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# Cubic sets

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Abstract. The notions of (internal, external) cubic sets, P-(R-)order, P-(R-)union and P-(R-)intersection are introduced, and related properties are investigated. We show that the P-union and the P-intersection of internal cubic sets are also internal cubic sets. We provide examples to show that the P-union and the P-intersection of external cubic sets need not be external cubic sets, and the R-union and the R-intersection of internal (resp. external) cubic sets need not be internal (resp. external) cubic sets. We provide conditions for the P-union (resp. P-intersection) of two external cubic sets to be an internal cubic set. We give conditions for the P-union (resp. R-union and R-intersection) of two external cubic sets to be an external cubic set. We consider conditions for the R-intersection (resp. P-intersection) of two cubic sets to be both an external cubic set and an internal cubic set.

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## 1. Introduction

Fuzzy sets are initiated by Zadeh [6]. In [7], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval $[0, 1]$ are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [2] in Medical diagnosis in thyroidian pathology, Kohout [1] also in Medicine, in a system CLINAID, Gorzalczany [10] in Approximate reasoning, Turksen [3, 4] in Interval-valued logic, in preferences modelling [5], etc. These works and others show the importance of these sets. Fuzzy sets deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences.

In this paper, using a fuzzy set and an interval-valued fuzzy set, we introduce a new notion, called a (internal, external) cubic set, and investigate several properties. We deal with P-union, P-intersection, R-union and R-intersection of cubic sets, and investigate several related properties.

## 2. Preliminaries

A *fuzzy set* in a set $X$ is defined to be a function $\lambda : X \to I$ where $I = [0,1]$. Denote by $I^X$ the collection of all fuzzy sets in a set $X$. Define a relation $\leq$ on $I^X$ as follows:

$$(\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of $\lambda$ and $\mu$ are defined by

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\},$$

$$(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\},$$

respectively, for all $x \in X$. The complement of $\lambda$, denoted by $\lambda^c$, is defined by

$$(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x)).$$

For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in $X$, we define the join ($\vee$) and meet ($\wedge$) operations as follows:

$$\Big(\bigvee_{i\in\Lambda} \lambda_i\Big)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\},$$

$$\Big(\bigwedge_{i\in\Lambda} \lambda_i\Big)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\},$$

respectively, for all $x \in X$.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $I$, where $0 \leq a^- \leq a^+ \leq 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by $\mathbf{a}$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

$$\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [I]$, its *complement*, denoted by $\tilde{a}^c$, is defined be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let $X$ be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in $X$. Let $[I]^X$ stand for the set of all IVF sets in $X$. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element $x$ to $A$, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in $X$ which are called a *lower fuzzy set* and an *upper fuzzy set* in $X$, respectively. For simplicity, we denote $A = [A^-, A^+]$. For every $A, B \in [I]^X$, we define

$$A \subseteq B \Leftrightarrow A(x) \preceq B(x) \text{ for all } x \in X,$$

and

$$A = B \Leftrightarrow A(x) = B(x) \text{ for all } x \in X.$$

The complement $A^c$ of $A \in [I]^X$ is defined as follows: $A^c(x) = A(x)^c$ for all $x \in X$, that is,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)] \text{ for all } x \in X.$$

For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets in $X$ where $\Lambda$ is an index set, the *union* $G = \bigcup_{i\in\Lambda} A_i$ and the *intersection* $F = \bigcap_{i\in\Lambda} A_i$ are defined as follows:

$$G(x) = \left(\bigcup_{i\in\Lambda} A_i\right)(x) = \operatorname{rsup}_{i\in\Lambda} A_i(x)$$

and

$$F(x) = \left(\bigcap_{i\in\Lambda} A_i\right)(x) = \operatorname{rinf}_{i\in\Lambda} A_i(x)$$

for all $x \in X$, respectively. For a point $p \in X$ and for $\tilde{a} = [a^-, a^+] \in [I]$ with $a^+ > 0$, the IVF set which takes the value $\tilde{a}$ at $p$ and $\mathbf{0}$ elsewhere in $X$ is called an *interval-valued fuzzy point* (briefly, an *IVF point*) and is denoted by $\tilde{a}_p$. The set of all IVF points in $X$ is denoted by $IVFP(X)$. For any $\tilde{a} \in [I]$ and $x \in X$, the IVF point $\tilde{a}_x$ is said to *belong* to an IVF set $A$ in $X$, denoted by $\tilde{a}_x \tilde{\in} A$, if $A(x) \succeq \tilde{a}$. It can be easily shown that $A = \cup\{\tilde{a}_x \mid \tilde{a}_x \tilde{\in} A\}$.

## 3. Cubic sets

**Definition 3.1.** Let $X$ be a nonempty set. By a *cubic set* in $X$ we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

in which $A$ is an IVF set in $X$ and $\lambda$ is a fuzzy set in $X$.

A cubic set $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $\mathscr{A} = \langle A, \lambda\rangle$. Denote by $C^X$ the collection of all cubic sets in $X$.

A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in which $A(x) = \mathbf{0}$ and $\lambda(x) = 1$ (resp. $A(x) = \mathbf{1}$ and $\lambda(x) = 0$) for all $x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

A cubic set $\mathscr{B} = \langle B, \mu\rangle$ in which $B(x) = \mathbf{0}$ and $\mu(x) = 0$ (resp. $B(x) = \mathbf{1}$ and $\mu(x) = 1$) for all $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$).

**Definition 3.2.** Let $X$ be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in $X$ is said to be an *internal cubic set* (briefly, ICS) if $A^-(x) \le \lambda(x) \le A^+(x)$ for all $x \in X$.

