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On some convergence structures in L-semi-uniform spaces

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Abstract. The notion of completeness on L-semi-uniform spaces is introduced in this paper. We have established that completeness is weakly hereditary and preserved under L-semi-uniform isomorphism. Finally, we have shown that in a totally bounded L-semi-uniform space, completeness is equivalent to compactness.

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1. Introduction

Uniform spaces play a very important role as a bridge between metric spaces and general topological spaces. Many generalizations of uniform spaces, viz. quasi-uniform spaces [5], semi-uniform spaces, semi-quasi-uniform spaces, locally uniform spaces [22] and locally quasi-uniform spaces [9, 15] have been developed in general topology. Semi-quasi-uniform spaces were introduced and a necessary and sufficient condition under which semi-quasi-uniform spaces are topological was obtained in [20]. The problem of completeness for a closure space in terms of semi-quasi-uniform space was considered in [20]. Closure spaces in relation to semi-uniform spaces have been studied and various results on semi-uniformly continuous and semi-pseudo-metrization have been obtained in [3]. Different types of completeness on t-semi-uniformity were considered in [2].

Uniform spaces and quasi-uniform spaces in the fuzzy setting have been studied by several authors [10, 11, 13, 14, 17, 18]. Garcia et. al. [21] introduced uniform spaces in a unifying framework of GL-monoid to include both the categories of Lowen Uniformity and Hutton type uniformities. In their recent work, Hazarika and Mitra [6, 19] developed L-local uniformity and L-local quasi-uniformity as generalizations of Hutton's uniformity and quasi-uniformity respectively in the category ***L*-TOP**. Further, in their subsequent work [7, 8], the same authors developed L-semi-quasi-uniformity and L-semi-uniformity in the same category as generalizations of L-local quasi-uniformity and L-local uniformity respectively. A sufficient condition for an L-semi-quasi-uniformity for generating an L-topological space was obtained. Various results on L-semi-uniformly continuous functions were obtained. The notion of L-semi-pseudo-metric as a generalization of Hutton's metric was introduced and it was established that every L-semi-uniform space with countable base is L-semi-pseudo-metrizable.

In this paper we take up the problem of completeness for the L-semi-uniformity $\mathscr{U}$ on L^X which satisfies the condition $x_\alpha \in U(x_\alpha)$, instead of our earlier assumption that $x_\alpha \subseteq U(x_\alpha)$, $\forall\, U \in \mathscr{U}, x_\alpha \in \mathrm{Pt}(L^X)$. In the process we have obtained a subclass of the class of the L-semi-uniformities developed in the earlier papers. This change has been necessitated in order to accommodate the system of Q-nbd at an L-fuzzy point x_α. The Q-nbd system, as may be noted, plays an important role in the theory of convergence. We shall continue to call this subclass of L-semi-uniform spaces as L-semi-uniform spaces for sake of convenience.

Throughout the paper $(L, \leq, \bigwedge, \bigvee, ')$ is a completely distributive lattice with order reversing involution $'$; 0_L and 1_L are respectively inf and sup in L. X is an arbitrary set and L^X will denote the collection of all mappings $A : X \to L$. Any member of L^X is an L-fuzzy set. The L-fuzzy sets $x_\alpha : X \to L$ defined by $x_\alpha(y) = 0_L$ if $x \neq y$ and $x_\alpha(y) = \alpha$ if $x = y$ are the L-fuzzy points. The set of all L-fuzzy points on X is denoted by $\mathrm{Pt}(L^X)$. The mappings $A : X \to L$ and $B : X \to L$ defined by $A(x) = 1_L$, $\forall x \in X$ and $B(x) = 0_L$, $\forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. For any $A, B \in L^X$, the union and intersection of A and B are defined as $A \bigcup B = A(x) \bigvee_{x\in X} B(x)$ and $A \bigcap B = A(x) \bigwedge_{x\in X} B(x)$ respectively; we say $A \subseteq B$ iff $A(x) \leq B(x)$ and $x_\alpha \in A$ iff $\alpha < A(x)$, where x_α is an L-fuzzy point; complement of A is defined as $A'(x) = A(x)'$ An L-topology $\mathbb{F}$ on L^X is a subset of L^X closed under finite intersection and arbitrary union. The elements of $\mathbb{F}$ are called open sets and their complements are the closed sets. For basic fuzzy topological definitions we refer to [4]. We consider Hutton's uniformity and quasi-uniformity [11]. Our definition of totally boundedness generalizes the notion of totally boundedness of G. Artico and R. Moresco [1]. We adopt Hutton's compactness in an interior space.

2. Preliminaries

Definition 2.1 ([16])**.** For any x_α, A, $B \in L^X$, x_α is said to be *quasi-coincident* with A, denoted as $x_\alpha \ll A$ if $x_\alpha \nsubseteq A'$ i.e., $\alpha \nleq A'(x)$.

A is called *quasi-coincident* with B at y if $A(y) \nleq B'(y)$. A is called *quasi-coincident* with B, denoted as $A\widehat{q}B$, if A quasi-coincides with B at some $y \in X$.

Definition 2.2 ([16])**.** For any ordinary mapping $f : X \to Y$, the induced *L-fuzzy mapping* $f^{\to} : L^X \to L^Y$ and its *L-fuzzy reverse mapping* $f^{\leftarrow} : L^Y \to L^X$ respectively are defined as:

$f^{\to}(A)(y) = \bigvee\{A(x) \mid x \in X,\ f(x) = y\},\ \forall\, A \in L^X,\ \forall\, y \in Y.$
$f^{\leftarrow}(B)(x) = B(f(x)),\ \forall\, B \in L^Y,\ \forall\, x \in X.$

Definition 2.3 ([16]). Let L^X and L^Y be L-fuzzy spaces, $f : X \to Y$ an ordinary mapping. Then

(i) $f^{\to} : L^X \to L^Y$ is said to be *injective*, if for any $A, B \in L^X, f^{\to}(A) = f^{\to}(B)$ implies $A = B$.

