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$\mathcal{N}$-structures applied to ideals in semigroups

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Abstract. The notions of (regular) $\mathcal{N}$-subsemigroups, left (right) $\mathcal{N}$-ideals, and (generalized) $\mathcal{N}$-bi-ideals are introduced, and several properties are investigated. Conditions for an $\mathcal{N}$-structure to be a regular $\mathcal{N}$-subsemigroup are provided, and conditions for a generalized $\mathcal{N}$-bi-ideal to be an $\mathcal{N}$-bi-ideal are considered. Characterizations of a regular $\mathcal{N}$-subsemigroup, a left (right) $\mathcal{N}$-ideal and a generalized $\mathcal{N}$-bi-ideal are displayed.

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1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Hur et al. [2] studied fuzzy sub-semigroups and fuzzy ideals with operators in semigroups, and Kim et al [7] discussed ideal theory of sub-semigroups based on the bipolar valued fuzzy set theory. Shabir and Ahmad [9] applied the soft set theory to ternary semigroups. Aslam et al. [1] investigated properties of rough (m, n)-bi-ideals and generalized rough (m, n)-bi-ideals in semigroups. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [5] introduced and used a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. The important achievement of the paper [5] was that one can deal with positive and negative information simultaneously by combining ideas in [5] and already well known positive information. Jun et al. discussed the $\mathcal{N}$-structure in hyper BCK-algebras, subtraction algebras and BCH-algebras (see [3, 4, 6]).

In this article, we introduce the notion of (regular) $\mathcal{N}$-subsemigroups, left (right) $\mathcal{N}$-ideals, and (generalized) $\mathcal{N}$-bi-ideals. We provide conditions for an $\mathcal{N}$-structure to be a regular $\mathcal{N}$-subsemigroup, and for a generalized $\mathcal{N}$-bi-ideal to be an $\mathcal{N}$-bi-ideal. We discuss characterizations of a regular $\mathcal{N}$-subsemigroup, a left (right) $\mathcal{N}$-ideal and a generalized $\mathcal{N}$-bi-ideal.

2. Preliminaries

Let S be a semigroup. For any subsets A and B of S, the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

An element $x \in S$ is said to be *regular* if there exists an element $a \in S$ such that $x = xax$. A semigroup S is said to be *regular* if every element of S is regular. For any $x \in S$, we write

$$R_x := \{a \in S \mid x = xax\}.$$

A nonempty set subset A of S is called a *subsemigroup* of S if $AA \subseteq A$. A nonempty set subset A of S is called a *left* (*right*) *ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). Further, A is called a two-sided ideal of S if it is both a left and a right ideal of S. A subsemigroup A of S is called a *bi-ideal* of S if $ASA \subseteq A$. A nonempty set subset A of S is called a *generalized bi-ideal* of S if $ASA \subseteq A$. For the undefined notions we refer to the book [8].

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(S, [-1, 0])$ the collection of functions from a set S to $[-1, 0]$. We say that an element of $\mathcal{F}(S, [-1, 0])$ is a *negative-valued function* from S to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on S). By an *$\mathcal{N}$-structure* we mean an ordered pair (S, f) of S and an $\mathcal{N}$-function f on S. For any $\mathcal{N}$-structure (S, f) and $\alpha \in [-1, 0]$, the set

$$C(f; \alpha) := \{x \in S \mid f(x) \leq \alpha\}$$

is called the closed support of (S, f) related to α.

3. $\mathcal{N}$-subsemigroups

In what follows, let S denote a semigroup unless otherwise specified.

