Annals of Fuzzy Mathematics and Informatics
Volume 4, No. 1, (July 2012), pp. 73- 81
ISSN 2093–9310
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

# Roughness in the field of quotients of an integral domain

Haci Aktaş, Filiz Çitak

Received 12 August 2011; Revised 29 September 2011; Accepted 07 October 2011

Abstract. In this paper, by using a relation on an integral domain we defined rough subgroup, rough ring (ideal) and rough field. We also introduced some properties of lower and the upper approximations in the field of quotients of an integral domain.

2010 AMS Classification: 03E72, 08A72

Keywords: Rough set, Lower approximations, Upper approximations, Rough subgroup, Rough ring, Rough field.

## 1. Introduction

The concept of rough sets was introduced by Pawlak [14]. Since then a large amount of papers devoted to development and fruitful applications of the rough set theory ([2], [5], [8]) have appeared. Subsequently, various algebraic and topological aspects and several applications have been discussed by Pawlak ( [15], [16], [17]). Bonikowaski [4], Iwinski [9], and Pomykala and Pomykala [18] studied algebraic properties of rough sets. Bismas and Nanda [2] described the notion of rough subgroups, but their notion depends on the upper approximations and does not depend on the lower approximations. Kuroki [10] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [11] gave also some properties of the lower and upper approximations with respect to the normal subgroups. Nanda and Majumdar [13] and Bismas [3] studied fuzzy rough sets. Dubois and Prade [7] investigated the problem of fuzzification of a rough set. They defined fuzzy rough sets in a little different way. They used a fuzzy partition instead of a crisp partition into disjoint equivalence classes of the universe for their definitions. Davvaz [5] introduced the notion of rough subring (ideal) with respect to an ideal of a ring and studied notion of fuzzy rough ideal and fuzzy rough subring. In this paper, we studied basic notions of the rough subgroup, rough subring, rough subfield theory. We also concerned a relationship between rough sets and groups as well as rings and fields. We introduced the notion of rough subgroup, rough subring and rough subfield with respect to a relation defined on an integral domain, and give some properties of the rough set in algebraic structures.

## 2. Preliminaries

Throughout this work, $D$ is an integral domain, and $S$ is a subset of the cartesian product $D \times D$ given by

$$S = \{(a, b) : a, b \in D, b \neq 0\}.$$

Two elements $(a, b)$ and $(c, d)$ in $S$ are equivalent, denoted by $(a, b)R(c, d)$, if and only if $ad = bc$.

The relation $R$ between elements of the set $S$ as just described is an equivalence class of $(a, b)$ in $S$ under the relation $R$.

$F$ is a set of all equivalence classes $[(a, b)]_R$ for $(a, b) \in S$ , that is

$$F = \{[(a, b)]_R : (a, b) \in S\}$$

**Lemma 2.1** ([8])**.** *For $[(a, b)]_R$ and $[(c, d)]_R$ in $F$, the equations*

$$[(a, b)]_R + [(c, d)]_R = [(ad + bc, bd)]_R$$

*and*

$$[(a, b)]_R[(a, b)]_R = [(ac, bd)]_R$$

*give well-defined operations of addition and multiplication on $F$.*

**Theorem 2.2** ([8])**.** *Any integral domain $D$ can be enlarged to (or embedded in) a field $F$ such that every element of $F$ can be expressed as a quotient of two elements of $D$. (Such a field $F$ is a field of quotients of $D$.)*

## 3. The lower and upper approximations on a fuzzy field

The concept of fuzzy subgroup of a given group was introduced by Rosenfeld in [19]. Since then many researchers have been studying for extending the concepts of abstract algebra to the broader framework of the fuzzy setting (e.g. [1], [6], [12]). In 1986, Nanda [12] defined and studied fuzzy subfields of a field.

**Definition 3.1.** Let $X$ be a nonempty subset of $S$. Then, the sets

$$\underline{A}_R(X) = \bigcup\{[(a, b)]_R \in F : [(a, b)]_R \subseteq X\}$$

and

$$\overline{A}_R(X) = \bigcup\{[(a, b)]_R \in F : [(a, b)]_R \cap X \neq \varnothing\}$$

called the lower and upper approximations of $X$. For a nonempty subset $X$ of $S$,

$$A_R(X) = (\underline{A}_R(X), \overline{A}_R(X))$$

is called a rough set with respect to $R$.

