Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 4, No. 2, (October 2012), pp. 235–242
ISSN 2093–9310
http://www.afmi.or.kr

Fuzzy weakly prime ideals of near-subtraction semigroups

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Received 30 August 2011; Revised 16 November 2011; Accepted 27 November 2011

Abstract. In this paper, we introduce the notion of fuzzy ideal,fuzzy weak ideal,fuzzy weakly prime left ideal and fuzzy prime left ideal system of a near-subtraction semigroup. We characterize fuzzy weak ideal and fuzzy ideal of a near-subtraction semigroup X through weak ideal and ideal of X respectively. We have shown that a fuzzy left ideal μ of a near-subtraction semigroup X is a fuzzy weakly prime left ideal of X if and only if $Im\ \mu = \{1, t\}$ and μ_1 is a weakly prime left ideal of X where $1 > t \geq 0$.

2010 AMS Classification: 08A72, 06F35, 03E72

Keywords: Near-subtraction semigroup, Fuzzy ideal, Fuzzy weak ideal, Fuzzy prime left ideal, Fuzzy weakly prime left ideal.

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1. Introduction

B. M. Schein [8] considered the systems of the form $(\phi; \circ, \backslash)$where ϕ is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" and (hence $(\phi; \backslash)$ is a subtraction algebra in the sense of [1].) B. Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebra of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [6] introduced the notion of ideals in subtraction algebras and obtained significant results. P. Dheena et al. [2, 3] introduced prime and fuzzy prime ideals of subtraction algebras and and near-subtraction semigroups. P. Dheena et al. [4] introduced prime left ideal and weakly prime left ideal. Y. B. Jun et al. [5] discussed prime and irreducible ideals in subtraction algebras. E. H. Roh et al. [7] considered prime and semiprime ideals in subtraction semigroups.

In this paper, we introduce the notion of fuzzy ideal, fuzzy weak ideal, fuzzy prime left ideal and fuzzy weakly prime left ideal of near-subtraction semigroup. Similar to fuzzy set theory, we have obtained significant results in a near-subtraction semigroup.

2. Preliminaries

A nonempty set X together with a binary operation "$-$" is said to be a subtraction algebra if it satisfies the following:

(1) $x-(y-x)=x$.
(2) $x-(x-y)=y-(y-x)$.
(3) $(x-y)-z=(x-z)-y$, for every $x,\ y,\ z\in X$.

Example 2.1. Let A be any nonempty set. Then $(P(A),\backslash)$ is a subtraction algebra, where "$P(A)$" denotes the power set of A and "$\backslash$" denotes the set theoretic subtraction.

In a subtraction algebra the following holds :

(1) $x-0=x$ and $0-x=0$.
(2) $(x-y)-x=0$.
(3) $(x-y)-y=x-y$.
(4) $x-(x-y)\leq y$.
(5) $(x-y)-(y-x)=x-y$.
(6) $x-(x-(x-y))=x-y$.
(7) $(x-y)-(z-y)\leq x-z$.
(8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$.
(9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$.
(10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$.

Definition 2.2. A nonempty set X together with two binary operations "-" and " $\cdot$ " is said to be a *subtraction semigroup* if it satisfies the following:

(1) $(X;-)$ is a subtraction algebra.
(2) $(X;\cdot)$ is a semigroup.
(3) $x(y-z)=xy-xz$ and $(x-y)z=xz-yz$ for every $x,y,z\in X$.

Note that it is clear that $0x=0$ and $x0=0$ for every $x\in X$.

Definition 2.3. Let $(X,-,\cdot)$ be a subtraction semigroup. A nonempty subset I of X is called a *left* (*right*) *ideal* if $x-y\in I$, for every $x\in I,y\in X$ and $XI\subseteq I(IX\subseteq I)$. If I is both a left and right ideal then I is an ideal, where $AB=\{ab|a\in A,b\in B\}$for any nonempty subsets A,B of X.

Example 2.4. Let Γ be a subtraction algebra. Then the set $M_h(\Gamma)$ of all homomorphisms of Γ into Γ is a subtraction semigroup under point wise subtraction and composition of mappings.

Definition 2.5. A nonempty set X together with two binary operations "-" and " $\cdot$ " is said to be a *near-subtraction semigroup* if it satisfies the following:

(1) $(X;-)$ is a subtraction algebra.
(2) $(X;\cdot)$ is a semigroup.
(3) $(x-y)z=xz-yz$ for every $x,y,z\in X$.