Let (S, f) and (S, g) be two $\mathcal{N}$-structures. The *$\mathcal{N}$-product* of (S, f) and (S, g) is an $\mathcal{N}$-structure $(S, f \,\tilde{\circ}\, g)$ in which

$$(f \,\tilde{\circ}\, g)(x) = \begin{cases} \bigwedge\limits_{x=ab} \{\bigvee \{f(a), g(b)\}\} & \text{if } \; x = ab \text{ for some } a, b \in S \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the operation $\tilde{\circ}$ is associative. For a nonempty subset A of S, the *characteristic* $\mathcal{N}$*-function* κ_A is defined as follows:

$$\kappa_A : S \to [-1,0], \quad x \mapsto \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.1. By an $\mathcal{N}$*-subsemigroup* of S we mean an $\mathcal{N}$-structure (S, f) in which f satisfies:

$$(\forall x, y \in S) \left(f(xy) \leq \bigvee \{f(x), f(y)\} \right). \tag{3.1}$$

Example 3.2. Let $S = \{a, b, c, x, y, z\}$ be a semigroup with the following multiplication table:

	a	b	c	x	y	z
a	a	a	a	x	a	a
b	a	b	b	x	b	b
c	a	b	c	x	y	y
x	a	a	x	x	x	x
y	a	b	c	x	y	y
z	a	b	c	x	y	z

Let (S, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} a & b & c & x & y & z \\ -0.9 & -0.7 & -0.4 & -0.5 & -0.2 & -0.1 \end{pmatrix}.$$

Then (S, f) is an $\mathcal{N}$-subsemigroup of S.

Theorem 3.3. *An* $\mathcal{N}$*-structure* (S, f) *is an* $\mathcal{N}$*-subsemigroup of* S *if and only if the nonempty closed support of* (S, f) *related to* α *is a subsemigroup of* S *for all* $\alpha \in [-1, 0]$.

Proof. Let $\alpha \in [-1, 0]$ be such that $C(f; \alpha) \neq \varnothing$. Let $x, y \in C(f; \alpha)$. Then $f(x) \leq \alpha$ and $f(y) \leq \alpha$. It follows from (3.1) that $f(xy) \leq \bigvee \{f(x), f(y)\} \leq \alpha$ so that $xy \in C(f, \alpha)$. Hence $C(f, \alpha)$ is a subsemigroup of S.

Conversely, assume that the nonempty closed support of (S, f) related to α is a subsemigroup of S for all $\alpha \in [-1, 0]$. If there exist $a, b \in S$ such that

$$f(ab) > \beta = \bigvee \{f(a), f(b)\},$$

then $a, b \in C(f, \beta)$ but $ab \notin C(f, \beta)$. This is a contradiction, and so

$$f(xy) \leq \bigvee \{f(x), f(y)\}$$

for all $x, y \in S$. Therefore (S, f) is an $\mathcal{N}$-subsemigroup of S. □

Theorem 3.4. *An* $\mathcal{N}$*-structure* (S, f) *is an* $\mathcal{N}$*-subsemigroup of* S *if and only if* $f \subseteq f \,\tilde{\circ}\, f$.

Proof. Straightforward. □

Definition 3.5. An $\mathcal{N}$-subsemigroup (S, f) of S is said to be *regular* if the following condition is valid:

$$(\forall x \in S)\,[f(x) \neq 0 \;\Rightarrow\; (\exists a \in R_x)\,(f(x) \geq f(a))]\,. \tag{3.2}$$

Theorem 3.6. *An $\mathcal{N}$-structure (S, f) is a regular $\mathcal{N}$-subsemigroup of S if and only if the nonempty closed support of (S, f) related to α is a regular subsemigroup of S for all $\alpha \in [-1, 0]$.*

Proof. With Theorem 3.3, it is sufficient to show that (S, f) satisfies the condition (3.2) if and only if each element of $C(f, \alpha)$, $\alpha \in [-1, 0]$, is regular. Assume that (S, f) satisfies the condition (3.2). Then there exists $a \in R_x$ such that $f(x) \geq f(a)$, and furthermore, for every $x \in C(f, \alpha)$, $f(a) \leq f(x) \leq \alpha$. This implies that $a \in C(f, \alpha)$, and so $C(f, \alpha)$ is a regular subsemigroup of S.