**Example 3.2.** $\mathbb{Z}_5$ is an integral domain, and $S$ is a subset of the cartesian product $\mathbb{Z}_5 \times \mathbb{Z}_5$ given by

$$S = \{(a, b) : a, b \in \mathbb{Z}_5, b \neq 0\}.$$

Two elements $(a, b)$ and $(c, d)$ in $\mathbb{Z}_5 \times \mathbb{Z}_5$ are equivalent, denoted by $(a, b)R(c, d)$, if and only if $ad = bc$. Then,

$$\begin{aligned} F &= \{[(0,1)]_R, [(0,2)]_R, [(0,3)]_R, [(0,4)]_R, [(1,1)]_R, [(1,2)]_R, [(1,3)]_R, \\ &[(1,4)]_R, [(2,1)]_R, [(2,2)]_R, [(2,3)]_R, [(2,4)]_R, [(3,1)]_R, [(3,2)]_R, \\ &[(3,3)]_R, [(3,4)]_R, [(4,1)]_R, [(4,2)]_R, [(4,3)]_R, [(4,4)]_R\}. \end{aligned}$$

Let $X = \{(0,1), (0,2), (0,3), (0,4), (1,2)\}$ be a subset of $S$. Thus,

$$\underline{A}_R(X) = \{(0,1), (0,2), (0,3), (0,4)\}$$

and

$$\overline{A}_R(X) = \{(0,1), (0,2), (0,3), (0,4), (1,2), (2,4), (3,1), (4,3)\}$$

**Definition 3.3.** Let $A_R(X) = (\underline{A}_R(X), \overline{A}_R(X))$ be a rough set of $X$ in $S$. A nonempty subset $X$ of a group $S$ is called an $\overline{A}_R-$ rough subgroup of $S$ if the upper approximation $\overline{A}_R(X)$ of $X$ is a subgroup of $S$. Similarly, a nonempty subset $X$ of a group $S$ is called an $\underline{A}_R-$ rough subgroup of $S$ if the lower approximation $\underline{A}_R(X)$ of $X$ is a subgroup of $S$.

Let $S$ be a subset of $D \times D$ given by

$$S = \{(a,b) : a,\ b \in D,\ a \neq 0,\ b \neq 0,\ (a,b) = (1,1) \text{ if } a = b\}.$$

$S$ is a group with respect to the binary operation $(a,b)(c,d) = (ac, bd)$ for all $(a,b), (c,d) \in S$.

**Proposition 3.4.** *Let $R$ be an equivalence relation on a group $S$. If $X$ is a subgroup of $S$, then it is an $\overline{A}_R-$ rough subgroup of $S$.*

*Proof.* It is easily shown that $X$ is an $\overline{A}_R-$ rough subsemigroup of $S$. Now, we can show that each element of $\overline{A}_R(X)$ has an inverse. Let $(a,b)$ be an element of $\overline{A}_R(X)$. Then $(a_1,b_1) \in [(a,b)]_R \cap X$ for some $(a_1,b_1) \in S$, that is, $(a_1,b_1) \in [(a,b)]_R$ and $(a_1,b_1) \in X$. Since $X$ is a subgroup of $S$, $(b_1,a_1) \in X$. On the other hand, since $(a_1,b_1) \in [(a,b)]_R$ for some $(a_1,b_1) \in S$, $a_1 b = b_1 a$. Since $b_1 a = a_1 b$, $(b_1,a_1) \in [(b,a)]_R$. Hence $(b_1,a_1) \in [(b,a)]_R \cap X$, and so $(b,a) \in \overline{A}_R(X)$. These imply that $\overline{A}_R(X)$ is a subgroup of $S$. □

**Proposition 3.5.** *Let $R$ be an equivalence relation on a group $S$. If $X$ is a subgroup of $S$, then it is an $\underline{A}_R-$ rough subgroup of $S$.*

*Proof.* It is easily shown that $X$ is an $\underline{A}_R-$ rough subsemigroup of $S$. Now, we can show that each element of $\underline{A}_R(X)$ has an inverse. Let $(a,b)$ be an element of $\underline{A}_R(X)$. Then $(a_1,b_1) \in [(a,b)]_R \subseteq X$ for some $(a_1,b_1) \in S$ and then $(b_1,a_1) \in [(b,a)]_R$. Since $X$ is a subgroup of $S$, $(b_1,a_1) \in X$. Thus $(b_1,a_1) \in [(b,a)]_R \subseteq X$. Hence $(b,a) \in \underline{A}_R(X)$. Therefore $\underline{A}_R(X)$ is a subgroup of $S$. □

