_1,y_2)) - (x_1,x_2)\alpha(y_1,y_2))\\
&\quad = A((x_1,x_2)\alpha(z_1,z_2) + (x_1,x_2)\alpha(y_1,y_2) - (x_1,x_2)\alpha(y_1,y_2))
\end{aligned}
$$

$= A((x_1, x_2)\alpha(z_1, z_2)) = A(x_1\alpha z_1, x_2\alpha z_2)$
$= (A_1 \times A_2)(x_1\alpha z_1, x_2\alpha z_2) = T(A_1(x_1\alpha z_1), A_2(x_2\alpha z_2))$
$\geq T((A_1(z_1), A_2(z_2)) = (A_1 \times A_2)(z_1, z_2)$.

Hence $A = A_1 \times A_2$ is a T-fuzzy ideal of $R_1 \times R_2$. This completes the proof. □

Theorem 4.14. *Let $f : R_1 \to R_2$ be an onto homomorphism of a Γ-near-rings. If A is a T-fuzzy ideal of R_1, then $f(A)$ is a T-fuzzy ideal of R_2.*

Proof. Let A be a T-fuzzy ideal of a Γ-near-ring R_1. Let $A_1 = f^{-1}(y_1)$ and $A_2 = f^{-1}(y_2)$, where $y_1, y_2 \in R_2$ are non-empty subsets of R_2. Similarly, $A_3 = f^{-1}(y_1 - y_2)$. Consider the set $A_1 - A_2 = \{a_1 - a_2 : a_1 \in A_1, a_2 \in A_2\}$. If $x \in A_1 - A_2$, then $x = x_1 - x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = y_1 - y_2$, which implies $x \in f^{-1}(y_1 - y_2) = A_3$. Thus $A_1 - A_2 \subseteq A_3$. That is $\{x : x \in f^{-1}(y_1 - y_2)\} \supseteq \{x_1 - x_2 : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$. Let $\alpha \in \Gamma$ and $y_3 \in R_2$. Then

$$\begin{aligned} f(A)(y_1 - y_2) &= \sup\{A(x) : x \in f^{-1}(y_1 - y_2)\} \\ &\geq \sup\{A(x_1 - x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\min(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{T(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= T(\sup\{A(x_1) : x_1 \in f^{-1}(y_1)\}, \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\}) \\ &= T(f(A)(y_1), f(A)(y_2)), \end{aligned}$$

and since $\{x : x \in f^{-1}(y_1\alpha y_2)\} \supseteq \{x_1\alpha x_2 : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$.

$$\begin{aligned} f(A)(y_1\alpha y_2) &= \sup\{A(x) : f^{-1}(y_1\alpha y_2)\} \\ &\geq \sup\{A(x_1\alpha x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\min(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{T(A(x_1), A(x_2)) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= T(\sup\{A(x_1) : x_1 \in f^{-1}(y_1)\}, \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\}) \\ &= T(f(A)(y_1), f(A)(y_2)). \end{aligned}$$

This shows that $f(A)$ is a T-fuzzy sub Γ-near-ring of R_2.

$$\begin{aligned} f(A)(y_2 + y_1 - y_2) &= \sup\{A(x) : x \in f^{-1}(y_1 + y_2 - y_1)\} \\ &\geq \sup\{A(x_2 + x_1 - x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{A(x_1) : x_1 \in f^{-1}(y_1)\} \\ &= f(A)(y_1), \\ f(A)(y_1\alpha y_2) &= \sup\{A(x) : x \in f^{-1}(y_1\alpha y_2)\} \\ &\geq \sup\{A(x_1\alpha x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2), \alpha \in \Gamma\} \\ &\geq \sup\{A(x_2) : x_2 \in f^{-1}(y_2)\} \\ &= f(A)(y_2), \end{aligned}$$

$$\begin{aligned} &f(A)(y_1\alpha(y_3 + y_2) - y_1\alpha y_2) = \sup\{A(x) : x \in f^{-1}(y_1\alpha(y_3 + y_2) - y_1\alpha y_2)\} \\ &= \sup\{A(x_1\alpha(x_3 + x_2) - x_1\alpha x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2), x_3 \in f^{-1}(y_3)\} \\ &\geq \sup\{A(x_3) : x_3 \in f^{-1}(y_3)\} \\ &= f(A)(y_3). \end{aligned}$$