Example 2.6. Let Γ be a subtraction algebra. Then the set $M(\Gamma)$ of all mappings of Γ into Γ is a near-subtraction semigroup under point wise subtraction and composition of mappings.

Example 2.7. Let $X = \{0,\ a,\ b,\ 1\}$ in which " $-$ " and " $\cdot$ " are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

Then $(X, -, \cdot)$ is a near-subtraction semigroup.

Definition 2.8. A near-subtraction semigroup X is said to be *zero-symmetric* if $x0 = 0$ for every $x \in X$.

Definition 2.9 ([3]). Let $(X, -, \cdot)$ be a near-subtraction semigroup. A nonempty subset I of X is called a *weak ideal* if

(WI1) $x - y \in I$ for all $x, y \in I$.
(WI2) $xi - x(y - i) \in I$ for all $x, y \in X$ and for all $i \in I$.
(WI3) $IX \subseteq I$.

Note that a nonempty subset I of X is called a *left weak ideal* if it satisfies (WI1) and (WI2) and is called *right weak ideal* if it satisfies (WI1) and (WI3).

Definition 2.10 ([4]). Let $(X, -, \cdot)$ be a near-subtraction semigroup. A nonempty subset I of X is called an *ideal* if

(I1) $x - y \in I$ for all $x \in I$ and for all $y \in X$.
(I2) $xi - x(y - i) \in I$ for all $x, y \in X$ and for all $i \in I$.
(I3) $IX \subseteq I$.

Note that a nonempty subset I of X is called a *left ideal* if it satisfies $I1$ and $I2$ and is called *right ideal* if it satisfies $I1$ and $I3$.

Remark 2.11. Every ideal of a near-subtraction semigroup X is a weak ideal. But weak ideal of X need not be an ideal of X as shown by the Example 2.12.

Example 2.12. Consider near-subtraction semigroup X as in the Example 2.7. Clearly $I = \{0, a\}$ is a right weak ideal of X. But I is not a right ideal of X because $a - b = 1 \notin I$.

Definition 2.13 ([4]). A left ideal P of a near-subtraction semigroup X is called a *prime left ideal* if $L_1 \cdot L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$ for all left ideals L_1 and L_2 of X.

Definition 2.14 ([4]). A left ideal P of a near-subtraction semigroup X is said to be *weakly prime* if $L_1 \cdot L_2 \subseteq P$ implies $L_1 = P$ or $L_2 = P$ for all left ideals L_1 and L_2 of X which contains P.

Remark 2.15 ([4]). (1) A prime left ideal is always weakly prime left ideal. But the converse need not be true as the following Example 2.16 shows.

(2) Let X be a zero-symmetric near-subtraction semigroup. Let I be a subset of X such that $x - y \in I$ for every $x \in I, y \in X$. Then the following are equivalent:

(i) $XI \subseteq I$.

(ii) $xi - x(x_1 - i) \in I$ for all $x, x_1 \in X$ and $i \in I$.

Proof. (i) $\Rightarrow$ (ii) Let $i \in I, x, x_1 \in X$. By (i) $xi \in I$. Since I is a subset of X such that $x - y \in I$ for every $x \in I, y \in X$, we have $xi - x(x_1 - i) \in I$ for all $x, x_1 \in X$ and $i \in I$.

(ii) $\Rightarrow$ (i) Since X is zero-symmetric near-subtraction semigroup,

$$xi - x(0 - i) = xi - x0 = xi - 0 = xi \in I,$$

for all $x \in X$ and $i \in I$. □

Example 2.16 ([4])**.** Consider the following near-subtraction semigroup.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	c	b
b	b	0	0	b
c	c	0	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Here the left ideal $\{0, c\}$ is weakly prime left ideal but not prime left ideal since $\{0, b\}$ is a left ideal such that $\{0, b\} \cdot \{0, b\} \subseteq \{0, c\}$.

Definition 2.17. Let X be a nonempty set. A mapping $\mu : X \to [0, 1]$ is called a *fuzzy set* of X.

Definition 2.18. Let $(X, -, \cdot)$ be a near-subtraction semigroup and μ be a fuzzy set of near-subtraction semigroup. The *level subset* of μ denoted by μ_t for all $t \in [0, 1]$ is defined as $\mu_t = \{x \in X \mid \mu(x) \geq t\}$.