**Definition 3.3.** Let $X$ be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in $X$ is said to be an *external cubic set* (briefly, ECS) if $\lambda(x) \notin (A^-(x), A^+(x))$ for all $x \in X$.

**Example 3.4.** (1) Let $X$ be a nonempty set. Let $A$ be an IVF set in $X$. Then $\mathscr{A} = \{\langle x, A(x), 1(x)\rangle \mid x \in X\}$, $\mathscr{B} = \{\langle x, A(x), 0(x)\rangle \mid x \in X\}$ and

$$\mathscr{C} = \left\{\langle x, A(x), \lambda(x)\rangle \mid x \in X\right\} \text{ where } \lambda(x) = \tfrac{A^-(x)+A^+(x)}{2}$$

are cubic sets in $X$.

(2) Let $\mathscr{A} = \{\langle x, A(x), \lambda(x) \mid x \in I\}$ be a cubic set in $I$. If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.4$ for all $x \in I$, then $\mathscr{A}$ is an ICS. If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.8$ for all $x \in I$, then $\mathscr{A}$ is an ECS. If $A(x) = [0.3, 0.7]$ and $\lambda(x) = x$ for all $x \in I$, then $\mathscr{A}$ is neither an ICS nor an ECS.

**Theorem 3.5.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in $X$ which is not an ECS. Then there exist $x \in X$ such that $\lambda(x) \in (A^-(x), A^+(x))$.*

*Proof.* Straightforward. □

**Theorem 3.6.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in $X$. If $\mathscr{A}$ is both an ICS and an ECS, then*

$$(\forall x \in X)\ (\lambda(x) \in U(A) \cup L(A))$$

*where $U(A) = \{A^+(x) \mid x \in X\}$ and $L(A) = \{A^-(x) \mid x \in X\}$.*

*Proof.* Assume that $\mathscr{A}$ is both an ICS and an ECS. Using Definitions 3.2 and 3.3, we have $A^-(x) \leq \lambda(x) \leq A^+(x)$ and $\lambda(x) \notin (A^-(x), A^+(x))$ for all $x \in X$. Thus $\lambda(x) = A^-(x)$ or $A^+(x) = \lambda(x)$, and so $\lambda(x) \in U(A) \cup L(A)$. □

**Remark 3.7.** Every intuitionistic fuzzy set $A = \{\langle x, \mu(x), \gamma(x)\rangle \mid x \in X\}$ in $X$ is considered as a cubic set in $X$.

**Definition 3.8.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X$. Then we define

(a) (Equality) $\mathscr{A} = \mathscr{B} \Leftrightarrow A = B$ and $\lambda = \mu$.
(b) (P-order) $\mathscr{A} \subseteq_P \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \mu$.
(c) (R-order) $\mathscr{A} \subseteq_R \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \mu$.

**Definition 3.9.** For any $\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\}$ where $i \in \Lambda$, we define

(a) $$\bigcup_{P\, i\in\Lambda} \mathscr{A}_i = \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} A_i\right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\} \quad \text{(P-union)}$$

(b) $$\bigcap_{P\, i\in\Lambda} \mathscr{A}_i = \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} A_i\right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\} \quad \text{(P-intersection)}$$

(c) $$\bigcup_{R\, i\in\Lambda} \mathscr{A}_i = \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} A_i\right)(x), \left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\} \quad \text{(R-union)}$$

(d) $$\bigcap_{R\, i\in\Lambda} \mathscr{A}_i = \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} A_i\right)(x), \left(\bigvee_{i\in\Lambda} \lambda_i\right)(x)\right\rangle \mid x \in X\right\} \quad \text{(R-intersection)}$$

The complement of $\mathscr{A} = \langle A, \lambda\rangle$ is defined to be the cubic set

$$\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}.$$

Obviously, $(\mathscr{A}^c)^c = \mathscr{A}$, $\hat{0}^c = \hat{1}$, $\hat{1}^c = \hat{0}$, $\ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$. For any

$$\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\},\ i \in \Lambda,$$

we have $\left(\bigcup_{P\, i\in\Lambda} \mathscr{A}_i\right)^c = \bigcap_{P\, i\in\Lambda} (\mathscr{A}_i)^c$ and $\left(\bigcap_{P\, i\in\Lambda} \mathscr{A}_i\right)^c = \bigcup_{P\, i\in\Lambda} (\mathscr{A}_i)^c$. Also we have

$$\left(\bigcup_{i\in\Lambda} {}_R\mathscr{A}_i\right)^c = \bigcap_{i\in\Lambda} {}_R(\mathscr{A}_i)^c \quad \text{and} \quad \left(\bigcap_{i\in\Lambda} {}_R\mathscr{A}_i\right)^c = \bigcup_{i\in\Lambda} {}_R(\mathscr{A}_i)^c.$$

**Proposition 3.10.** *For any cubic sets $\mathscr{A} = \langle A, \lambda\rangle$, $\mathscr{B} = \langle B, \mu\rangle$, $\mathscr{C} = \langle C, \gamma\rangle$, and $\mathscr{D} = \langle D, \rho\rangle$, we have*