Let $S$ be a subset of $D \times D$ given by

$$S = \{(a,b) : a,\ b \in D,\ b \neq 0,\ (a,b) = (0,1) \text{ if } a = 0\}.$$

$S$ is a group with respect to binary operation $(a,b) + (c,d) = (ad + bc, bd)$ for all $(a,b), (c,d) \in S$.

**Proposition 3.6.** *Let $R$ be an equivalence relation on a group $S$. If $X$ is a subgroup of $S$, then it is an $\overline{A}_R-$ rough subgroup of $S$.*

*Proof.* It is easily shown that $X$ is an $\overline{A}_R-$ rough subsemigroup of $S$. Now, we can show that each element of $\overline{A}_R(X)$ has an inverse. Let $(a, b)$ be an element of $\overline{A}_R(X)$. Then $(a_1, b_1) \in [(a, b)]_R \cap X$ for some $(a_1, b_1) \in S$, that is, $(a_1, b_1) \in [(a, b)]_R$ and $(a_1, b_1) \in X$. Since $X$ is a subgroup of $S$, $(-a_1, b_1) \in X$. On the other hand, since $(a_1, b_1) \in [(a, b)]_R$ for some $(a_1, b_1) \in S$, $a_1 b = b_1 a$. Since -$a_1 b = b_1(-a)$, $(-a_1, b_1) \in [(-a, b)]_R$. Thus $(-a_1, b_1) \in [(-a, b)]_R \cap X$, and so $(-a, b) \in \overline{A}_R(X)$. These imply that $\overline{A}_R(X)$ is a subgroup of $S$. □

**Proposition 3.7.** *Let $R$ be an equivalence relation on a group $S$. If $X$ is a subgroup of $S$, then it is an $\underline{A}_R-$ rough subgroup of $S$.*

*Proof.* It is easily shown that $X$ is an $\underline{A}_R-$ rough subsemigroup of $S$. Now, we can show that each element of $\underline{A}_R(X)$ has an inverse. Let $(a, b)$ be an element of $\underline{A}_R(X)$. Then $(a_1, b_1) \in [(a, b)]_R \subseteq X$ for some $(a_1, b_1) \in S$ and then $(-a_1, b_1) \in [(-a, b)]_R$. Since $X$ is a subgroup of $S$, $(-a_1, b_1) \in X$. Thus $(-a_1, b_1) \in [(-a, b)]_R \subseteq X$. Hence $(-a, b) \in \underline{A}_R(X)$. Therefore $\underline{A}_R(X)$ is a subgroup of $S$. □

**Definition 3.8.** Let $A_R(X) = (\underline{A}_R(X), \overline{A}_R(X))$ be a rough set of $X$ in $S$. A nonempty subset $X$ of a ring $S$ is called an $\overline{A}_R-$ rough subring of $S$ if the upper approximation $\overline{A}_R(X)$ of $X$ is a subring of $S$. Similarly, a nonempty subset $X$ of a group $S$ is called an $\underline{A}_R-$ rough subring of $S$ if the lower approximation $\underline{A}_R(X)$ of $X$ is a subring of $S$.

Let $S$ be a subset of $D \times D$ given by

$$S = \{(a, b) : a,\ b \in D,\ b \neq 0,\ (a, b) = (0, 1) \text{ if } a = 0\}.$$

$S$ is a ring with respect to binary operations $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in S$.

**Proposition 3.9.** *Let $R$ be an equivalence relation on a ring $S$. If $X$ is a subring of $S$, then $X$ is an $\overline{A}_R-$ rough subring of $S$.*

*Proof.* Let $(a, b)$ and $(c, d)$ be any elements of $\overline{A}_R(X)$. Then there exist elements $(a_1, b_1)$ and $(c_1, d_1)$ in $S$ such that $(a_1, b_1) \in [(a, b)]_R \cap X$ and $(c_1, d_1) \in [(c, d)]_R \cap X$. Thus $(a_1, b_1) \in [(a, b)]_R$, $(a_1, b_1) \in X$ and $(c_1, d_1) \in [(c, d)]_R$, $(c_1, d_1) \in X$. Since $X$ is a subring of $S$, $(a_1 d_1 - b_1 c_1, b_1 d_1) \in X$. Hence $(a_1 d_1 - b_1 c_1, b_1 d_1) \in [(ad - bc, bd)]_R = [(a, b) - (c, d)]_R$. Therefore $(a_1 d_1 - b_1 c_1, b_1 d_1) \in [(a, b) - (c, d)]_R \cap X$, and so $(a, b) - (c, d) \in \overline{A}_R(X)$.