Conversely, suppose that each element of $C(f, \alpha)$, $\alpha \in [-1, 0]$, is regular. Assume that (3.2) does not hold, i.e, there exists $x \in S$ such that $f(x) \neq 0$ and $f(x) < f(a)$ for all $a \in R_x$. Set $\alpha = f(x)$. Clearly, $x \in C(f, \alpha)$ and $a \notin C(f, \alpha)$ for all $a \in R_x$. This contradicts the fact that $C(f, \alpha)$ is regular. Thus (3.2) is true. Consequently, (S, f) is a regular $\mathcal{N}$-subsemigroup of S. □

Theorem 3.7. *For a nonempty subset A of S, the following are equivalent:*

(1) *(S, κ_A) is a regular $\mathcal{N}$-subsemigroup of S.*
(2) *A is a regular subsemigroup of S.*

Proof. Assume that (S, κ_A) is a regular $\mathcal{N}$-subsemigroup of S and let $x, y \in A$. Then $\kappa_A(xy) \leq \bigvee \{\kappa_A(x), \kappa_A(y)\} = -1$, and so $xy \in A$. Moreover, if $x \in A$ then $f(x) = -1 \neq 0$ and so there exists $a \in R_x$ such that $f(x) \geq f(a)$ by (3.2). Thus $f(a) = -1$, i.e., $a \in A$. Therefore A is a regular subsemigroup of S.

Conversely, let A be a regular subsemigroup of S and let $x, y \in S$. If $x, y \in A$, then $xy \in A$ and so $\kappa_A(xy) = -1 = \bigvee \{\kappa_A(x), \kappa_A(y)\}$. If $x \notin A$ or $y \notin A$, then $\kappa_A(x) = 0$ or $\kappa_A(y) = 0$. Hence $\kappa_A(xy) \leq \bigvee \{\kappa_A(x), \kappa_A(y)\}$. From the regularity of A, we know that there exists $a \in R_x$ such that $a \in A$, i.e., $\kappa_A(a) = -1$. Thus $\kappa_A(x) \geq -1 = \kappa_A(a)$, i.e., (3.2) holds. This shows that (S, κ_A) is a regular $\mathcal{N}$-subsemigroup of S. □

Proposition 3.8. *If (S, f) is a regular $\mathcal{N}$-subsemigroup of S, then $f \,\tilde{\circ}\, f = f$.*

Proof. By Theorem 3.4, we have that $f \subseteq f \,\tilde{\circ}\, f$. Now, for any $x \in S$, if $f(x) = 0$, then $f(x) \leq (f \,\tilde{\circ}\, f)(x)$ which implies that $(f \,\tilde{\circ}\, f)(x) = f(x)$. If $f(x) \neq 0$, then there exists $a \in R_x$ such that $f(x) \geq f(a)$ because (S, f) is regular. Hence

$$\begin{aligned}
(f \,\tilde{\circ}\, f)(x) &= \bigwedge_{yz=x} \left\{ \bigvee \{f(y), f(z)\} \right\} \\
&\leq \bigvee \{f(xa), f(x)\} \\
&\leq \bigvee \{f(a), f(x)\} = f(x),
\end{aligned}$$

i.e., $(f \,\tilde{\circ}\, f) \subseteq f$. Thus $f \,\tilde{\circ}\, f = f$. □

Let $S^e = S \cup \{e\}$ and $xe = ex = x$ for all $x \in S^e$. Then S^e is a semigroup with identity e. For any $\mathcal{N}$-structure (S, f), we define an $\mathcal{N}$-structure (S^e, f^e) in which f^e is defined as follows:

$$f^e(x) = \begin{cases} -1 & \text{if } x = e, \\ f(x) & x \in S. \end{cases}$$

Clearly, $e \in C(f^e, \alpha)$ for all $\alpha \in [-1, 0]$.

We provide a condition for an $\mathcal{N}$-structure to be a regular $\mathcal{N}$-subsemigroup.

Theorem 3.9. *Consider an $\mathcal{N}$-structure (S, f) which satisfies the following condition:*

$$(3.3) \qquad (\forall x \in S) \left(\begin{array}{l} f(x) \neq 0 \ \Rightarrow \ \exists \beta \in [-1, 0) \text{ and} \\ \text{an idempotent element } w \in C(f, \beta) \\ \text{such that } xC(f^e, \beta) = wC(f, \beta), \\ \text{where } \beta = f(x) \end{array} \right).$$

Then (S, f) is a regular $\mathcal{N}$-subsemigroup of S.