(ii) $f^{\to} : L^X \to L^Y$ is said to be *surjective*, if for any $B \in L^Y$ there exists $A \in L^X$ such that $f^{\to}(A) = B$.

(iii) $f^{\to} : L^X \to L^Y$ is said to be *bijective*, if it is both injective and surjective.

Theorem 2.4 ([16]). *Let L^X and L^Y be L-fuzzy spaces, $f : X \to Y$ an ordinary mapping. Then*

(i) *$f^{\to}$ is injective if and only if f is injective.*

(ii) *$f^{\to}$ is surjective if and only if f is surjective.*

(iii) *$f^{\to}$ is bijective if and only if f is bijective.*

We shall now adopt all the necessary definitions and results of L-semi-uniform spaces in this setting.

Definition 2.5. Let $i : L^X \to L^X$ be a mapping on L^X. Then i is called an *interior operator* on L^X if it fulfills the following conditions:

(IO1) $i(\underline{1}) = \underline{1}$.

(IO2) $i(A) \subseteq A, \ \forall A \in L^X$.

(IO3) $i(A \bigcap B) = i(A) \bigcap i(B), \ \forall A, B \in L^X$.

L^X together with an interior operator 'i' shall be called an *interior space*. For any $A \in L^X$, we shall call $(i(A'))'$ is the *closure* of A with respect to the interior operator 'i' [denoted by $\mathrm{cl}(A)$] and A is called *closed* or *open* with respect to that interior operator according as $A = \mathrm{cl}(A)$ or $A = i(A)$ respectively.

Obviously, for any interior operator 'i' and $A \in L^X$, we have A is open with respect to 'i' iff A' is closed with respect to that interior operator.

An interior operator 'i' is said to be an *L-topological interior operator* if in addition it satisfies the following:

(IO4) $i(i(A)) = i(A), \ \forall A \in L^X$.

We shall call a closure operator an *L-topological closure operator* iff the relative interior operator is L-topological.

Definition 2.6 ([11]). Let $\mathscr{U}^*$ be the collection of all maps $U : L^X \to L^X$ which satisfy:

(s1) $\Delta \subsetneq U$.

(s2) $U(\bigcup_\lambda V_\lambda) = \bigcup_\lambda U(V_\lambda), V_\lambda \in L^X$.

Here $\Delta : L^X \to L^X$ such that $\Delta(A) = A, A \in L^X$. For any $U, V \in \mathscr{U}^*$, $U \circ V$ is the composition of functions. Obviously, $\Delta \circ U = U = U \circ \Delta$, $\Delta \circ U$ and $U \circ \Delta$ are the composition of functions.

Definition 2.7 ([11]). For any $U \in \mathscr{U}^*$, $U^r(x_\alpha) = \bigcap\{y_\beta \mid U(y'_\beta) \subseteq x'_\alpha\}$. Then $U^r \in \mathscr{U}^*$ and $(U^r)^r = U$, by proposition 10.2 in [11]. If $U = U^r$, then U is said to be symmetric.

Definition 2.8 ([7, 8]). An *L-semi-quasi-uniformity* $\mathscr{U}$ on L^X is a non empty subfamily of $\mathscr{U}^*$ satisfying the following:

(SQ1) $U \bigcap V \in \mathscr{U}, \ \ \forall U, V \in \mathscr{U}$.

(SQ2) If $V \in \mathscr{U}^*$ such that $U \subseteq V$, for some $U \in \mathscr{U}$, then $V \in \mathscr{U}$.

The pair $(L^X, \mathscr{U})$ is called an *L-semi-quasi-uniform space*.

Definition 2.9 ([7, 8]). A non empty subfamily $\mathscr{B}$ of $\mathscr{U}^*$ is called a *base* for some L-semi-quasi-uniformity $\mathscr{U}$ if for any $U \in \mathscr{U}$, there is $B \in \mathscr{B}$ such that $B \subseteq U$.

A non empty subfamily $\mathscr{B}$ of $\mathscr{U}^*$ is a base for some L-semi-quasi-uniformity $\mathscr{U}$ if it satisfies the following:
(SQ1′) For any U, $V \in \mathscr{U}$, there is $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 2.10 ([7, 8]). An L-semi-quasi-uniformity $\mathscr{U}$ on L^X is said to be an *L-semi-uniformity* if $\mathscr{U}$ has a base $\mathscr{B}$ such that:
(SQ3) For any $B \in \mathscr{B}$ implies $B^r \in \mathscr{B}$.
The pair $(L^X, \mathscr{U})$ is then called an *L-semi-uniform space*.

Also, the collection of symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$.

Theorem 2.11 ([7]). *Let $(L^X, \mathscr{U})$ be an L-semi-quasi-uniform space and $\mathscr{B}$ be any base of $\mathscr{U}$. Then the mapping* $\mathrm{int} : L^X \to L^X$ *defined by,* $\mathrm{int}(A) = \bigcup\{x_\alpha \mid \exists\ V \in \mathscr{B} \text{ s.t. } V(x_\alpha) \subseteq A\}$*, is an interior operator on L^X.*

Every fuzzy semi-quasi-uniformity therefore generates an interior space. Further, for any L-semi-uniform space $(L^X, \mathscr{U})$, since the interior space is generated by 'int', so, in particular, for any $x_\alpha \in \mathrm{Pt}(L^X)$, the collection $\mathscr{N}_{x_\alpha} = \{U(x_\alpha) \mid U \in \mathscr{U}\}$ is the neighborhood system at x_α in the generated interior space. If the family $\{\mathscr{N}_{x_\alpha} \mid x_\alpha \in \mathrm{Pt}(L^X)\}$ is a neighborhood system for some L-topology $\mathbb{F}$, we say that $\mathbb{F}$ is the L-topology generated by $\mathscr{U}$.