(1) *if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{B} \subseteq_P \mathscr{C}$ then $\mathscr{A} \subseteq_P \mathscr{C}$.*
(2) *if $\mathscr{A} \subseteq_P \mathscr{B}$ then $\mathscr{B}^c \subseteq_P \mathscr{A}^c$.*
(3) *if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{A} \subseteq_P \mathscr{C}$ then $\mathscr{A} \subseteq_P \mathscr{B} \cap_P \mathscr{C}$.*
(4) *if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{C} \subseteq_P \mathscr{B}$ then $\mathscr{A} \cup_P \mathscr{C} \subseteq_P \mathscr{B}$.*
(5) *if $\mathscr{A} \subseteq_P \mathscr{B}$ and $\mathscr{C} \subseteq_P \mathscr{D}$ then $\mathscr{A} \cup_P \mathscr{C} \subseteq_P \mathscr{B} \cup_P \mathscr{D}$ and $\mathscr{A} \cap_P \mathscr{C} \subseteq_P \mathscr{B} \cap_P \mathscr{D}$*
(6) *if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{B} \subseteq_R \mathscr{C}$ then $\mathscr{A} \subseteq_R \mathscr{C}$.*
(7) *if $\mathscr{A} \subseteq_R \mathscr{B}$ then $\mathscr{B}^c \subseteq_R \mathscr{A}^c$.*
(8) *if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{A} \subseteq_R \mathscr{C}$ then $\mathscr{A} \subseteq_R \mathscr{B} \cap_R \mathscr{C}$.*
(9) *if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{C} \subseteq_R \mathscr{B}$ then $\mathscr{A} \cup_R \mathscr{C} \subseteq_R \mathscr{B}$.*
(10) *if $\mathscr{A} \subseteq_R \mathscr{B}$ and $\mathscr{C} \subseteq_R \mathscr{D}$ then $\mathscr{A} \cup_R \mathscr{C} \subseteq_R \mathscr{B} \cup_R \mathscr{D}$ and $\mathscr{A} \cap_R \mathscr{C} \subseteq_R \mathscr{B} \cap_R \mathscr{D}$.*

*Proof.* Straightforward. □

**Theorem 3.11.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in $X$. If $\mathscr{A}$ is an ICS* (resp, *ECS*), *then $\mathscr{A}^c$ is an ICS* (resp, *ECS*).

*Proof.* Since $\mathscr{A} = \langle A, \lambda\rangle$ is an ICS (resp. ECS) in $X$, we have $A^-(x) \le \lambda(x) \le A^+(x)$ (resp. $\lambda(x) \notin (A^-(x), A^+(x))$ for all $x \in X$. This implies that

$$1 - A^+(x) \le 1 - \lambda(x) \le 1 - A^-(x)$$

(resp. $1 - \lambda(x) \notin (1 - A^+(x), 1 - A^-(x))$. Hence

$$\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}$$

is an ICS (resp. ECS) in $X$. □

**Theorem 3.12.** *Let $\{\mathscr{A}_i = \langle A_i, \lambda_i\rangle \mid i \in \Lambda\}$ be a family of ICSs in $X$. Then the P-union and the P-intersection of $\{\mathscr{A}_i = \langle A_i, \lambda_i\rangle \mid i \in \Lambda\}$ are ICSs in $X$.*

*Proof.* Since $\mathscr{A}_i$ is an ICS in $X$, we have ${A_i}^-(x) \le \lambda_i(x) \le {A_i}^+(x)$ for $i \in \Lambda$. This implies that

$$\left(\bigcup_{i\in\Lambda} A_i\right)^-(x) \le \left(\bigvee_{i\in\Lambda} \lambda_i\right)(x) \le \left(\bigcup_{i\in\Lambda} A_i\right)^+(x)$$

and

$$\left(\bigcap_{i\in\Lambda} A_i\right)^-(x) \le \left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x) \le \left(\bigcap_{i\in\Lambda} A_i\right)^+(x).$$

Hence $\bigcup_{i\in\Lambda} {}_P\mathscr{A}_i$ and $\bigcap_{i\in\Lambda} {}_P\mathscr{A}_i$ are ICSs in $X$. □

The following example shows that the P-union and P-intersection of ECSs need not be an ECS.

**Example 3.13.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $I = [0, 1]$ in which $A(x) = [0.3, 0.5]$, $\lambda(x) = 0.8$, $B(x) = [0.7, 0.9]$ and $\mu(x) = 0.4$ for all $x \in I$.

(1) We know that $\mathscr{A} \cup_P \mathscr{B} = \{\langle x, B(x), \lambda(x)\rangle \mid x \in I\}$ and $\lambda(x) \in (B^-(x), B^+(x))$ for all $x \in I$. Hence $\mathscr{A} \cup_P \mathscr{B}$ is not an ECS in $I$.

(2) We know that $\mathscr{A} \cap_P \mathscr{B} = \{\langle x, A(x), \mu(x)\rangle \mid x \in I\}$ and $\mu(x) \in (A^-(x), A^+(x))$ for all $x \in I$. Hence $\mathscr{A} \cap_P \mathscr{B}$ is not an ECS in $I$.

The following example shows that the R-union and R-intersection of ICSs need not be an ICS.

**Example 3.14.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $I = [0, 1]$ in which $A(x) = [0.3, 0.5]$, $\lambda(x) = 0.4$, $B(x) = [0.7, 0.9]$ and $\mu(x) = 0.8$ for all $x \in I$.

(1) We know that $\mathscr{A} \cup_R \mathscr{B} = \{\langle x, B(x), \lambda(x)\rangle \mid x \in I\}$ and $\lambda(x) \notin [B^-(x), B^+(x)]$ for all $x \in I$. Hence $\mathscr{A} \cup_R \mathscr{B}$ is not an ICS in $I$.

(2) We know that $\mathscr{A} \cap_R \mathscr{B} = \{\langle x, A(x), \mu(x)\rangle \mid x \in I\}$ and $\mu(x) \notin [A^-(x), A^+(x)]$ for all $x \in I$. Hence $\mathscr{A} \cap_R \mathscr{B}$ is not an ICS in $I$.

The following example shows that the R-union and R-intersection of ECSs need not be an ECS.