Hence $f(A)$ is a T-fuzzy ideal of R_2. This completes the proof. □

Theorem 4.15. *Let A and B be T-fuzzy left ideals of a Γ-near-ring R. Then $A+B$ is the smallest T-fuzzy left ideal of R containing both A and B.*

Proof. Let A and B be T-fuzzy ideals of a Γ-near-ring R. Let $x, y, z \in R$ and $\alpha \in \Gamma$. Let $x = a + b, y = c + d : a, b, c, d \in R$. Then we have

$$\begin{aligned}
x - y &= (a + b) - (c + d) = a + b - c - d \\
&= (b + a - b) - c + (c + b - c) - d = e + f, \\
(A + B)(x - y) &= \bigvee_{x-y=e+f} [A(e) \wedge B(f)] \\
&= \bigvee_{x=a+b, y=c+d} [A((b + a - b) - c) \wedge B((c + b - c) - d)] \\
&\geq \bigvee_{x=a+b, y=c+d} [T(A(b + a - b), A(c)) \wedge T(B(c + b - c), B(d))] \\
&\geq \bigvee_{x=a+b, y=c+d} [T(A(a), A(c)) \wedge T(B(b), B(d))] \\
&\geq \bigvee_{x=a+b, y=c+d} [T(A(a), B(b)) \wedge T(A(c), B(d))] \\
&= T[\bigvee_{x=a+b} (A(a) \wedge B(b)) \wedge \bigvee_{y=c+d} (A(c) \wedge B(d))] \\
&= T[(A + B)(x), (A + B)(y)].
\end{aligned}$$

Put $y = y_1 + y_2; \quad y_1, y_2 \in R$. Then

$$\begin{aligned}
(A + B)(x\alpha y) &= (A + B)(x\alpha(y_1 + y_2)) \\
&= (A + B)(x\alpha y_1 + x\alpha y_2) \\
&= \bigvee [A(x\alpha y_1) \wedge B(x\alpha y_2)] \\
&\geq \bigvee [T(A(x), A(y_1)) \wedge T(B(x), B(y_2))] \\
&\geq \bigvee [T(A(x), B(x)) \wedge T(A(y_1), B(y_2))] \\
&\geq T[\bigvee (A(x), B(x)) \wedge \bigvee (A(y_1), B(y_2))] \\
&\geq T[\bigvee (A(x) \wedge B(x)) \wedge \bigvee (A(y_1) \wedge B(y_2))] \\
&= T[(A + B)(x), (A + B)(y)], \\
(A + B)(x\alpha y) &= (A + B)(x\alpha y_1 + x\alpha y_2) = \bigvee [A(x\alpha y_1) \wedge B(x\alpha y_2)] \\
&\geq \bigvee [A(y_1) \wedge B(y_2)] \geq \bigvee_{y=y_1+y_2} [A(y_1) \wedge B(y_2)] = (A + B)(y),
\end{aligned}$$

and for any $x = a + b$, we have

$$y + x - y = y + a + b - y = (y + a - y) + (y + b - y);$$

and for each $y + x - y = c + d$, we have

$$\begin{aligned}
x &= -y + c + d + y = (-y + c + y) + (-y + d + y), \\
(A + B)(y + x - y) &= \bigvee_{y+x-y=c+d} [A(c), B(d)] \\
&= \bigvee_{x=a+b} [A(y + a - y), B(y + b - y)] \\
&\geq \bigvee_{x=a+b} [A(a) \wedge B(b)] = (A + B)(x).
\end{aligned}$$