3. Fuzzy weakly prime left ideal

Throughout this section X denote zero symmetric near-subtraction semigroup unless otherwise specified.

Definition 3.1. A fuzzy set μ of a near-subtraction semigroup X is called a *fuzzy weak ideal* if
(FWI1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$
(FWI2) $\mu(xz - x(y - z)) \geq \mu(z)$
(FWI3) $\mu(xy) \geq \mu(x)$ for all $x, y, z \in X$.

Note that a fuzzy set μ of X is called a *fuzzy left weak ideal* if it satisfies (FWI1) and (FWI2) and is called *fuzzy right weak ideal* if it satisfies (FWI1) and (FWI3).

Definition 3.2. A fuzzy set μ of a near-subtraction semigroup X is called a *fuzzy ideal* if

(FI1) $\mu(x - y) \geq \mu(x)$
(FI2) $\mu(xz - x(y - z)) \geq \mu(z)$
(FI3) $\mu(xy) \geq \mu(x)$ for all $x, y, z \in X$.

Note that a fuzzy subset μ of X is called a *fuzzy left ideal* if it satisfies (FI1) and (FI2) and is called *fuzzy right ideal* if it satisfies (FI1) and (FI3).

Theorem 3.3. *Let μ be a fuzzy subset of a a near-subtraction semigroup X. Then μ is fuzzy weak ideal if and only if the level subset μ_t is a weak ideal of X for all $t \in [0,1]$ whenever nonempty.*

Proof. Let μ be a fuzzy weak ideal of a a near-subtraction semigroup X. Let $t \in [0,1]$ and $x, y \in \mu_t$. Then $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq t$. Thus $x - y \in \mu_t$ for all $x, y \in \mu_t$. If $x \in \mu_t$, then $\mu(zx - z(y-x)) \geq \mu(x) \geq t$ and $\mu(xy) \geq \mu(x) \geq t$. Thus $zx - z(y-x), xy \in \mu_t$ for all $x \in \mu_t$ and for all$y, z \in X$. Therefore μ_t is a weak ideal for all $t \in [0,1]$ whenever nonempty.

Conversely,let $x, y, z \in X$. Then $x, y \in \mu_t$ for $t = \min\{\mu(x), \mu(y)\}$. Thus $x-y \in \mu_t$. Therefore $\mu(x-y) \geq \min\{\mu(x), \mu(y)\} = t$. Let $s = \mu(x)$. Now, $x \in \mu_s$. Then $zx - z(y-x), xy \in \mu_s$ for all $y, z \in X$. Thus $\mu(zx - z(y-x)) \geq \mu(x) = t$ and $\mu(xy) \geq \mu(x) = t$. Therefore μ is a fuzzy weak ideal of X. □

Corollary 3.4. *χ_I is a fuzzy weak ideal of X if and only if I is a weak ideal of X.*

Proof. The proof follows from Theorem 3.3. □

Theorem 3.5. *Let μ be a fuzzy subset of a a near-subtraction semigroup X. Then μ is fuzzy ideal if and only if the level subset μ_t is an ideal of X for all $t \in [0,1]$ whenever nonempty.*

Proof. Let μ be a fuzzy ideal of a a near-subtraction semigroup X. Let $t \in [0,1]$. If $x \in \mu_t$, then $\mu(x-y) \geq \mu(x) \geq t$, $\mu(zx - z(y-x)) \geq \mu(x) \geq t$ and $\mu(xy) \geq \mu(x) \geq t$. Thus $x - y, zx - z(y-x)$ and $xy \in \mu_t$ for all $x \in \mu_t$ and for all$y, z \in X$. Therefore μ_t is an ideal for all $t \in [0,1]$ whenever nonempty.

Conversely,let $x, y, z \in X$. Let $s = \mu(x)$. Now, $x \in \mu_s$. Then $x - y, zx - z(y-x), xy \in \mu_s$ for all $y, z \in X$. Thus $\mu(x-y) \geq \mu(x) = t, \mu(zx - z(y-x)) \geq \mu(x) = t$ and $\mu(xy) \geq \mu(x) = t$. Therefore μ is a fuzzy ideal of X. □

Corollary 3.6. *χ_I is a fuzzy ideal of X if and only if I is an ideal of X.*

Proof. The proof follows from Theorem 3.5. □

Corollary 3.7. *Every fuzzy ideal is a fuzzy weak ideal.*

Proof. The proof follows from Remark 2.11, and Theorems 3.3 and 3.5. □

The converse of above Corollary 3.7 need not be true as shown by following Example 3.8.