**Example 3.15.** (1) Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $I = [0, 1]$ in which $A(x) = [0.2, 0.4]$, $\lambda(x) = 0.7$, $B(x) = [0.6, 0.8]$ and $\mu(x) = 0.9$ for all $x \in I$. We know that $\mathscr{A} \cup_R \mathscr{B} = \{\langle x, B(x), \lambda(x)\rangle \mid x \in I\}$ and $\lambda(x) \in (B^-(x), B^+(x))$ for all $x \in I$. Hence $\mathscr{A} \cup_R \mathscr{B}$ is not an ECS in $I$.

(2) Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $I = [0, 1]$ in which $A(x) = [0.2, 0.4]$, $\lambda(x) = 0.1$, $B(x) = [0.6, 0.8]$ and $\mu(x) = 0.3$ for all $x \in I$. Then $\mathscr{A} \cap_R \mathscr{B} = \{\langle x, A(x), \mu(x)\rangle \mid x \in I\}$ and $\mu(x) \in (A^-(x), A^+(x))$ for all $x \in I$. Thus $\mathscr{A} \cap_R \mathscr{B}$ is not an ECS in $I$.

We provide a condition for the R-union of two ICSs to be an ICS.

**Theorem 3.16.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$ such that*

$$\max\left\{A^-(x), B^-(x)\right\} \le (\lambda \wedge \mu)(x) \tag{3.1}$$

*for all $x \in X$. Then the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ICS in $X$.*

*Proof.* Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$ which satisfy the condition (3.1). Then $A^-(x) \le \lambda(x) \le A^+(x)$ and $B^-(x) \le \mu(x) \le B^+(x)$, which implies that $(\lambda \wedge \mu)(x) \le (A \cup B)^+(x)$. It follows from the condition (3.1) that

$$(A \cup B)^-(x) = \max\left\{A^-(x), B^-(x)\right\} \le (\lambda \wedge \mu)(x) \le (A \cup B)^+(x)$$

so that $\mathscr{A} \cup_R \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \wedge \mu)(x)\rangle \mid x \in X\}$ is an ICS in $X$. □

We provide a condition for the R-intersection of two ICSs to be an ICS.

**Theorem 3.17.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$ satisfying the following inequality*

$$\min\left\{A^+(x), B^+(x)\right\} \ge (\lambda \vee \mu)(x) \tag{3.2}$$

*for all $x \in X$. Then the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ICS in $X$.*

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.2, 0.3]$ | $0.1$ |
| $b$ | $[0.5, 0.6]$ | $0.7$ |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.4, 0.5]$ | $0.9$ |
| $b$ | $[0.7, 0.9]$ | $0.4$ |

TABLE 1. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.3, 0.5]$ | $0.7$ |
| $b$ | $[0.2, 0.4]$ | $0.65$ |
| $c$ | $[0.35, 0.45]$ | $0.75$ |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.6, 0.8]$ | $0.35$ |
| $b$ | $[0.25, 0.55]$ | $0.1$ |
| $c$ | $[0.7, 0.85]$ | $0.4$ |

TABLE 2. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

*Proof.* Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$ which satisfy the condition (3.2.) Then $A^-(x) \leq \lambda(x) \leq A^+(x)$ and $B^-(x) \leq \mu(x) \leq B^+(x)$, and therefore $(A \cap B)^-(x) \leq (\lambda \vee \mu)(x)$. Using the condition (3.2,) we have

$$(A \cap B)^-(x) \leq (\lambda \vee \mu)(x) \leq \min\left\{A^+(x), B^+(x)\right\} = (A \cap B)^+(x)$$

and so $\mathscr{A} \cap_R \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \vee \mu)(x)\rangle \mid x \in X\}$ is an ICS in $X$. □

Given two cubic sets $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ in $X$, if we exchange $\mu$ for $\lambda$, we denote the cubic sets by $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$, respectively.

For two ECSs $\mathscr{A}$ and $\mathscr{B}$ in $X$, two cubic sets $\mathscr{A}^*$ and $\mathscr{B}^*$ may not be ICSs in $X$ as seen in the following example.

**Example 3.18.** (1) Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $I = [0, 1]$ in which $A(x) = [0.6, 0.7]$, $\lambda(x) = 0.8$, $B(x) = [0.3, 0.4]$ and $\mu(x) = 0.2$ for all $x \in I$. Then we know that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are not ICSs in $X$ because $\mu(0.5) = 0.2 \notin [0.6, 0.7] = A(0.5)$ and $\lambda(0.5) = 0.8 \notin [0.3, 0.4] = B(0.5)$.

(2) Let $X = \{a, b\}$ be a set. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ defined by Table 1. Then we know that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are not ICSs in $X$ because $\mu(a) = 0.9 \notin [0.2, 0.3] = A(a)$ and $\lambda(a) = 0.1 \notin [0.4, 0.5] = B(a)$.

The following example shows that the P-union of two ECSs in $X$ need not be an ICS in $X$.

**Example 3.19.** Let $X = \{a, b, c\}$ be a set. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ defined by Table 2. Then we know that $\mathscr{A} \cup_P \mathscr{B} = \langle A \cup B, \lambda \vee \mu\rangle$ is not an ICS in $X$ because $(\lambda \vee \mu)(b) = 0.65 \notin [0.25, 0.55] = (A \cup B)(b)$.

We provide a condition for the P-union of two ECSs to be an ICS.