Hence $A+B$ is a T-fuzzy left ideal of R. As $x = x+0$ and $x = 0+x$, so $(A+B) \geq A(x)$ and $(A + B)(x) \geq B(x)$. If C is a fuzzy ideal of R such that $C(x) \geq A(x)$ and $C(x) \geq B(x)$ for all $x \in R$, then

$$\begin{aligned}
(A + B)(x) &= \bigvee_{x=a+b} [A(a) \wedge B(b)] \leq \bigvee_{x=a+b} [C(a) \wedge C(b)] \\
&= \bigvee_{x=a+b} [C(a) \wedge C(-b)] \leq \bigvee_{x=a+b} C(a + b) = C(x).
\end{aligned}$$

Thus $A + B \leq C$. This completes the proof. □

5. T-fuzzy ideals of quotient Γ-near-rings

Example 5.1. Let $R = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$. Then R is a Γ-near-ring as in Example 3.5. Clearly $I = \{0,\ b\}$ is an ideal of R. Now $R/I = \{x + I \mid x \in R\} = \{0 + I, a + I, b + I, c + I\}$.
We define

$(a + I) + (b + I) = (a + b) + I$,
$(a + I) \cdot (b + I) = ab + I$, and
$(a + I)\alpha(b + I) = a\alpha b + I$

for all $(a + I), (b + I) \in R/I$ and $\alpha \in \Gamma$. Define a binary operation "$+$" on R/I by the following table:

$+$	$0+I$	$a+I$	$b+I$	$c+I$
$0+I$	$0+I$	$a+I$	$b+I$	$c+I$
$a+I$	$a+I$	$0+I$	$c+I$	$b+I$
$b+I$	$b+I$	$c+I$	$0+I$	$a+I$
$c+I$	$c+I$	$b+I$	$a+I$	$0+I$

Clearly, $(R/I,\ +)$ is a group. Let $X, Y \in R/I$ and $\alpha \in \Gamma$ then $X\alpha Y \in R/I$. Then the map $R \times \Gamma \times R \to R$ satisfies the following:

(i) $(X + Y)\alpha Z = X\alpha Z + Y\alpha Z$ for every $X, Y, Z \in R/I$, $\alpha \in \Gamma$,
(ii) $(X\alpha Y)\beta Z = X\alpha(Y\beta Z)$ for every $X, Y, Z \in R/I$ and $\alpha, \beta \in \Gamma$.

Thus R/I is a Γ-near-ring. Let T be a t-norm defined by

$$T(p,\ q) = \max(p + q - 1, 0) \text{ for all } p, q \in [0, 1].$$

Define a fuzzy subset $A : R/I \to [0, 1]$ by

$$A(0 + I) = 0.9 \text{ and } A(a + I) = A(b + I) = A(c + I) = 0.4$$

for all $0, a, b, c \in R$. The routine calculation shows that A is a T-fuzzy ideal of the quotient Γ-near-ring R/I.

The following results were obtained by Bh. Satyanarayana and Kuncham Syam prasad.

Theorem 5.2 ([11])**.** *Let I be a fuzzy ideal of a Γ-near-ring R. Then the set R/I of all fuzzy co-sets of I is a Γ-near-ring with respect to the operations defined by $(x + I) + (y + I) = (x + y) + I$ and $(x + y)\alpha(y + I) = (x\alpha y) + I$ for all $x, y \in R$ and $\alpha \in \Gamma$.*

Notation ([11]). Let μ be a fuzzy ideal of a Γ-near-ring R. We define
$\theta_\mu : R/\mu \to [0, 1]$ by $\theta_\mu(x + \mu) = \mu(x)$
for all $x \in R$. Using these we prove the following results.