Example 3.8. Consider near-subtraction semigroup X as in the Example 2.7.

$$\mu(x) = \begin{cases} 0.9 & \text{if } x \in \{0, a\} \\ 0.3 & \text{otherwise} \end{cases}$$

Clearly μ is a fuzzy right weak ideal of X. But μ is not a fuzzy right ideal of X because $\mu(a - b = 1) = 0.3 < 0.9 = \mu(a)$.

Definition 3.9. The *product* between two fuzzy sets λ and μ of a near-subtraction semigroup X is defined as usual:

$$(\lambda \cdot \mu)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\lambda(y), \mu(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.10. A fuzzy left ideal μ of a near-subtraction semigroup X is called *prime* if $\lambda_1 \cdot \lambda_2 \subseteq \mu$ implies $\lambda_1 \subseteq \mu$ or $\lambda_2 \subseteq \mu$ for all fuzzy left ideals λ_1 and λ_2 of X.

Definition 3.11. A fuzzy left ideal μ of a near-subtraction semigroup X is called *fuzzy weakly prime* if $\lambda_1 \cdot \lambda_2 \subseteq \mu$ implies $\lambda_1 = \mu$ or $\lambda_2 = \mu$ for all fuzzy left ideals λ_1 and λ_2 of X which contain μ.

Remark 3.12. A fuzzy prime left ideal is always fuzzy weakly prime ideal. But the converse need not be true as the following example shows.

Example 3.13. Consider the near-subtraction semigroup as in Example 2.16 We define fuzzy sets λ, σ as follows:

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in \{0, c\} \\ 0.2 & \text{otherwise} \end{cases} \qquad \sigma(x) = \begin{cases} 0.8 & \text{if } x \in \{0, b\} \\ 0 & \text{otherwise} \end{cases}$$

$$(\sigma \cdot \sigma)(x) = \begin{cases} 1 & \text{if } x \in \{0, b\} \cdot \{0, b\} = \{0\} \\ 0 & \text{otherwise} \end{cases}$$

Here the fuzzy left ideal λ is fuzzy weakly prime left ideal but not fuzzy prime left ideal since $\sigma \cdot \sigma \subseteq \lambda$ but $\sigma \not\subseteq \lambda$.

Lemma 3.14. *If μ is a nonconstant fuzzy weakly prime left ideal of a near-subtraction semigroup X, then $Im\mu = \{1, t\}$ where $1 > t \geq 0$.*

Proof. Let μ be a nonconstant fuzzy weakly prime left ideal of X. If $Im\mu = \{t_1, t_2, t_3\}$ for $1 > t_1 > t_2 > t_3 \geq 0$, then there exists $a, b, c \in X$ such that $\mu(a) = t_1, \mu(b) = t_2, \mu(c) = t_3$. Choose s_1, s_2 in such a way that $1 > s_1 > t_1 > s_2 > t_2$. Now, we define fuzzy sets λ_1, λ_2 as follows:

$$\lambda_1(x) = \begin{cases} s_1 & \text{if } x \in \mu_{t_1} \\ t_2 & \text{otherwise} \end{cases} \qquad \lambda_2(x) = \begin{cases} t_1 & \text{if } x \in \mu_{t_1} \\ s_2 & \text{if } x \in \mu_{t_2} - \mu_{t_1} \\ t_3 & \text{otherwise} \end{cases}$$

Clearly $\mu \subseteq \lambda_1$ and $\mu \subseteq \lambda_2$.

$$(\lambda_2 \cdot \lambda_1)(x) = \begin{cases} t_1 & \text{if } x = yz, y, z \in \mu_{t_1} \\ s_2 & \text{if } x = yz, y \in \mu_{t_2} - \mu_{t_1}, z \in \mu_{t_1} \\ t_2 & \text{if } x = yz, y, z \in \mu_{t_2} - \mu_{t_1} \\ t_3 & \text{if } x = yz, y \in \mu_{t_3} - \mu_{t_2} \\ 0 & \text{otherwise} \end{cases}$$