**Theorem 3.20.** *For two ECSs $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ in $X$, if $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in $X$, then the P-union $\mathscr{A} \cup_P \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is an ICS in $X$.*

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.2, 0.3]$ | 0.5 |
| $b$ | $[0.3, 0.6]$ | 0.7 |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.4, 0.6]$ | 0.9 |
| $b$ | $[0.7, 0.9]$ | 0.4 |

TABLE 3. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

*Proof.* Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in $X$. Then $\lambda(x) \notin (A^-(x), A^+(x))$, $\mu(x) \notin (B^-(x), B^+(x))$, $B^-(x) \leq \lambda(x) \leq B^+(x)$ and $A^-(x) \leq \mu(x) \leq A^+(x)$ for all $x \in X$. Thus, for a given $x \in X$, we can consider the following cases:

(i) $\lambda(x) \leq A^-(x) \leq \mu(x) \leq A^+(x)$ and $\mu(x) \leq B^-(x) \leq \lambda(x) \leq B^+(x)$.
(ii) $A^-(x) \leq \mu(x) \leq A^+(x) \leq \lambda(x)$ and $B^-(x) \leq \lambda(x) \leq B^+(x) \leq \mu(x)$.
(iii) $\lambda(x) \leq A^-(x) \leq \mu(x) \leq A^+(x)$ and $B^-(x) \leq \lambda(x) \leq B^+(x) \leq \mu(x)$.
(iv) $A^-(x) \leq \mu(x) \leq A^+(x) \leq \lambda(x)$ and $\mu(x) \leq B^-(x) \leq \lambda(x) \leq B^+(x)$.

We consider the first case only. For remaining cases, it is similar to the first case. For the first case, we have $\mu(x) = A^-(x) = B^-(x) = \lambda(x)$. Since $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in $X$, we have $\mu(x) \leq A^+(x)$ and $\lambda(x) \leq B^+(x)$. It follows that

$$\begin{aligned}(A \cup B)^-(x) &= \max\left\{A^-(x), B^-(x)\right\} = (\lambda \vee \mu)(x) \\ &\leq \max\left\{A^+(x), B^+(x)\right\} = (A \cup B)^+(x).\end{aligned}$$

Hence $\mathscr{A} \cup_P \mathscr{B}$ is an ICS in $X$. □

We provide a condition for the P-intersection of two ECSs to be an ICS.

**Theorem 3.21.** *Let $\mathscr{A}$ and $\mathscr{B}$ be ECSs in $X$ such that $\mathscr{A}^*$ and $\mathscr{B}^*$ are ICSs. Then the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ICS in $X$.*

*Proof.* It is similar to the proof of Theorem 3.20. □

For two ECSs $\mathscr{A}$ and $\mathscr{B}$ in $X$, two cubic sets $\mathscr{A}^*$ and $\mathscr{B}^*$ may not be ECSs in $X$ as shown by the following example.

**Example 3.22.** Let $X = \{a, b\}$ be a set. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ defined by Table 3. Then we know that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are not ECSs in $X$ because $\mu(b) = 0.4 \in (0.3, 0.6) = A(b)$ and $\lambda(a) = 0.5 \in (0.4, 0.6) = B(a)$.

We provide a condition for the P-union of two ECSs to be an ECS.

**Theorem 3.23.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ECSs in $X$. Then the P-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* For any $x \in X$, we have $\lambda(x) \notin (A^-(x), A^+(x))$, $\mu(x) \notin (B^-(x), B^+(x))$, $\mu(x) \notin (A^-(x), A^+(x))$ and $\lambda(x) \notin (B^-(x), B^+(x))$. Hence

$$(\lambda \vee \mu)(x) \notin \left(\max\left\{A^-(x), B^-(x))\right\}, \max\left\{A^+(x), B^+(x)\right\}\right)$$

which means that $(\lambda \vee \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$. Hence $\mathscr{A} \cup_P \mathscr{B}$ is an ECS in $X$. □

Note that the P-intersection of two ECSs may not be an ECS (see Example 3.13(2)). We give a condition for the P-intersection of two ECSs to be an ECS.

**Theorem 3.24.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ such that*

$$\begin{aligned}&\min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\} \geq (\lambda \wedge \mu)(x)\\ &> \max\left\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\right\}\end{aligned} \tag{3.3}$$

*for all $x \in X$. Then the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* For each $x \in X$, take

$$\alpha_x := \min\left\{\max\left\{A^+(x), B^-(x)\right\}, \max\left\{A^-(x), B^+(x)\right\}\right\}$$

and

$$\beta_x := \max\left\{\min\left\{A^+(x), B^-(x)\right\}, \min\left\{A^-(x), B^+(x)\right\}\right\}.$$

Then $\alpha_x$ is one of $A^-(x)$, $B^-(x)$, $A^+(x)$ and $B^+(x)$. We consider $\alpha_x = A^-(x)$ or $\alpha_x = A^+(x)$ only. For the remaining cases, it is similar to this case.

If $\alpha_x = A^-(x)$, then

$$B^-(x) \leq B^+(x) \leq A^-(x) \leq A^+(x)$$

and so $\beta_x = B^+(x)$. Thus

$$B^-(x) = (A \cap B)^-(x) \leq (A \cap B)^+(x) = B^+(x) = \beta_x < (\lambda \wedge \mu)(x),$$

and hence $(\lambda \wedge \mu)(x) \notin ((A \cap B)^-(x), (A \cap B)^+(x))$.

If $\alpha_x = A^+(x)$ then $B^-(x) \leq A^+(x) \leq B^+(x)$ and so $\beta_x = \max\{A^-(x), B^-(x)\}$. Assume that $\beta_x = A^-(x)$. Then

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) \leq A^+(x) \leq B^+(x). \tag{3.4}$$

From the inequality (3.4), we have

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) < A^+(x) \leq B^+(x)$$

or

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) = A^+(x) \leq B^+(x).$$

For the case $B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) < A^+(x) \leq B^+(x)$, it is a contradiction to the fact that $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. For the case

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) = A^+(x) \leq B^+(x),$$

we have $(\lambda \wedge \mu)(x) \notin ((A \cap B)^-(x), (A \cap B)^+(x))$ since $(\lambda \wedge \mu)(x) = A^+(x) = (A \cap B)^+(x)$.