Let $(a, b)$ and $(c, d)$ be any elements of $\overline{A}_R(X)$. Then there exist elements $(a_1, b_1)$ and $(c_1, d_1)$ in $S$ such that $(a_1, b_1) \in [(a, b)]_R \cap X$ and $(c_1, d_1) \in [(c, d)]_R \cap X$. Thus $(a_1, b_1) \in [(a, b)]_R$, $(a_1, b_1) \in X$ and $(c_1, d_1) \in [(c, d)]_R$, $(c_1, d_1) \in X$. Since $X$ is a subring of $S$, $(a_1, b_1)(c_1, d_1) = (a_1 c_1, b_1 d_1) \in X$. And $(a_1 c_1, b_1 d_1) \in [(ac, bd)]_R = [(a, b)(c, d)]_R$. Hence $(a_1 c_1, b_1 d_1) \in [(a, b)(c, d)]_R \cap X$ and so $(a, b)(c, d) \in \overline{A}_R(X)$. Therefore $\overline{A}_R(X)$ is a subring of $S$. □

**Proposition 3.10.** *Let $R$ be an equivalence relation on a ring $S$. If $X$ is a subring of $S$, then $X$ is an $\underline{A}_R-$ rough subring of $S$.*

*Proof.* Let $(a, b)$ and $(c, d)$ be any elements of $\underline{A}_R(X)$. Then $[(a, b)]_R \subseteq X$ and $[(c, d)]_R \subseteq X$. Hence $[(a, b)]_R - [(c, d)]_R = [(ad - bc, bd)]_R \subseteq X$ and so $(ad - bc, bd) = (a, b) - (c, d) \in \underline{A}_R(X)$.

Let $(a, b)$ and $(c, d)$ be any elements of $\underline{A}_R(X)$. Then there exist elements $(a_1, b_1)$ and $(c_1, d_1)$ in $S$ such that $(a_1, b_1) \in [(a, b)]_R \subseteq X$ and $(c_1, d_1) \in [(c, d)]_R \subseteq X$. Thus $(a_1b_1, c_1d_1) \in [(ac, bd)]_R$. Since $X$ is a subring of $S$, $(a_1b_1, c_1d_1) \in X$. Hence $(a_1b_1, c_1d_1) \in [(ac, bd)]_R \subseteq X$. Then $(a, b)(c, d) = (ac, bd) \in \overline{A}_R(X)$. Therefore $\underline{A}_R(X)$ is a subring of $S$. □

**Definition 3.11.** Let $A_R(X) = (\underline{A}_R(X), \overline{A}_R(X))$ be a rough set of $X$ in $S$. A nonempty subset $X$ of a field $S$ is called an $\overline{A}_R-$ rough subfield of $S$ if the upper approximation $\overline{A}_R(X)$ of $X$ is a subfield of $S$. Similarly, a nonempty subset $X$ of a field $S$ is called an $\underline{A}_R-$ rough subfield of $S$ if the lower approximation $\underline{A}_R(X)$ of $X$ is a subfield of $S$.

Let $S$ be the subset of $D \times D$ given by

$$S = \{(a, b) : a,\ b \in D,\ b \neq 0,\ (a, b) = (0, 1) \text{ if } a = 0\ \wedge\ (xa, xb) = (a, b)\}.$$

$S$ is a field with respect to binary operations $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in S$.