Theorem 5.3. *If μ is a T-fuzzy ideal of a Γ-near-ring R, then θ_μ is a T-fuzzy ideal of R/μ.*

Proof. Let μ be a T-fuzzy ideal of a Γ-near-ring R and $x, y \in R$. Suppose that $x+\mu = y+\mu$. Then $\mu(x-y) = \mu(0)$. This implies $\mu(x) = \mu(y)$. That is, $\theta_\mu(x+\mu) = \theta_\mu(y + \mu)$. Hence θ_μ is well defined. Let $x + \mu, y + \mu, z + \mu \in R/\mu$ and $\alpha \in \Gamma$. Then

$$\theta_\mu\{(x + \mu) - (y + \mu)\} = \theta_\mu\{(x - y) + \mu\} = \mu(x - y) \geq T(\mu(x), \mu(y))$$
$$= T(\theta_\mu(x + \mu), \theta_\mu(y + \mu)),$$

and

$$\theta_\mu\{(x+\mu)\alpha(y+\mu)\} = \theta_\mu\{(x\alpha y)+\mu\} = \mu(x\alpha y) \geq T(\mu(x),\mu(y))$$
$$= T(\theta_\mu(x+\mu),\theta_\mu(y+\mu)).$$

This shows that θ_μ is a T-fuzzy sub Γ-near-ring of R/μ.

$$\theta_\mu\{(y+\mu)+(x+\mu)-(y+\mu)\} = \theta_\mu\{(y+x-y)+\mu\} = \mu(y+x-y)$$
$$\geq \mu(x) = \theta_\mu(x+\mu),$$
$$\theta_\mu\{(x+\mu)\alpha(y+\mu)\} = \theta_\mu\{(x\alpha y)+\mu\} = \mu(x\alpha y)$$
$$\geq \mu(y) = \theta_\mu(y+\mu)$$

and

$$\theta_\mu\{(x+\mu)\alpha((z+\mu)+(y+\mu)) - (x+\mu)\alpha(y+\mu)\}$$
$$= \theta_\mu\{(x+\mu)\alpha((z+y)+\mu)) - (x\alpha y+\mu)\}$$
$$= \theta_\mu\{(x\alpha(z+y)+\mu) - (x\alpha y+\mu)\}$$
$$= \theta_\mu\{x\alpha(z+y) - (x\alpha y)+\mu\}$$
$$= \mu\{x\alpha(z+y) - (x\alpha y)\} \geq \mu(z) = \theta_\mu(z+\mu).$$

Hence θ_μ is a T-fuzzy ideal of R/μ. This completes the proof. □

Theorem 5.4. *Let I be an ideal of a Γ-near-ring R. If A is a T-fuzzy ideal of R, then the fuzzy set $\bar{A}$ of R/I defined by $\bar{A}(a+I) = \sup_{x\in I} A(a+x)$ is a T-fuzzy ideal of the quotient Γ-near-ring R/I of R with respect to I.*

Proof. Let R be a Γ-near-ring and A be a T-fuzzy ideal of R. Let $a, b \in R$ such that $a+I = b+I$. Then $b = a+y$ for some $y \in I$. Thus

$$\bar{A}(b+I) = \sup_{x\in I} A(b+x) = \sup_{x\in I} A(a+y+x) = \sup_{x+y=z\in I} A(a+z) = \bar{A}(a+I).$$

This shows that $\bar{A}$ is well defined. Let $x+I, y+I \in R/I$. Then we have

$$\bar{A}((x+I)-(y+I)) = \bar{A}((x-y)+I) = \sup_{z\in I} A((x-y)+z)$$
$$= \sup_{z=u-v\in I} A((x-y)+(u-v)) = \sup_{u,v\in I} A((x+u)-(y+v))$$
$$\geq \sup_{u,v\in I} T(A(x+u),A(y+v)) = T(\sup_{u\in I} A(x+u), \sup_{v\in I} A(y+v))$$
$$= T(\bar{A}(x+I),\bar{A}(y+I))$$

and

$$\bar{A}((x+I)\alpha(y+I)) = \bar{A}(x\alpha y+I) = \sup_{t\in I} A(x\alpha y+t) = \sup_{t\in I} A[(x+t)\alpha(y+t)]$$
$$\geq \sup_{t\in I}[\min((x+t)\alpha(y+t))] \geq \sup_{t\in I} T(A(x+t),A(y+t))$$
$$= T(\sup_{t\in I} A(x+t), \sup_{t\in I} A(y+t)) = T(\bar{A}(x+I),\bar{A}(y+I)).$$