Assume that $\beta_x = B^-(x)$. Then

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) \leq A^+(x) \leq B^+(x). \tag{3.5}$$

From the inequality (3.5), we have

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) < A^+(x) \leq B^+(x)$$

or

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) = A^+(x) \leq B^+(x).$$

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.2, 0.6]$ | $0.7$ |
| $b$ | $[0.3, 0.7]$ | $0.3$ |
| $c$ | $[0.2, 0.6]$ | $0.9$ |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.3, 0.7]$ | $0.3$ |
| $b$ | $[0.2, 0.6]$ | $0.7$ |
| $c$ | $[0.4, 0.7]$ | $0.4$ |

TABLE 4. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

For the case $A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) < A^+(x) \leq B^+(x)$, it contradicts to the fact that $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. For the case

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) = A^+(x) \leq B^+(x),$$

we get $(\lambda \wedge \mu)(x) \notin ((A \cap B)^-(x), (A \cap B)^+(x))$ since

$$(\lambda \wedge \mu)(x) = A^+(x) = (A \cap B)^+(x).$$

Hence the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$. □

The following example shows that for two ECSs $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ which satisfy the condition

$$\begin{aligned} &\min\big\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\big\} > (\lambda \wedge \mu)(x) \\ &= \max\big\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\big\}, \end{aligned}$$

for all $x \in X$, the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ may not be an ECS in $X$.

**Example 3.25.** Let $X = \{a, b, c\}$ be a set. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ECS in $X$ defined by Table 4. Then we know that $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ satisfy the following condition:

$$\begin{aligned} &\min\big\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\big\} > (\lambda \wedge \mu)(x) \\ &= \max\big\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\big\}. \end{aligned}$$

But $\mathscr{A} \cap_P \mathscr{B} = \langle A \cap B, \lambda \wedge \mu \rangle$ is not an ECS in $X$ because $(\lambda \wedge \mu)(a) = 0.3 \in (0.2, 0.6) = ((A \cap B)^-(a), (A \cap B)^+(a))$.

Now, we provide a condition for the P-intersection of two cubic sets to be both an ECS and an ICS.

**Theorem 3.26.** *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X$ such that*

$$\begin{aligned} &\min\big\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\big\} = (\lambda \wedge \mu)(x) \\ &= \max\big\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\big\}, \end{aligned} \tag{3.6}$$

*for all $x \in X$. Then the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ is both an ECS and an ICS in $X$.*

*Proof.* For each $x \in X$, take

$$\alpha_x := \min\big\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\big\}$$

and

$$\beta_x := \max\big\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\big\}.$$

Then $\alpha_x$ is one of $A^-(x)$, $B^-(x)$, $A^+(x)$ and $B^+(x)$. We consider $\alpha_x = A^-(x)$ or $\alpha_x = A^+(x)$ only. For remaining cases, it is similar to this cases.

If $\alpha_x = A^-(x)$, then

$$B^-(x) \leq B^+(x) \leq A^-(x) \leq A^+(x)$$

and so $\beta_x = B^+(x)$. This implies that $A^-(x) = \alpha_x = (\lambda \wedge \mu)(x) = \beta_x = B^+(x)$. Thus

$$B^-(x) \leq B^+(x) = (\lambda \wedge \mu)(x) = A^-(x) \leq A^+(x).$$

This implies that $(\lambda \wedge \mu)(x) = B^+(x) = (A \cap B)^+(x)$. Hence

$$(\lambda \wedge \mu)(x) \notin ((A \cap B)^-(x), (A \cap B)^+(x))$$

and $(A \cap B)^-(x) \leq (\lambda \wedge \mu)(x) \leq (A \cap B)^+(x)$.

If $\alpha_x = A^+(x)$, then $B^-(x) \leq A^+(x) \leq B^+(x)$ and so $(\lambda \wedge \mu)(x) = A^+(x) = (A \cap B)^+(x)$. Hence $(\lambda \wedge \mu)(x) \notin ((A \cap B)^-(x), (A \cap B)^+(x))$ and $(A \cap B)^-(x) \leq (\lambda \wedge \mu)(x) \leq (A \cap B)^+(x)$. Consequently, we know that the P-intersection of $\mathscr{A}$ and $\mathscr{B}$ is both an ECS and an ICS in $X$. □

The following example shows that the P-union of two ECSs $\mathscr{A}$ and $\mathscr{B}$ may not be an ECS.

**Example 3.27.** Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ECSs in $I$ defined by

$$A(x) = \begin{cases} [0.15, 0.25] & \text{if } 0 \leq x < 0.5 \text{ ,} \\ [0.6, 0.7] & \text{if } 0.5 \leq x \leq 1, \end{cases} \quad \lambda(x) = \begin{cases} 0.5x + 0.5 & \text{if } 0 \leq x < 0.5 \text{ ,} \\ 0.3 & \text{if } 0.5 \leq x \leq 1, \end{cases}$$

$$B(x) = \begin{cases} [0.8, 0.9] & \text{if } 0 \leq x < 0.5 \text{ ,} \\ [0.1, 0.2] & \text{if } 0.5 \leq x \leq 1, \end{cases} \quad \mu(x) = \begin{cases} 0.4 & \text{if } 0 \leq x < 0.5 \text{ ,} \\ x & \text{if } 0.5 \leq x \leq 1. \end{cases}$$