**Proposition 3.12.** *Let $R$ be an equivalence relation on a field $S$. If $X$ is a subfield of $S$, then $X$ is an $\overline{A}_R-$ rough subfield of $S$.*

*Proof.* It can be easily seen that $\overline{A}_R(X)$ is a commutative ring with unity. Let $(a, b)$ be any element of $\overline{A}_R(X)$. Then there exists element $(a_1, b_1)$ in $S$ such that $(a_1, b_1) \in [(a, b)]_R \cap X$. Thus $(a_1, b_1) \in [(a, b)]_R$ and $(a_1, b_1) \in X$. Since $X$ is a subfield of $S$, $(b_1, a_1) \in X$. And $(b_1, a_1) \in [(b, a)]_R$. Hence $(b_1, a_1) \in ([(b, a)]_R \cap X$, and so $(b, a) \in \overline{A}_R(X)$. Therefore $\overline{A}_R(X)$ is a subfield of $S$. □

**Proposition 3.13.** *Let $R$ be an equivalence relation on a field $S$. If $X$ is a subfield of $S$, then $X$ is an $\underline{A}_R-$ rough subfield of $S$.*

*Proof.* It can be easily seen that $\overline{A}_R(X)$ is a commutative ring with unity. Let $(a, b)$ be any element of $\underline{A}_R(X)$. Then there exists element $(a_1, b_1)$ in $S$ such that $(a_1, b_1) \in [(a, b)]_R \subseteq X$. Thus $(a_1, b_1) \in [(a, b)]_R$ and $(a_1, b_1) \in X$. Since $X$ is a subfield of $S$, $(b_1, a_1) \in X$. And $(b_1, a_1) \in [(b, a)]_R$. Hence $(b_1, a_1) \in ([(b, a)]_R \subseteq X$, and so $(b, a) \in \underline{A}_R(X)$. Therefore $\underline{A}_R(X)$ is a subfield of $S$. □

**Definition 3.14.** Let $\mu$ be a fuzzy field of $F$. For each $t \in [0, 1]$, the set

$$\mu_t = \{([(x, y)]_R, [(a, b)]_R) : \mu([(x, y)]_R[(a, b)]_R^{-1}) \geq t \text{ for}$$

$$a \neq 0 \text{ and } \mu([(x, y)]_R - [(a, b)]_R) \geq t)\}$$

is called a $t-$level relation of $\mu$.

**Proposition 3.15.** *Let $\mu$ be a fuzzy field of a field $F$, and $t \in [0, 1]$. Then $\mu_t$ is a congruence relation on $F$.*

*Proof.* For any element $[(x,y)]_R$ of $F$,

$$\begin{aligned}\mu([(x,y)]_R[(x,y)]_R^{-1}) &= \mu([(x,y)]_R[(y,x)]) \\ &= \mu([1,1]) \\ &\geq t,\end{aligned}$$

and so $([(x,y)]_R,[(x,y)]_R) \in \mu_t$. For any element $[(x,y)]_R$ of $F$,

$$\begin{aligned}\mu([(x,y)]_R - [(x,y)]_R) &= \mu([(x,y)]_R + [(-x,y)]) \\ &= \mu([0,1]) \\ &\geq t,\end{aligned}$$

and so $([(x,y)]_R,[(x,y)]_R) \in \mu_t$.
If $([(x,y)]_R,[(a,b)]_R) \in \mu_t$, then $\mu([(x,y)]_R,[(a,b)]_R^{-1}) \geq t$. Since $\mu$ is a fuzzy field of $F$,

$$\begin{aligned}\mu([(a,b)]_R[(x,y)]_R^{-1}) &= \mu(([(a,b)]_R[(x,y)]_R^{-1})^{-1}) \\ &= \mu([(x,y)]_R[(a,b)]_R^{-1}) \\ &\geq t,\end{aligned}$$

which yields $([(a,b)]_R,[(x,y)]_R) \in \mu_t$. If $([(x,y)]_R,[(a,b)]_R) \in \mu_t$, then
$\mu([(x,y)]_R,[(a,b)]_R^{-1}) \geq t$. Since $\mu$ is a fuzzy field of $F$,

$$\begin{aligned}\mu([(a,b)]_R - [(x,y)]_R) &= \mu(([(a,b)]_R - [(x,y)]_R)^{-1}) \\ &= \mu([(x,y)]_R - [(a,b)]_R) \\ &\geq t,\end{aligned}$$

which yields $([(a,b)]_R,[(x,y)]_R) \in \mu_t$.
If $([(x,y)]_R,[(a,b)]_R) \in \mu_t$ and $([(a,b)]_R,[(z,w)]_R) \in \mu_t$, then since $\mu$ is a fuzzy field of $F$,