Theorem 2.12 ([7]). *Every L-semi-quasi-uniformity generates an L-topological space under the following condition:*
For any $U \in \mathscr{U}$ and $x_\alpha \in \mathrm{Pt}(L^X)$, there exists $V \in \mathscr{U}$ such that to each $y_\beta \in V(x_\alpha)$ there corresponds $W \in \mathscr{U}$ with $W(y_\beta) \subseteq U(x_\alpha)$.

Definition 2.13 ([8]). Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces. A function $f^{\to} : (L^X, \mathscr{U}) \to (L^Y, \mathscr{V})$ is called *L-semi-uniformly continuous* iff for every $V \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $\widehat{f^{\to}}(U) \subseteq V$, where $\widehat{f^{\to}}(x_\alpha, y_\beta) = (f^{\to}(x_\alpha), f^{\to}(y_\beta))$. The function $f^{\to}$ is said to be an *L-semi-uniformly isomorphism* iff $f^{\to}$ is bijective and both $f^{\to}$ and $f^{\leftarrow}$ are L-semi-uniformly continuous.

Now since $\widehat{f^{\to}}(U) \subseteq V$ implies $\widehat{f^{\to}}(U)(f^{\to}(x_\alpha)) \subseteq V(f^{\to}(x_\alpha))$, $\forall x_\alpha \in \mathrm{Pt}(L^X)$, therefore we have the following:

Theorem 2.14 ([8]). *L-semi-uniformly continuous functions on L-semi-uniform spaces are continuous with respect to the relative interior spaces.*

Proof. Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces. Let $f^{\to} : (L^X, \mathscr{U}) \to (L^Y, \mathscr{V})$ be L-semi-uniformly continuous, $\mathrm{int}_{\mathscr{U}}$ and $\mathrm{int}_{\mathscr{V}}$ respectively be the interior operators generated by $\mathscr{U}$ and $\mathscr{V}$. For $x_\alpha \in \mathrm{Pt}(L^X)$ and for each neighborhood N of $f^{\to}(x_\alpha)$ in the interior space generated by $\mathscr{V}$, we may choose $V \in \mathscr{V}$ and $U \in \mathscr{U}$ so that $V(f^{\to}(x_\alpha)) \subseteq N$ and $\widehat{f^{\to}}(U) \subseteq V$. Therefore, $f^{\to}(U(x_\alpha)) = \widehat{f^{\to}}(U)(f^{\to}(x_\alpha)) \subseteq V(f^{\to}(x_\alpha)) \subseteq N$. □

Corollary 2.15 ([8]). *Every L-semi-uniformly isomorphism is an L-semi- homeomorphism.*

3. Completeness and Compactness

In this section by characterizing completeness and compactness in terms of Cauchy ultrafilter and ultrafilter respectively, we show that in a totally bounded L-semi-uniform space, the notions of completeness and compactness are equivalent.

Definition 3.1. Let 'i' be an interior operator on L^X. Then for any $x_\alpha \in \mathrm{Pt}(L^X)$, we shall call an L-fuzzy set N to be a *neighborhood* (nbd) at x_α with respect to 'i', if there is $G \in L^X$ such that $i(G) \nsubseteq x_\alpha$ and $i(G) \subseteq N$. The family of all nbds at x_α in the interior space is denoted by, $\mathscr{N}_i(x_\alpha)$. We shall call an L-fuzzy set F to be a *quasi-coincident neighborhood* (Q-nbd) at x_α with respect to 'i', if there is an L-fuzzy set B such that $x_\alpha \ll i(B)$ and $B \subseteq F$. The family of all Q-nbd at x_α in the interior space is denoted by, $\mathscr{Q}_i(x_\alpha)$.

Definition 3.2. Let 'i' be an interior operator on L^X. Then a subfamily $\mathscr{A}$ of L^X is said to be a

(i) *nbd base* if $\mathscr{A} \subseteq \mathscr{N}_i(x_\alpha)$ and for every $N \in \mathscr{N}_i(x_\alpha)$, $\exists\, A \in \mathscr{A}$ such that $A \subseteq N$.

(ii) *Q-nbd base* if $\mathscr{A} \subseteq \mathscr{Q}_i(x_\alpha)$ and for every $F \in \mathscr{Q}_i(x_\alpha)$, $\exists\, B \in \mathscr{A}$ such that $B \subseteq F$.

Lemma 3.3. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space. Then for any $A \in L^X$,*

$$\mathrm{cl}(A) = \bigcap\{V(A) \mid V \in \mathscr{U}\}.$$

Proof. Let $\mathscr{B}$ be a base for $\mathscr{U}$ consisting of symmetric members of $\mathscr{U}$. Now,

$$\begin{aligned}\mathrm{int}(A') &= \bigcup\{x_\alpha \mid \exists\, U \in \mathscr{B} \text{ s.t. } U(x_\alpha) \subseteq A'\}\\ &= \bigcup\{\bigcup\{x_\alpha \mid U(x_\alpha) \subseteq A'\}, U \in \mathscr{B}\}.\\ &= \bigcup\{[U^r(A)]' \mid U \in \mathscr{B}\}\\ &= \bigcup\{[U(A)]' \mid U \in \mathscr{B}\}, \text{ since } U \in \mathscr{B} \Rightarrow U^r = U.\\ &= \bigcup\{[V(A)]' \mid V \in \mathscr{U}\}, \text{ since } \mathscr{B} \text{ is a base for } \mathscr{U}.\\ &= [\bigcap\{V(A) \mid V \in \mathscr{B}\}]'.\end{aligned}$$

Hence, $\mathrm{cl}(A) = (\mathrm{int}(A'))' = \bigcap\{V(A) \mid V \in \mathscr{U}\}$. □

Theorem 3.4. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space. Then for any $A \in L^X$, $\mathrm{cl}(A)$ satisfies the following:*

(CO1) $\mathrm{cl}(\underline{0}) = \underline{0}$.