Then

$$(A \cup B)(x) = \begin{cases} [0.8, 0.9] & \text{if } 0 \leq x < 0.5 \text{ ,} \\ [0.6, 0.7] & \text{if } 0.5 \leq x \leq 1, \end{cases}$$

$$(\lambda \vee \mu)(x) = \begin{cases} 0.5x + 0.5 & \text{if } 0 \leq x < 0.5 \text{ ,} \\ x & \text{if } 0.5 \leq x \leq 1. \end{cases}$$

But $\mathscr{A} \cup_P \mathscr{B}$ is not an ECS because

$$(\lambda \vee \mu)(0.65) = 0.65 \in (0.65, 0.7) = \big((A \cup B)^-(0.65), (A \cup B)^+(0.65)\big) .$$

We provide a condition for the P-union of two ECSs to be an ECS.

**Theorem 3.28.** *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ECSs in $X$ such that*

$$\begin{aligned} &\min\big\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\big\} > (\lambda \vee \mu)(x) \\ &\geq \max\big\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\big\}, \end{aligned} \tag{3.7}$$

*for all $x \in X$. Then the P-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* For each $x \in X$, take

$$\alpha_x := \min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\}$$

and

$$\beta_x := \max\left\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\right\}.$$

Then $\alpha_x$ is one of $A^-(x)$, $B^-(x)$, $A^+(x)$ and $B^+(x)$. We consider $\alpha_x = A^-(x)$ or $\alpha_x = A^+(x)$ only. For remaining cases, it is similar to this cases.

If $\alpha_x = A^-(x)$, then

$$B^-(x) \leq B^+(x) \leq A^-(x) \leq A^+(x)$$

and so $\beta_x = B^+(x)$. Thus

$$(A \cup B)^-(x) = A^-(x) = \alpha_x > (\lambda \vee \mu)(x)$$

and hence $(\lambda \vee \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$.

If $\alpha_x = A^+(x)$ then $B^-(x) \leq A^+(x) \leq B^+(x)$ and so $\beta_x = \max\{A^-(x), B^-(x)\}$. Assume that $\beta_x = A^-(x)$. Then

$$B^-(x) \leq A^-(x) \leq (\lambda \vee \mu)(x) < A^+(x) \leq B^+(x), \tag{3.8}$$

and so

$$B^-(x) \leq A^-(x) < (\lambda \vee \mu)(x) < A^+(x) \leq B^+(x)$$

or

$$B^-(x) \leq A^-(x) = (\lambda \vee \mu)(x) \leq A^+(x) \leq B^+(x).$$

For the first case, it contradicts to the fact that $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. The second case implies that $(\lambda \vee \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$ since $(A \cup B)^-(x) = A^-(x) = (\lambda \vee \mu)(x)$.

Assume that $\beta_x = B^-(x)$. Then

$$A^-(x) \leq B^-(x) \leq (\lambda \vee \mu)(x) \leq A^+(x) < B^+(x), \tag{3.9}$$

which implies that

$$A^-(x) \leq B^-(x) < (\lambda \vee \mu)(x) < A^+(x) \leq B^+(x)$$

or

$$A^-(x) \leq B^-(x) = (\lambda \vee \mu)(x) < A^+(x) \leq B^+(x).$$

For the case $A^-(x) \leq B^-(x) < (\lambda \vee \mu)(x) < A^+(x) \leq B^+(x)$, it contradicts to the fact that $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. For the case

$$A^-(x) \leq B^-(x) = (\lambda \vee \mu)(x) \leq A^+(x) \leq B^+(x),$$

we have $(\lambda \vee \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$ since $(A \cup B)^-(x) = B^-(x) = (\lambda \vee \mu)(x)$. Hence the P-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$. □

We provide a condition for the R-union of two ECSs to be an ECS.

**Theorem 3.29.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$. If for each $x \in X$,*

$$\begin{aligned} &\min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\} > (\lambda \wedge \mu)(x) \\ &\geq \max\left\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\right\}, \end{aligned} \tag{3.10}$$

*then the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* For each $x \in X$, take

$$\alpha_x := \min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\}$$

and

$$\beta_x := \max\left\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\right\}.$$

Then $\alpha_x$ is one of $A^-(x)$, $B^-(x)$, $A^+(x)$ and $B^+(x)$. We consider $\alpha_x = B^-(x)$ or $\alpha_x = B^+(x)$ only. For remaining cases, it is similar to this cases.

If $\alpha_x = B^-(x)$, then

$$A^-(x) \leq A^+(x) \leq B^-(x) \leq B^+(x)$$

and so $\beta_x = A^+(x)$. Thus by inequality 3.10,

$$(A \cup B)^-(x) = B^-(x) = \alpha_x > (\lambda \wedge \mu)(x)$$

and hence $(\lambda \wedge \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$.

If $\alpha_x = B^+(x)$ then $A^-(x) \leq B^+(x) \leq A^+(x)$ and so $\beta_x = \max\{A^-(x), B^-(x)\}$. Assume that $\beta_x = A^-(x)$. Then

$$B^-(x) \leq A^-(x) \leq (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x). \tag{3.11}$$

which implies that

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$$

or

$$B^-(x) \leq A^-(x) = (\lambda \wedge \mu)(x) \leq B^+(x) \leq A^+(x).$$

For the case $B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$, it contradicts to the fact that $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. For the case

$$B^-(x) \leq A^-(x) = (\lambda \wedge \mu)(x) \leq B^+(x) \leq A^+(x),$$

we get $(\lambda \wedge \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$ since $(A \cup B)^-(x) = A^-(x) = (\lambda \wedge \mu)(x)$.