For any $x, y, z \in R$ and $\alpha \in \Gamma$, we get

$$\bar{A}[(y+I)+(x+I)-(y+I)] = \bar{A}[(y+x-y)+I] = \bar{A}[((y+x)-y)+I]$$
$$= \sup_{z\in I} A[((y+x)-y)+z] = \sup_{z=u+v-w\in I} A[((y+x)-y)+u+v-w]$$
$$= \sup_{u,v,w\in I} A[(y+u)+(x+v)-(y+w)] \geq \sup_{v\in I} A[x+v] = \bar{A}(x+I),$$
$$\bar{A}((x+I)\alpha(y+I)) = \bar{A}(x\alpha y+I) = \sup_{t\in I} A(x\alpha y+t) = \sup_{t=x\alpha z\in I} A(x\alpha y+x\alpha z)$$
$$= \sup_{x,z\in I} A(x\alpha(y+z)) \geq \sup_{z\in I} A(y+z) = \bar{A}(y+I),$$

and

$$\bar{A}\{(x+I)\alpha((z+I)+(y+I)) - (x+I)\alpha(y+I)\} = \bar{A}(x\alpha(z+y)+I-(x\alpha y+I))$$
$$= \bar{A}((x\alpha(z+y)-x\alpha y)+I) = \sup_{t\in I} A((x\alpha(z+y)-x\alpha y)+t)$$

$\geq \sup_{t\in I} A(z+t) = \bar{A}(z+I)$.

Hence $\bar{A}$ is a T-fuzzy ideal of R/I. This completes the proof. □

Theorem 5.5. *Let I be an ideal of a Γ-near-ring R. If $\bar{A}$ with $\bar{A}(a+I) = A(a)$ where $a \in R$, is a T-fuzzy ideal of R/I, then the fuzzy set A is a T-fuzzy ideal of R.*

Proof. Let I be an ideal of a Γ-near-ring R and $\bar{A}$ be a T-fuzzy ideal of R/I. Let $x, y, z \in R$ and $\alpha \in \Gamma$. Then

$$A(x-y) = \bar{A}[(x-y)+I] = \bar{A}[(x+I)-(y+I)]$$
$$\geq T(\bar{A}(x+I), \bar{A}(y+I)) = T(A(x), A(y)),$$
$$A(x\alpha y) = \bar{A}[x\alpha y+I] = \bar{A}((x+I)\alpha(y+I)) \geq T(\bar{A}(x+I), \bar{A}(y+I)) = T(A(x), A(y)),$$
$$A(y+x-y) = \bar{A}[(y+x-y)+I] = \bar{A}[(y+I)+(x+I)-(y+I)] \geq \bar{A}(x+I) = A(x),$$
$$A(x\alpha y) = \bar{A}[x\alpha y+I] = \bar{A}[(x+I)\alpha(y+I)] \geq \bar{A}(y+I) = A(y),$$

and

$$A[x\alpha(z+y) - x\alpha y] = \bar{A}[x\alpha(z+y) - x\alpha y + I]$$
$$= \bar{A}([x\alpha(z+y) + I - (x\alpha y + I)])$$
$$= \bar{A}[(x+I)\alpha((z+y)+I) - (x+I)\alpha(y+I)]$$
$$= \bar{A}[(x+I)\alpha[(z+I)+(y+I)] - (x+I)\alpha(y+I)]$$
$$\geq \bar{A}(z+I) = A(z).$$

Hence A is a T-fuzzy ideal of R. This completes the proof. □

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