Thus $\lambda_2 \cdot \lambda_1 \subseteq \mu$. But $\lambda_1(a) = s_1 > t_1 = \mu(a)$ and $\lambda_2(b) = s_2 > t_2 = \mu(b)$. Then $\lambda_1 \neq \mu$ and $\lambda_2 \neq \mu$ is a contradiction. If $Im\mu = \{t_1, t_2\}$ for $1 > t_1 > t_2 \geq 0$, then there exists $a_1, b_1 \in X$ such that $\mu(a_1) = t_1, \mu(b_1) = t_2$ Choose s_1, s_2 in such a way that $1 > s_1 > t_1 > s_2 > t_2$. Now, we define fuzzy sets λ_1, λ_2 as follows:

$$\lambda_1(x) = \begin{cases} s_1 & \text{if } x \in \mu_{t_1} \\ t_2 & \text{otherwise} \end{cases} \qquad \lambda_2(x) = \begin{cases} t_1 & \text{if } x \in \mu_{t_1} \\ s_2 & \text{otherwise} \end{cases}$$

Clearly $\mu \subseteq \lambda_1$ and $\mu \subseteq \lambda_2$.

$$(\lambda_2 \cdot \lambda_1)(x) = \begin{cases} t_1 & \text{if } x = yz, y, z \in \mu_{t_1} \\ s_2 & \text{if } x = yz, y \in \mu_{t_2} - \mu_{t_1}, z \in \mu_{t_1} \\ t_2 & \text{if } x = yz, y \notin \mu_{t_1}, z \notin \mu_{t_1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\lambda_2 \cdot \lambda_1 \subseteq \mu$. But $\lambda_1(a_1) = s_1 > t_1 = \mu(a)$ and $\lambda_2(b_1) = s_2 > t_2 = \mu(b)$. Then $\lambda_1 \neq \mu$ and $\lambda_2 \neq \mu$ is a contradiction. Therefore $Im\mu = \{1, t\}$ where $1 > t \geq 0$. □

Lemma 3.15. *If μ is a nonconstant weakly fuzzy prime left ideal of a near-subtraction semigroup X, then μ_1 is a weakly prime left ideal of X.*

Proof. Let μ be a nonconstant fuzzy weakly prime left ideal of X. Let L_1 and L_2 be left ideals of X such that $L_1 \cdot L_2 \subseteq \mu_1$, $\mu_1 \subseteq L_1$ and $\mu_1 \subseteq L_2$. Now, we define fuzzy sets λ and σ as follows:

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in L_1 \\ t & \text{otherwise} \end{cases} \qquad \sigma(x) = \begin{cases} 1 & \text{if } x \in L_2 \\ t & \text{otherwise} \end{cases}$$

Clearly λ and σ are fuzzy left ideals of X, $\mu \subseteq \lambda$ and $\mu \subseteq \sigma$.

$$(\lambda \cdot \sigma)(x) = \begin{cases} 1 & \text{if } x \in L_1 \cdot L_2 \\ t & \text{if } x = yz, y \notin L_1 \text{ or } z \notin L_2 \\ 0 & \text{otherwise} \end{cases}$$

Then $\lambda \cdot \sigma \subseteq \mu$ implies $\lambda = \mu$ or $\sigma = \mu$. Therefore $L_1 = \mu_1$ or $L_2 = \mu_1$. Hence μ_1 is a weakly prime left ideal of X. □

Theorem 3.16. *Let μ be a nonconstant fuzzy left ideal of a near-subtraction semigroup X. Then μ is a fuzzy weakly prime left ideal of X if and only if*

(1) *$Im\mu = \{1, t\}$ where $1 > t \geq 0$.*
(2) *μ_1 is a weakly prime left ideal of X.*

Proof. Let μ be a fuzzy weakly prime left ideal of X. Then (1) and (2) follow from Lemmas 3.14 and 3.15.