Proof. Let (S, f) be an $\mathcal{N}$-structure which satisfies the condition (3.3). According to Theorem 3.6, it is sufficient to show that the nonempty closed support $C(f, \alpha)$ of (S, f) related to $\alpha \in [-1, 0)$ is a regular subsemigroup of S. If $C(f, \alpha) \neq \varnothing$ for every $\alpha \in [-1, 0)$, then $f(x) \leq \alpha$ for all $x \in C(f, \alpha)$. We set $f(x) = \alpha$ and have $\alpha_0 < \alpha$. By assumption, there is an idempotent element $w \in C(f, \alpha_0)$ such that $xC(f^e, \alpha_0) = wC(f, \alpha_0)$. Therefore $\exists\, y \in C(f, \alpha_0)$ with $x = wy$ and $\exists\, z \in C(f^e, \alpha_0)$ with $xz = w$. Now $wx = w^2 y = wy = x$, i.e., $xzx = x$, and so $z \in R_x$ and $f(z) \leq \alpha_0 < \alpha$, i.e., $z \in C(f, \alpha)$. Consequently,

$$(\forall x \in C(f, \alpha))\, (\exists\, z \in R_x)\, (z \in C(f, \alpha)).$$

Hence $C(f, \alpha)$ is a regular subsemigroup of S. This completes the proof. □

4. $\mathcal{N}$-ideals

Definition 4.1. By a *left $\mathcal{N}$-ideal* (resp. *right $\mathcal{N}$-ideal*) of S we mean an $\mathcal{N}$-structure (S, f) in which f satisfies:

$$(4.1) \qquad (\forall x, y \in S)\, (f(xy) \leq f(y) \ (\text{resp. } f(xy) \leq f(x))).$$

Example 4.2. Consider the semigroup $S = \{a, b, c, x, y, z\}$ as in Example 3.2. Let (S, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} a & b & c & x & y & z \\ -0.6 & -0.5 & -0.3 & -0.7 & -0.1 & 0 \end{pmatrix}.$$

Then (S, f) is a left $\mathcal{N}$-ideal of S, but not a right $\mathcal{N}$-ideal of S.

Obviously, every left (right) $\mathcal{N}$-ideal is an $\mathcal{N}$-subsemigroup, but the converse is not true as shown in the following example.

Example 4.3. Consider the semigroup $S = \{a, b, c, x, y, z\}$ as in Example 3.2. Let (S, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} a & b & c & x & y & z \\ -0.7 & -0.6 & -0.5 & -0.4 & -0.4 & -0.3 \end{pmatrix}.$$

Then (S, f) is an $\mathcal{N}$-subsemigroup of S. But it is not a left $\mathcal{N}$-ideal of S since $f(dc) = f(d) = -0.4 > -0.5 = f(c)$.

We note that the semigroup S can be considered an $\mathcal{N}$-function on S which is given by $S(x) = -1$ for all $x \in S$.

Theorem 4.4. *An $\mathcal{N}$-structure (S, f) is a left* (resp. *right*) *$\mathcal{N}$-ideal of S if and only if $f \subseteq S \tilde{\circ} f$* (resp. $f \subseteq f \tilde{\circ} S$).

Proof. Assume that (S, f) is a left $\mathcal{N}$-ideal of S. Let $x \in S$. If $(S \tilde{\circ} f)(x) = 0$, then it is clear that $f \subseteq S \tilde{\circ} f$. Otherwise, there exist elements $a, b \in S$ such that $x = ab$. Hence

$$\begin{aligned}(S \tilde{\circ} f)(x) &= \bigwedge_{x=ab} \left\{ \bigvee \{S(a), f(b)\} \right\} \geq \bigwedge_{x=ab} \left\{ \bigvee \{-1, f(ab)\} \right\} \\ &= \bigwedge \left\{ \bigvee \{-1, f(x)\} \right\} = f(x),\end{aligned}$$

and so $f \subseteq S \tilde{\circ} f$.