(CO2) $A \subseteq \mathrm{cl}(A)$.

(CO3) $\mathrm{cl}(A \bigcup B) = \mathrm{cl}(A) \bigcup \mathrm{cl}(B), \quad \forall\, B \in L^X$.

Proof. Since, by Lemma 3.3, $\mathrm{cl}(A) = (\mathrm{int}(A'))', \quad \forall\, A \in L^X$, therefore,

(CO1) $\mathrm{cl}(\underline{0}) = (\mathrm{int}(\underline{0}'))' = (\mathrm{int}(\underline{1}))'$. Then by (IO1) $\mathrm{cl}(\underline{0}) = (\underline{1})' = \underline{0}$.

(CO2) $(\mathrm{int}(A'))' = \mathrm{cl}(A) \Rightarrow ((A'))' \subseteq \mathrm{cl}(A)$, by (IO2). Therefore, $A \subseteq \mathrm{cl}(A)$.

(CO3) $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A \bigcup B)')' = (\mathrm{int}(A' \bigcap B'))'$. Then, by (IO3), $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A') \bigcap \mathrm{int}(B'))'$. Then, $\mathrm{cl}(A \bigcup B) = (\mathrm{int}(A'))' \bigcup (\mathrm{int}(B'))' = \mathrm{cl}(A) \bigcup \mathrm{cl}(B)$. □

Remark 3.5. Obviously, 'cl' on L^X is an L-topological operator iff it satisfies the following axiom: (CO4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A), \quad \forall\, A \in L^X$.

Definition 3.6. For any $x_\alpha \in \mathrm{Pt}(L^X)$ we define its *dual point* as an L-fuzzy point x^*_α such that

$$x^*_\alpha(y) = \begin{cases} \alpha' & \text{if } y = x, \\ 0_L & \text{if } y \neq x. \end{cases}$$

In view of Theorem 2.3.24 in [16], we have the following:

Theorem 3.7. *Let 'i' be an interior operator on L^X and $A \in L^X$. Then $x_\alpha \in (i(A'))'$ iff each neighborhood of its dual point x^*_α is quasi-coincident with A.*

The following definitions are from [16] which are adapted to an interior space:

Definition 3.8. A non empty sub collection $\mathscr{F}$ of L^X is said to be a *filter* in an interior space, if:
(F1) $\underline{0} \notin \mathscr{F}$.
(F2) $U, V \in \mathscr{F} \Rightarrow U \bigcap V \in \mathscr{F}$.
(F3) $U \in \mathscr{F}$ and $G \in L^X$ such that $U \subseteq G$ then $G \in \mathscr{F}$.

Let $A \in L^X$ such that for any $F \subseteq A$, $F \notin \mathscr{F}$. Then $\mathscr{F}$ is said to be a filter *relative to* A.

Definition 3.9. A subfamily $\mathscr{B}$ of L^X is called a *filter base* in an interior space if
(B1) $\underline{0} \notin \mathscr{B}$
(B2) for any $U, V \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 3.10. A filter $\mathscr{F}$ is said to be *closed* with respect to some interior operator 'i' if for any $F \in \mathscr{F}$ implies $F = c(F)$.

Definition 3.11. Let $x_\alpha \in \mathrm{Pt}(L^X)$ and $\mathscr{F}$ be a filter. Then $\mathscr{F}$ is said to be *convergent to* x_α with respect to some interior operator 'i', denoted by $\mathscr{F} \to x_\alpha$, if for any $U \in \mathscr{Q}_i(x_\alpha)$ there exists $F \in \mathscr{F}$ such that $F \subseteq U$, that is, $\mathscr{Q}_i(x_\alpha) \subseteq \mathscr{F}$.

Definition 3.12. *Cluster set* of $\mathscr{F}$ with respect to some interior operator 'i', is given by $\bigcap\{\mathrm{cl}(F) \mid F \in \mathscr{F}\}$.

For any $x_\alpha \in \mathrm{Pt}(L^X)$, if x_α is in the cluster set of $\mathscr{F}$, then we denote it by $\mathscr{F} \rightsquigarrow x_\alpha$.

Remark 3.13. Cluster set of a filter with respect to an L-topological space was defined by Hutton in an analogous way.

Remark 3.14. If x_α is in the cluster set of $\mathscr{F}$ with respect to some interior operator 'i', then for any $F \in \mathscr{F}$, $x_\alpha \subseteq (i(F'))'$. But $x_\alpha \subseteq (i(F'))' \Rightarrow \alpha \leq (i(F'))'(x) \Rightarrow (i(F'))'(x) \not< \alpha \Rightarrow \alpha' \not< i(F')(x)$. Now $\alpha' \not< i(F')(x) \Rightarrow x^*_\alpha \notin i(F') \Rightarrow G \nsubseteq F'$, $\forall\, G \in \mathscr{N}_i(x^*_\alpha)$. This implies that $G\widehat{q}F$, $\forall\, G \in \mathscr{N}_i(x^*_\alpha)$. But for any $A, B \in L^X$, $A\widehat{q}B$ implies $A \bigcap B \neq \underline{0}$. For if $A\widehat{q}B$, then there exists $x \in X$ such that $A(x) \nleq B'(x)$. So, $0_L < A(x)$ and $B'(x) < 1_L$. Hence, $0_L < A(x)$ and $0_L < B(x)$. We then have, $A(x) \bigwedge B(x) \neq 0_L$. So, $A \bigcap B \neq \underline{0}$. Hence, $G \bigcap F \neq \underline{0}$, $\forall G \in \mathscr{N}_i(x^*_\alpha)$. Again since G is a nbd at x^*_α iff G is a Q-nbd at x_α. Therefore, $\mathscr{F} \rightsquigarrow x_\alpha$ implies that x_α is a cluster point of $\mathscr{F}$ in the sense of [16].