Conversely, if there exists fuzzy left ideals σ and θ of X both containing μ such that $\sigma \cdot \theta \subseteq \mu$ with $\sigma \neq \mu$ and $\theta \neq \mu$. Then there exists $a, b \in X$ such that $\sigma(a) = s_1 > t = \mu(a)$ and $\theta(b) = s_2 > t = \mu(b)$. Thus $a \in \sigma_{s_1}$ but $a \notin \mu_1$ and $b \in \theta_{s_2}$ but $b \notin \mu_1$. Clearly σ_{s_1} and θs_2 are left ideals. Let $x \in \mu_1$. Since $\mu \subseteq \sigma$, $\mu(x) = 1$ implies $\sigma(x) = 1$. Thus $\sigma(x) \geq s_1$. Therefore $\mu_1 \subseteq \sigma_{s_1}$. Similarly,$\mu_1 \subseteq \theta_{s_2}$. If $\sigma_{s_1} \cdot \theta_{s_2} \subseteq \mu_1$, then $\sigma_{s_1} = \mu_1$ or $\theta_{s_2} = \mu_1$ which is a contradiction. Thus $\sigma_{s_1} \cdot \sigma_{s_2} \nsubseteq \mu_1$. Then $a_1 b_1 \notin \mu_1$ for some $a_1 \in \sigma_{s_1}$ and for some $b_1 \in \theta_{s_2}$. Thus $\mu(a_1 b_1) = t$.

$$\begin{aligned} (\sigma \cdot \theta)(a_1 b_1) &\geq \sigma(a_1) \wedge \theta(b_1) \\ &= s_1 \wedge s_2 \\ &> t = \mu(a_1 b_1) \end{aligned}$$

This is a contradiction to $\sigma.\theta \subseteq \mu$. Therefore μ is a fuzzy weakly prime left ideal of X. □

Theorem 3.17. *Let μ be a nonconstant fuzzy left ideal of a near-subtraction semigroup X. Then μ is a fuzzy prime left ideal of X if and only if*
(1) $Im\mu = \{1, t\}$ *where* $1 > t \geq 0$.
(2) μ_1 *is a prime left ideal of* X.

Proof. Let μ be a fuzzy prime left ideal of X. Then μ is a fuzzy weakly prime left ideal of X. By Lemma 3.14, $Im\mu = \{1, t\}$ where $1 > t \geq 0$. Let L_1 and L_2 be left ideals of X such that $L_1 \cdot L_2 \subseteq \mu_1$. Clearly χ_{L_1} and χ_{L_2} are fuzzy left ideals of X.

$$(\chi_{L_1} \cdot \chi_{L_2})(x) = \begin{cases} 1 & \text{if } x \in L_1 \cdot L_2 \\ 0 & \text{otherwise} \end{cases}$$

Then $\chi_{L_1} \cdot \chi_{L_2} \subseteq \mu$ implies $\chi_{L_1} \subseteq \mu$ or $\chi_{L_2} \subseteq \mu$. Therefore $L_1 \subseteq \mu_1$ or $L_2 \subseteq \mu_1$. Hence μ_1 is a prime left ideal of X.

Conversely,if there exists fuzzy left ideals λ and σ of X such that $\lambda \cdot \sigma \subseteq \mu$ with $\lambda \nsubseteq \mu$ and $\sigma \nsubseteq \mu$. Then there exists $a, b \in X$ such that $\lambda(a) = s_1 > t = \mu(a)$ and $\sigma(b) = s_2 > t = \mu(b)$. Thus $a \in \lambda_{s_1}$ but $a \notin \mu_1$ and $b \in \sigma_{s_2}$ but $b \notin \mu_1$. Clearly λ_{s_1} and σ_{s_2} are left ideals. If $\lambda_{s_1} \cdot \sigma_{s_2} \subseteq \mu_1$, then $\lambda_{s_1} \subseteq \mu_1$ or $\sigma_{s_2} \subseteq \mu_1$ which is a contradiction. Thus $\lambda_{s_1} \cdot \sigma_{s_2} \nsubseteq \mu_1$. Then $a_1 b_1 \notin \mu_1$ for $a_1 \in \lambda_{s_1}$ and $b_1 \in \sigma_{s_2}$. Thus $\mu(a_1 b_1) = t$.

$$\begin{aligned} (\lambda \cdot \sigma)(a_1 b_1) &\geq \lambda(a_1) \wedge \sigma(b_1) \\ &= s_1 \wedge s_2 > t = \mu(a_1 b_1) \end{aligned}$$

This contradicts $\lambda \cdot \sigma \subseteq \mu$. Therefore μ is a fuzzy prime left ideal of X. □

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