Assume that $\beta_x = B^-(x)$. Then

$$A^-(x) \leq B^-(x) \leq (\lambda \wedge \mu)(x) \leq B^+(x) < A^+(x). \tag{3.12}$$

Hence

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$$

or

$$A^-(x) \leq B^-(x) = (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x).$$

For the case $A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$, it is a contradiction because $\mathscr{A}$ and $\mathscr{B}$ are ECSs in $X$. For the case

$$A^-(x) \leq B^-(x) = (\lambda \wedge \mu)(x) \leq B^+(x) \leq A^+(x),$$

we obtain $(\lambda \wedge \mu)(x) \notin ((A \cup B)^-(x), (A \cup B)^+(x))$ since $(A \cup B)^-(x) = B^-(x) = (\lambda \wedge \mu)(x)$. Hence the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$. □

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.1, 0.8]$ | 0.9 |
| $b$ | $[0.3, 0.6]$ | 0.6 |
| $c$ | $[0.4, 0.5]$ | 0.5 |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.2, 0.7]$ | 0.7 |
| $b$ | $[0.1, 0.7]$ | 0.8 |
| $c$ | $[0.3, 0.8]$ | 0.9 |

Table 5. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

The following example shows that for two ECSs $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ which satisfy the condition

$$\min\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\} = (\lambda \wedge \mu)(x)$$
$$> \max\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\},$$

for all $x \in X$, the R-union of $\mathscr{A}$ and $\mathscr{B}$ may not be an ECS in $X$.

**Example 3.30.** Let $X = \{a, b, c\}$ be a set. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ defined by Table 5. Then we know that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ satisfy the following condition:

$$\min\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\} = (\lambda \wedge \mu)(x)$$
$$> \max\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\},$$

But $\mathscr{A} \cup_R \mathscr{B} = \langle A \cup B, \lambda \wedge \mu\rangle$ is not an ECS in $X$ because $(\lambda \wedge \mu)(c) = 0.5 \in (0.4, 0.8) = ((A \cup B)^-(c), (A \cup B)^+(c))$.

Now, we provide a condition for the R-intersection of two ECSs to be ECS.

**Theorem 3.31.**

$$\min\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\} \geq (\lambda \vee \mu)(x)$$
$$> \max\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\}, \tag{3.13}$$

*then the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* By similar way to Theorem 3.29, we can obtain the result. □

The following example shows that for two ECSs $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ which satisfy the condition

$$\min\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\} > (\lambda \vee \mu)(x)$$
$$= \max\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\},$$

for all $x \in X$, the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ may not be an ECS in $X$.

**Example 3.32.** Let $X = \{a, b, c\}$ be a set. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ defined by Table 6. Then we know that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ satisfy the condition

$$\min\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\} > (\lambda \vee \mu)(x)$$
$$= \max\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\},$$

But $\mathscr{A} \cap_R \mathscr{B} = \langle A \cap B, \lambda \vee \mu\rangle$ is not an ECS in $X$ because $(\lambda \vee \mu)(b) = 0.5 \in (0.4, 0.7) = ((A \cap B)^-(b), (A \cap B)^+(b))$.

| $X$ | $A(x)$ | $\lambda(x)$ |
|---|---|---|
| $a$ | $[0.2, 0.4]$ | 0.1 |
| $b$ | $[0.5, 0.8]$ | 0.5 |
| $c$ | $[0.6, 0.8]$ | 0.4 |

| $X$ | $B(x)$ | $\mu(x)$ |
|---|---|---|
| $a$ | $[0.3, 0.6]$ | 0.3 |
| $b$ | $[0.4, 0.7]$ | 0.2 |
| $c$ | $[0.7, 0.9]$ | 0.7 |

TABLE 6. Cubic sets $\mathscr{A}$ and $\mathscr{B}$ respectively

Now, we provide a condition for the $R$-intersection of two cubic sets to be both an ECS and an ICS.

**Theorem 3.33.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X$ such that*

$$\begin{aligned}&\min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\} = (\lambda \vee \mu)(x)\\&= \max\left\{\min\{A^+(x), B^-(x)\}, \min\{A^-(x), B^+(x)\}\right\},\end{aligned} \tag{3.14}$$

*for all $x \in X$. Then the $R$-intersection of $\mathscr{A}$ and $\mathscr{B}$ is both an ECS and an ICS in $X$.*

*Proof.* By the similar way to Theorem 3.26, it is straightforward. □

We provide a condition for the R-union of two ICSs to be an ECS.

**Theorem 3.34.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$. If*

$$(\lambda \wedge \mu)(x) \leq \max\{A^-(x), B^-(x)\}$$

*for all $x \in X$, then the $R$-union of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* Straightforward. □

We provide a condition for the R-intersection of two ICSs to be an ECS.

**Theorem 3.35.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in $X$. If*

$$(\lambda \vee \mu)(x) \geq \min\{A^+(x), B^+(x)\}$$

*for all $x \in X$, then the $R$-intersection of $\mathscr{A}$ and $\mathscr{B}$ is an ECS in $X$.*

*Proof.* Straightforward. □

We provide a condition for the R-union of two ECSs to be an ICS.

**Theorem 3.36.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in $X$ such that*

$$\begin{aligned}&\min\left\{\max\{A^+(x), B^-(x)\}, \max\{A^-(x), B^+(x)\}\right\} \leq (\lambda \wedge \mu)(x)\\&\leq \max\{A^+(x), B^+(x)\}\end{aligned} \tag{3.15}$$

*for all $x \in X$. Then the $R$-union of $\mathscr{A}$ and $\mathscr{B}$ is an ICS in $X$.*

*Proof.* Straightforward. □

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