Definition 3.15. Let 'i' be an interior operator on L^X. A subset $\mathscr{F}$ of L^X is said to satisfy the *F. I. P. relative to an open set* G with respect to the interior operator 'i' if $F_1, ..., F_n \in \mathscr{F} \Rightarrow \bigcap_{i=1}^n F_i \nsubseteq G$.

Obviously, every subset $\mathscr{F}$ of L^X which satisfies the F. I. P. relative to G is contained in a filter relative to G.

Definition 3.16. Let L^X and L^Y be interior spaces with respect to the interior operators i_X and i_Y respectively. Then a function $f^{\rightarrow} : L^X \rightarrow L^Y$ is said to be *open* with respect to the interior operators, if for any $G \in L^X$ such that $i_X(G) = G$ implies $i_Y(f^{\rightarrow}(G)) = f^{\rightarrow}(G)$.

Definition 3.17. Let L^X and L^Y be interior spaces with respect to the interior operators i_X and i_Y respectively. Then a function $f^{\rightarrow} : L^X \rightarrow L^Y$ is said to be *continuous* with respect to the interior operators iff for each $x_\alpha \in \mathrm{Pt}(L^X)$ and each neighborhood V of $f^{\rightarrow}(x_\alpha)$ with respect to the interior operator i_Y, there is a neighborhood U of x_α with respect to the interior operator i_X such that $f^{\rightarrow}(U) \subseteq V$.

The following result follows from Theorem 5.2.27 in [16].

Theorem 3.18. *Let i_X and i_Y be two interior operators on L^X and L^Y respectively. Then a function $f^{\rightarrow} : L^X \rightarrow L^Y$ is continuous with respect to the interior operators if and only if for any filter $\mathscr{F}$ converging on L^X with respect to the interior operator i_X implies that $f^{\rightarrow}(\mathscr{F}) = \{f^{\rightarrow}(F) \mid F \in \mathscr{F}\}$ converges on L^Y with respect to the interior operator i_Y.*

In view of Theorem 5.2.9 in [16], we obtain the following:

Theorem 3.19. *Let ' i ' be an interior operator on L^X and $A \in L^X$. Then $x_\alpha \in (i(A'))'$ iff there is a filter $\mathscr{F}$ relative to A' such that $\mathscr{F} \rightarrow x_\alpha$, with respect to that interior operator 'i'.*

Definition 3.20. Let $A \in L^X$. We shall call the maximal filter (with respect to partial ordering by set inclusion) $\mathscr{F}_\text{и}$ relative to A as an *ultrafilter relative to* A. If $A = \underline{0}$, then we simply call $\mathscr{F}_\text{и}$ to be an *ultrafilter*.

Theorem 3.21. *Let 'i' be an interior operator and $\mathscr{F}$ be a filter on L^X. Let $x_\alpha \in \mathrm{Pt}(L^X)$ such that $\mathscr{F} \rightarrow x_\alpha$ with respect to 'i'. Then $\mathscr{F} \rightsquigarrow x_\alpha$.*

Proof. Let F be any member of $\mathscr{F}$. Now we consider the following two cases:

Case I. Let $F' \notin \mathscr{F}$. Then, $\mathscr{F}$ is a filter relative to F' and $\mathscr{F} \rightarrow x_\alpha$. So, by Theorem 3.19, $x_\alpha \in (i(F'))'$.

Case II. Let $F' \in \mathscr{F}$ and N be any Q-nbd at x_α. Then $N \in \mathscr{F}$ and hence $F' \bigcap N \neq \underline{0}$. Then, there exists $y_\beta \in F$ such that $y^*_\beta \in N$. This implies that $N(y) \nleq (F(y))'$ and hence $N\widehat{q}F$. Therefore, by Theorem 3.7, $x_\alpha \in (i(F'))'$. Thus, in either case $x_\alpha \in (i(F'))', \ \forall\, F \in \mathscr{F}$. Hence, $\mathscr{F} \rightsquigarrow x_\alpha$. □

The following result can be obtained from Theorem 5.2.16 in [16] and Theorem 3.21.

Theorem 3.22. *Let 'i' be an interior operator and $\mathscr{F}_\text{и}$ be an ultrafilter on L^X. Then $\mathscr{F}_\text{и} \rightsquigarrow x_\alpha$ iff $\mathscr{F}_\text{и} \rightarrow x_\alpha$.*

In view of Lemma 11 in [12], we have the following:

Lemma 3.23. *For any ultrafilter $\mathscr{F}_\text{и}$ and A, $B \in L^X$ such that $A \bigcup B \in \mathscr{F}_\text{и}$, either $A \in \mathscr{F}_\text{и}$ or $B \in \mathscr{F}_\text{и}$.*

Definition 3.24. Let 'i' be an interior operator on L^X, an *open cover* $\mathscr{C}$ of an L-fuzzy set A is a collection of open sets with respect to the interior operator 'i' such that $A \subseteq \bigcup_{G\in\mathscr{C}} G$.

In view of Definition 5 in [12], we adopt the following definition for an interior space.

Definition 3.25. An interior space is said to be *compact* if it satisfies any of the following equivalent conditions:
(1) Every open cover $\mathscr{C}$ of a closed set has a finite subcover.
(2) Every collection of closed sets $\mathscr{F}$ satisfying the F. I. P. relative to an open set G has $\bigcap_{F\in\mathscr{F}} F \nsubseteq G$.