Conversely, assume that $f \subseteq S \tilde{\circ} f$. Let x and y be any elements of S. Let $a = xy$. Then we have

$$\begin{aligned}f(xy) = f(a) \leq (S \tilde{\circ} f)(a) &= \bigwedge_{a=bc} \left\{ \bigvee \{S(b), f(c)\} \right\} \\ &\leq \bigvee \{S(x), f(y)\} = \bigvee \{-1, f(y)\} = f(y).\end{aligned}$$

Hence (S, f) is a left $\mathcal{N}$-ideal of S. For the case of right $\mathcal{N}$-ideal, it can be seen in a similar manner. $\square$

Definition 4.5. By a *generalized $\mathcal{N}$-bi-ideal* of S we mean an $\mathcal{N}$-structure (S, f) in which f satisfies:

$$(\forall x, a, y \in S) \left(f(xay) \leq \bigvee \{f(x), f(y)\} \right). \tag{4.2}$$

A generalized $\mathcal{N}$-bi-ideal which is also an $\mathcal{N}$-subsemigroup is called an $\mathcal{N}$-bi-ideal of S.

Example 4.6. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let (S, f) be an $\mathcal{N}$-structure in which f is defined by

$$f = \begin{pmatrix} a & b & c & d \\ -0.6 & -0.1 & -0.3 & -0.1 \end{pmatrix}.$$

Then (S, f) is a generalized $\mathcal{N}$-bi-ideal of S. But, it is not an $\mathcal{N}$-bi-ideal of S since $f(cc) = f(b) = -0.1 > -0.3 = \bigvee \{f(c), f(c)\}$.

We now provide a condition for a generalized $\mathcal{N}$-bi-ideal to be an $\mathcal{N}$-bi-ideal.

Theorem 4.7. *Every generalized $\mathcal{N}$-bi-ideal of a regular semigroup S is an $\mathcal{N}$-bi-ideal of S.*

Proof. Let (S, f) be a generalized $\mathcal{N}$-bi-ideal of a regular semigroup S. Let $a, b \in S$. Since S is regular, there exists an element x in S such that $b = bxb$. Then we have

$$f(ab) = f\left(a(bxb)\right) = f\left(a(bx)b\right) \leq \bigvee \left\{f(a), f(b)\right\},$$

and so (S, f) is an $\mathcal{N}$-subsemigroup of S. Hence (S, f) is an $\mathcal{N}$-bi-ideal of S. □

Theorem 4.8. *Let A be a nonempty subset of a semigroup S. Then A is a generalized bi-ideal of S if and only if (S, κ_A) is a generalized $\mathcal{N}$-bi-ideal of S.*

Proof. Suppose that A is a generalized bi-ideal of S. Let x, y and a be any elements of S. If $x, y \in A$, then $\kappa_A(x) = \kappa_A(y) = -1$ and $xay \in ASA \subseteq A$. Hence $\kappa_A(xay) = -1 = \bigvee \left\{\kappa_A(x), \kappa_A(y)\right\}$. If $x \notin A$ or $y \notin A$, then $\kappa_A(x) = 0$ or $\kappa_A(y) = 0$ and so $\kappa_A(xay) \leq 0 = \bigvee \left\{\kappa_A(x), \kappa_A(y)\right\}$. Hence (S, κ_A) is a generalized $\mathcal{N}$-bi-ideal of S.

Conversely, assume that (S, κ_A) is a generalized $\mathcal{N}$-bi-ideal of S. Let $z \in ASA$. Then $z = xay$ for some $x, y \in A$ and $a \in S$. Then

$$\kappa_A(z) = \kappa_A(xay) \leq \bigvee \left\{\kappa_A(x), \kappa_A(y)\right\} = -1,$$

and thus $\kappa_A(z) = -1$. Hence $z \in A$, and so $ASA \subseteq A$. Therefore A is a generalized bi-ideal of S. □

Note that, for a nonempty subset A of S, A is a subsemigroup of S if and only if (S, κ_A) is an $\mathcal{N}$-subsemigroup of S. Therefore, we have the following corollary.