We now state the following lemma:

Lemma 3.26. *For any $U \in \mathscr{U}^*$ and $x_\alpha, y_\beta \in \mathrm{Pt}(L^X)$ we get*

$$y_\beta \subseteq U(x_\alpha) \text{ iff } x_\alpha \subseteq U^r(y_\beta).$$

Proof. Since, $(U^r)^r = U$. So, we need to prove only one way implication. Here, $U^r(y_\beta) = \bigcap\{z_\gamma \mid U(z'_\gamma) \subseteq y'_\beta\}$. Let $y_\beta \subseteq U(x_\alpha)$. Then $[U(x_\alpha)]' \subseteq y'_\beta$. Let $A : X \to L$ be a mapping defined by

$$\forall\, z \in X, \quad A(z) = \begin{cases} \gamma & \text{if } U(z'_\gamma) \subseteq y'_\beta, \\ 0_L & \text{if } U(z'_\gamma) \nsubseteq y'_\beta. \end{cases}$$

Then $U^r(y_\beta) = \bigcap A$. Let $B : X \to L$ be a mapping defined by

$$\forall\, w \in X, \quad B(w) = \begin{cases} \eta & \text{if } U(w_\eta) \subseteq [U(x_\alpha)]', \\ 0_L & \text{if } U(w_\eta) \nsubseteq [U(x_\alpha)]'. \end{cases}$$

Let w be any element of X. Then $B'(w) = \eta \Rightarrow B(w) = \eta' \Rightarrow U(w'_\eta) \subseteq [U(x_\alpha)]' \Rightarrow U(w'_\eta) \subseteq y'_\beta \Rightarrow A(w) = \eta$. Therefore $B' \subseteq A$ and hence $\bigcup A' \subseteq \bigcup B$. Again $b_\mu \subseteq B \Rightarrow U(b_\mu) \subseteq [U(x_\alpha)]' \Rightarrow U(b_\mu) \subseteq x'_\alpha \Rightarrow b_\mu \subseteq x'_\alpha$. It follows that $\bigcup B \subseteq x'_\alpha \Rightarrow \bigcup A' \subseteq x'_\alpha \Rightarrow x_\alpha \subseteq \bigcap A$. Hence, $x_\alpha \subseteq U^r(y_\beta)$. □

Theorem 3.27. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space and 'int' be the induced interior operator on L^X. Then the respective interior space is compact iff every ultrafilter relative to an open set with respect to 'int' is convergent.*

Proof. Let the space be compact and $\mathscr{F}_и$ be an ultrafilter relative to an open set G on the space. Then by Theorem 3.4, $\mathscr{F} = \{\mathrm{cl}(F) \mid F \in \mathscr{F}_и\}$ is a collection of closed sets satisfying F. I. P. relative to the open set G. Consequently, by compactness, $\bigcap_{F\in\mathscr{F}_и} \mathrm{cl}(F) \nsubseteq G$. This implies that there is some $x_\alpha \in \mathrm{Pt}(L^X)$ such that $x_\alpha \subseteq \bigcap_{F\in\mathscr{F}_и} \mathrm{cl}(F)$. Thus by Theorem 3.22, we have $\mathscr{F}_и \to x_\alpha$.

Conversely, let $\mathscr{F}$ be a collection of closed sets satisfying F. I. P. relative to an open set G. Let $\mathscr{F}^*$ be a filter relative to the open set G and containing $\mathscr{F}$. Then $\bigcap_{F^*\in\mathscr{F}^*} F^* \subseteq \bigcap_{F\in\mathscr{F}} F$. Let $\mathscr{F}_и$ be an ultrafilter relative to the open set G. We then have,

$$\bigcap_{F_и\in\mathscr{F}_и} F_и \subseteq \bigcap_{F^*\in\mathscr{F}^*} F^* \subseteq \bigcap_{F\in\mathscr{F}} F. \tag{3.1}$$

Let $\mathscr{F}_{и} \to x_\alpha$. Then $\mathscr{Q}(x_\alpha) = \{U(x_\alpha^*) \mid U \in \mathscr{U}\} \subseteq \mathscr{F}_{и}$. Now let U be any symmetric member of $\mathscr{U}$. Let $F_{и}$ be any member of $\mathscr{F}_{и}$. Then $U(x_\alpha^*) \in \mathscr{F}_{и}$ implies that $F_{и} \bigcap U(x_\alpha^*) \neq \underline{0}$. Hence there exists $y_\beta \subseteq F_{и}$ such that $y_\beta \subseteq U(x_\alpha^*)$. This further implies $x_\alpha^* \subseteq U^r(y_\beta) = U(y_\beta)$, by Lemma 3.26. But $y_\beta \subseteq F_{и}$ implies $U(y_\beta) \subseteq U(F_{и})$. Hence, for any symmetric member U of $\mathscr{U}$, we get $x_\alpha^* \subseteq U(F_{и})$. Again since the collection of symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$, therefore by Lemma 3.3, $x_\alpha^* \subseteq \mathrm{cl}(F_{и}) = F_{и}, \ \ \forall\, F_{и} \in \mathscr{F}_{и}$. Hence,

$$x_\alpha^* \subseteq \bigcap_{F_{и} \in \mathscr{F}_{и}} F_{и}. \tag{3.2}$$

Now, if $x_\alpha^* \subseteq G$, then there is $U \in \mathscr{U}$ such that $U(x_\alpha^*) \subseteq G$, as G is open. But then $G \in \mathscr{F}_{и}$ and this contradicts the fact that $\mathscr{F}_{и}$ is an ultrafilter relative to G. So $x_\alpha^* \nsubseteq G$. This implies $\bigcap_{F_{и} \in \mathscr{F}_{и}} F_{и} \nsubseteq G$, by (3.2). Thus by (3.1), we have $\bigcap_{F \in \mathscr{F}} F \nsubseteq G$. Hence, the space is compact. □

Definition 3.28. A filter $\mathscr{F}$ in an L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *Cauchy* if for each $U \in \mathscr{U}$, $\exists x_\alpha \in \mathrm{Pt}(L^X)$ and $F \in \mathscr{F}$ such that $F \subseteq U(x_\alpha)$.

Definition 3.29. An L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *complete* if and only if for every Cauchy filter $\mathscr{F}$ relative to an open set with respect to the interior operator generated by $\mathscr{U}$, $\bigcap_{F \in \mathscr{F}} \mathrm{cl}(F) \neq \underline{0}$.

The following result follows from Theorem 3.22.

Theorem 3.30. *An L-semi-uniform space $(L^X, \mathscr{U})$ is complete iff every Cauchy ultrafilter relative to an open set with respect to the interior operator generated by $\mathscr{U}$ is convergent.*

Theorem 3.31. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be L-semi-uniform spaces and let $f^\to : L^X \to L^Y$ be L-semi-uniformly continuous. If $\mathscr{F}$ is a Cauchy filter in $(L^X, \mathscr{U})$, then $f^\to(\mathscr{F})$ is a Cauchy filter in $(L^Y, \mathscr{V})$.*

Proof. Let $\mathscr{F}$ be a Cauchy filter on L^X. Let $V \in \mathscr{V}$. Since $f^\to : L^X \to L^Y$ is L-semi-uniformly continuous, therefore there exists $U \in \mathscr{U}$ such that $\widehat{f^\to(U)} \subseteq V$. Now, $\mathscr{F}$ is a Cauchy filter on L^X. Hence, there exists $F \in \mathscr{F}$ and $x_\alpha \in \mathrm{Pt}(L^X)$ such that $F \subseteq U(x_\alpha)$. Then $f^\to(F) \subseteq V(f^\to(x_\alpha))$. Hence, $f^\to(\mathscr{F})$ is a Cauchy filter on $(L^Y, \mathscr{V})$. □

Theorem 3.32. *Let $(L^X, \mathscr{U})$ and $(L^Y, \mathscr{V})$ be two L-semi-uniform spaces and let $f^\to : L^X \to L^Y$ be an L-semi-uniformly isomorphism. Then $(L^X, \mathscr{U})$ is complete iff $(L^Y, \mathscr{V})$ is complete.*

Proof. Let $(L^Y, \mathscr{V})$ be complete and $\mathscr{F}$ be a Cauchy filter on L^X relative to an open set G. Let $V \in \mathscr{V}$. Then by Theorem 3.31, $f(\mathscr{F})$ is a Cauchy filter on $(L^Y, \mathscr{V})$. Again, since $f^\leftarrow$ is L-semi-uniformly continuous, therefore by Theorem 2.14, $f^\leftarrow$ is continuous and so $f^\to$ is open. Hence, $f^\to(G)$ is open in L^Y. Also, as $G \subseteq f^\leftarrow(f^\to(G))$ and $G \notin \mathscr{F}$, therefore $f^\to(\mathscr{F})$ is a Cauchy filter relative to the open set $f^\to(G)$. Thus, $f^\to(\mathscr{F})$ is convergent in $(L^Y, \mathscr{V})$, it being complete. But by Corollary 2.15, $f^\to$ is a homeomorphism. Consequently by Theorem 3.18, $\mathscr{F}$ converges in $(L^X, \mathscr{U})$. Hence, $(L^X, \mathscr{U})$ is complete. □

Definition 3.33. Let $(L^X, \mathscr{U})$ be an L-semi-uniform space and $A \in L^X$. Let for any $U \in \mathscr{U}$, $U_A : L^X \to L^X$ be a mapping such that

$$U_A(x_\alpha) = \begin{cases} U(x_\alpha) & \text{if } x_\alpha \subseteq A, \\ \underline{0} & \text{if } x_\alpha \nsubseteq A. \end{cases}$$

Then $\mathscr{U}_A = \{U_A \mid U \in \mathscr{U}\}$ is an L-semi-uniformity on A, which we call a *sub L-semi-uniformity* on A and $(A, \mathscr{U}_A)$ to be the *subspace*. $\mathscr{U}_A$ is called *open* or *closed sub L-semi-uniformity* provided $A = \text{int}_{\mathscr{U}_A}(A)$ or $A = (\text{int}_{\mathscr{U}_A}(A'))'$ respectively, where $\text{int}_{\mathscr{U}_A}$ is the interior operator generated by $\mathscr{U}_A$.

Theorem 3.34. *Every closed sub L-semi-uniformity in a complete L-semi-uniform space is complete.*

Proof. Let $(L^X, \mathscr{U})$ be a complete L-semi-uniform space and $A \in L^X$ such that $A = (\text{int}_{\mathscr{U}}(A'))'$, where $\text{int}_{\mathscr{U}}$ is the interior operator generated by $\mathscr{U}$. Let $\mathscr{F}_ɴ = \{F \mid F \subseteq A\}$ be a Cauchy ultrafilter relative to an open set B with respect to $\text{int}_{\mathscr{U}_A}$, where $\text{int}_{\mathscr{U}_A}$ is the interior operator generated by $\mathscr{U}_A$. Now if $B' \in \mathscr{F}_ɴ$, then from the definition of $\mathscr{F}_ɴ$, $B' \subseteq A$. But $B' \subseteq A$ implies $A' \subseteq B$ and consequently, $A' \notin \mathscr{F}_ɴ$, as $\mathscr{F}_ɴ$ is a filter relative to B. Also if, $B' \notin \mathscr{F}_ɴ$, then $A' \notin \mathscr{F}_ɴ$. Thus, in either case $\mathscr{F}_ɴ$ is an ultrafilter in $(L^X, \mathscr{U})$ relative to A'. Now, since for any $U \in \mathscr{U}$ $U_A \subseteq U$, therefore $\mathscr{F}_ɴ$ is also Cauchy in $(L^X, \mathscr{U})$. Thus, $\mathscr{F}_ɴ$ is a Cauchy ultrafilter in $(L^X, \mathscr{U})$ relative to the open set A' and consequently there exists $x_\alpha \in \text{Pt}(L^X)$ such that $\mathscr{F}_ɴ \to x_\alpha$. But as $A = (\text{int}_{\mathscr{U}}(A'))'$, so by Theorem 3.19, $x_\alpha \in A$. Hence, $(A, \mathscr{U}_A)$ is complete. □