Corollary 4.9. *Let A be a nonempty subset of a semigroup S. Then A is a bi-ideal of S if and only if (S, κ_A) is an $\mathcal{N}$-bi-ideal of S.*

Theorem 4.10. *Let (S, f) be an $\mathcal{N}$-structure. Then (S, f) is a generalized $\mathcal{N}$-bi-ideal of S if and only if $f \subseteq f \,\tilde{\circ}\, S \,\tilde{\circ}\, f$.*

Proof. Assume that (S, f) is a generalized $\mathcal{N}$-bi-ideal of S. Let a be any element of S. If $(f \,\tilde{\circ}\, S \,\tilde{\circ}\, f)(a) = 0$, then it is clear that $f \subseteq f \,\tilde{\circ}\, S \,\tilde{\circ}\, f$. If $(f \,\tilde{\circ}\, S \,\tilde{\circ}\, f)(a) \neq 0$, then there exist $x, y, u, v \in S$ such that $a = xy$ and $x = uv$. Since (S, f) is a generalized $\mathcal{N}$-bi-ideal of S, we have $f(uvy) \leq \bigvee \left\{f(u), f(y)\right\}$. Thus

$$\begin{aligned}
(f \,\tilde{\circ}\, S \,\tilde{\circ}\, f)(a) &= \bigwedge_{a=xy} \left\{\bigvee \left\{(f \,\tilde{\circ}\, S)(x), f(y)\right\}\right\} \\
&= \bigwedge_{a=xy} \left\{\bigvee \left\{\bigwedge_{x=uv} \left\{\bigvee \left\{f(u), S(v)\right\}\right\}, f(y)\right\}\right\} \\
&= \bigwedge_{a=xy} \left\{\bigvee \left\{\bigwedge_{x=uv} \left\{\bigvee \left\{f(u), -1\right\}\right\}, f(y)\right\}\right\} \\
&= \bigwedge_{a=uvy} \left\{\bigvee \left\{f(u), f(y)\right\}\right\} \\
&\geq \bigwedge_{a=uvy} f(uvy) = f(a),
\end{aligned}$$

and so $f \subseteq f \,\tilde{\circ}\, S \,\tilde{\circ}\, f$.

Conversely, suppose that $f \subseteq f \,\tilde{\circ}\, S \,\tilde{\circ}\, f$. Let x, y and z be any elements of S. Set $a = xyz$. Then

$$\begin{aligned} f(xyz) &= f(a) \leq (f \,\tilde{\circ}\, S \,\tilde{\circ}\, f)\,(a) \\ &= \bigwedge_{a=bc} \left\{ \bigvee \{(f \,\tilde{\circ}\, S)(b), f(c)\} \right\} \\ &\leq \bigvee \{(f \,\tilde{\circ}\, S)(xy), f(z)\} \\ &= \bigvee \left\{ \bigwedge_{xy=uv} \left\{ \bigvee \{f(u), S(v)\} \right\}, f(z) \right\} \\ &\leq \bigvee \left\{ \bigvee \{f(x), S(y)\}, f(z) \right\} \\ &= \bigvee \left\{ \bigvee \{f(x), -1\}, f(z) \right\} \\ &= \bigvee \{f(x), f(z)\}, \end{aligned}$$

and thus (S, f) is a generalized $\mathcal{N}$-bi-ideal of S. □

Theorem 4.11. *Every left* (resp. *right*) *$\mathcal{N}$-ideal of S is a generalized $\mathcal{N}$-bi-ideal of S.*

Proof. Let (S, f) be a left $\mathcal{N}$-ideal of S and $x, a, y \in S$ Then

$$f(xay) = f((xa)y) \leq f(y) \leq \bigvee \{f(x), f(y)\}.$$

Thus (S, f) is a generalized $\mathcal{N}$-bi-ideal of S. The right case is proved in an analogous way. □

Since every left (right) $\mathcal{N}$-ideal is an $\mathcal{N}$-subsemigroup, we have the following corollary.

Corollary 4.12. *Every left* (resp. *right*) *$\mathcal{N}$-ideal of S is an $\mathcal{N}$-bi-ideal of S.*

The converse of Theorem 4.11 is not true as seen in the following example.

Example 4.13. The generalized $\mathcal{N}$-bi-ideal (S, f) in Example 4.6 is not a left $\mathcal{N}$-ideal of S since $f(dc) = f(b) = -0.1 > -0.3 = f(c)$.

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