Definition 3.35. An L-semi-uniform space $(L^X, \mathscr{U})$ is said to be *totally bounded* if for any $U \in \mathscr{U}$ there is a finite $A \subseteq Pt(L^X)$ such that

$$\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}.$$

Theorem 3.36. *In a totally bounded space $(L^X, \mathscr{U})$, every ultrafilter is a Cauchy filter.*

Proof. Let $\mathscr{F}_ɴ$ be an ultrafilter and $U \in \mathscr{U}$. By totally boundedness there is a finite $A \subseteq Pt(L^X)$ such that $\underline{1} = U(A) = \bigcup\{U(x_\alpha) \mid x_\alpha \in A\}$. But as $\underline{1} \in \mathscr{F}_ɴ$, therefore by Lemma 3.23, $U(x_\alpha) \in \mathscr{F}_ɴ$, for some $x_\alpha \in A$. □

Theorem 3.37. *Let $(L^X, \mathscr{U})$ be an L-semi-uniform space. Then the space is compact iff* (i) *$(L^X, \mathscr{U})$ is totally bounded and* (ii) *$(L^X, \mathscr{U})$ is complete.*

Proof. Let $(L^X, \mathscr{U})$ be a compact space.

(i) Let $U \in \mathscr{U}$ and 'cl' be the closure operator generated by $\mathscr{U}$. Then $\{\text{int}(U(x_\alpha)) \mid x_\alpha \in \text{Pt}(L^X)\}$ is an open cover of $\underline{1}$. Since $\text{cl}(\underline{1}) = \underline{1}$, therefore by compactness, for this open cover there is a finite $A \subseteq Pt(L^X)$ such that $\underline{1} = \bigcup\{\text{int}(U(x_\alpha)) \mid x_\alpha \in A\}$. Hence $(L^X, \mathscr{U})$ is totally bounded.

(ii) Follows from Theorems 3.27 and 3.30.

Conversely, if the space is totally bounded and complete, then, by Theorems 3.36, 3.30 and 3.27 the space is compact. □

References

[1] G. Artico and R. Moresco, Fuzzy proximities and totally bounded fuzzy uniformities, J. Math. Anal. Appl. 92 (1984) 320–337.
[2] B. Batíková, Completion of semi-uniform spaces, Appl. Categor. Struct. 15 (2007) 483–491.
[3] E. Čech, Topological Space, Interscience, New York, 1966.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] P. Fletcher and W. Lindgren, Quasi-uniform Spaces, Marcel Dekker INC., 1982.
[6] D. Hazarika and D. K. Mitra, L-locally quasi-uniform spaces, J. Indian Acad. Math. 31(2) (2009) 317–324.
[7] D. Hazarika and D. K. Mitra, On L-semi-pseudo metrization, Int. Math. Forum 5(23) (2010) 1141–1148.
[8] D. Hazarika and D. K. Mitra, On some generalised classes of L-valued uniform structures, Abstracts (eds. P. Cintula, E. P. Klement and L. N. Stout), 31st Linz Seminar on Fuzzy Set Theory: Lattice Valued Logic and its Applications (9-13 February, 2010) 87–90, http://www.flll.uni–linz.ac.at/div/research/linz2010/LINZ2010Abstracts.pdf.
[9] T. L. Hicks and S. M. Huffman, A note on locally quasi-uniform spaces, J. Canad. Math. Bull. 19(4) (1976) 501–504.
[10] U. Höhle, Probabilistic uniformization of fuzzy topologies, Fuzzy Sets and Systems 1 (1978) 311–332.
[11] B. Hutton, Uniformities of fuzzy topological spaces, J. Math. Anal. Appl. 58 (1977) 559–571.
[12] B. Hutton, Products of fuzzy topological spaces, Topology Appl. 11 (1980) 59–67.
[13] A. K. Katsaras, On fuzzy uniform spaces, J. Math. Anal. Appl. 101 (1984) 97–113.
[14] W. Kotzé, Uniform spaces, in Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, 1999.
[15] W. Lindgren and P. Fletcher, Locally quasi-uniform spaces with countable bases, Duke Math. J. 41 (1974) 231–240.
[16] Y. M. Liu and M. K. Luo, Fuzzy Topology, Advances in Fuzzy Systems - Applications and Theory, vol. 9, World Scientific, 1997.
[17] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.
[18] H. C. Ming, Fuzzy topological spaces, J. Math. Anal. Appl. 110 (1985) 141–178.
[19] D. K. Mitra and D. Hazarika, L-locally uniform spaces, J. Fuzzy Math. 18(2) (2010) 505–516.
[20] H. W. Pu and H. H. Pu, Semi-quasi-uniform spaces, J. Portugal. Math. 33 (1974) 177–184.
[21] S. E. Rodabaugh and E. Klement (eds.), Topological and Algebraic Structures in Fuzzy Sets, A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, vol. 20, Kluwer Academic Publishers(Boston/ Dordrecht/ London), 2003.
[22] J. Williams, Locally uniform spaces, Trans. Amer. Math. Soc. 168 (1972) 435–469.

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