$$\begin{aligned}\mu([(x,y)]_R[(z,w)]_R^{-1}) &= \mu([(x,y)]_R[1,1]_R[(z,w)]_R^{-1}) \\ &= \mu([(x,y)]_R[(a,b)]_R[(a,b)]_R^{-1}[(z,w)]_R^{-1}) \\ &= \mu(([(x,y)]_R[(a,b)]_R^{-1})([(a,b)]_R[(z,w)]_R^{-1})) \\ &\geq \min\{\mu([(x,y)]_R[(a,b)]_R^{-1}),\mu([(a,b)]_R[(z,w)]_R^{-1})\} \\ &\geq \min\{t,t\} \\ &= t,\end{aligned}$$

and so $([(x,y)]_R,[(z,w)]_R) \in \mu_t$.
If $([(x,y)]_R,[(a,b)]_R) \in \mu_t$ and $([(a,b)]_R,[(z,w)]_R) \in \mu_t$, then since $\mu$ is a fuzzy field of $F$,

$$\begin{aligned}\mu([(x,y)]_R - [(z,w)]_R) &= \mu([(x,y)]_R + [0,1]_R - [(z,w)]_R) \\ &= \mu([(x,y)]_R + [(a,b)]_R - [(a,b)]_R - [(z,w)]_R) \\ &= \mu(([(x,y)]_R - [(a,b)]_R) + ([(a,b)]_R - [(z,w)]_R)) \\ &\geq \min\{\mu([(x,y)]_R - [(a,b)]_R),\mu([(a,b)]_R - [(z,w)]_R)\} \\ &\geq \min\{t,t\} \\ &= t,\end{aligned}$$

and so $([(x,y)]_R,[(z,w)]_R)\in\mu_t$. Therefore $\mu_t$ is an equivalence relation on $F$. We can show that $\mu_t$ is a congruence relation on $F$. Let $([(x,y)]_R,[(a,b)]_R)\in\mu_t$ and $[(z,w)]_R$ be any element of $F$. Then, since $\mu([(x,y)]_R[(a,b)]_R^{-1})\geq t$,

$$\begin{aligned}
& \mu(([(x,y)]_R[(z,w)]_R)([(a,b)]_R[(z,w)]_R)^{-1}) \\
= & \mu(([(x,y)]_R[(z,w)]_R)([(z,w)]_R^{-1}[(a,b)]_R^{-1})) \\
= & \mu([(x,y)]_R([(z,w)]_R[(z,w)]_R^{-1})[(a,b)]_R^{-1}) \\
= & \mu([(x,y)]_R[1,1]_R[(a,b)]_R^{-1}) \\
= & \mu([(x,y)]_R[(a,b)]_R^{-1}) \\
\geq & t,
\end{aligned}$$

and so $([(x,y)]_R[(z,w)]_R,[(a,b)]_R[(z,w)]_R)\in\mu_t$. Similarly

$$([(z,w)]_R[(x,y)]_R,[(z,w)]_R[(a,b)]_R)\in\mu_t.$$

Let $([(x,y)]_R,[(a,b)]_R),([(z,w)]_R,[m,n]_R)\in\mu_t$. Then, $\mu([(x,y)]_R,[(a,b)]_R)\geq t$ and $\mu([(z,w)]_R+[(m,n)]_R^{-1})\geq t$,

$$\begin{aligned}
& \mu(([(x,y)]_R+[(z,w)]_R)+([(a,b)]_R+[(m,n)]_R)^{-1}) \\
= & \mu(([(x,y)]_R+[(z,w)]_R)+([(-a,b)]_R-[(m,n)]_R)) \\
= & \mu(([(x,y)]_R-[(a,b)]_R)+([(z,w)]_R-[(m,n)]_R)) \\
\geq & \min\{\mu([(x,y)]_R-[(a,b)]_R),\mu([(z,w)]_R-[(m,n)]_R)\} \\
\geq & \min\{t,t\} \\
= & t,
\end{aligned}$$

and so $([(x,y)]_R+[(z,w)]_R,[(a,b)]_R+[(m,n)]_R)\in\mu_t$. Similarly

$$([(z,w)]_R+[(x,y)]_R,[(m,n)]_R+[(a,b)]_R)\in\mu_t.$$

Therefore, $\mu_t$ is a congruence relation on $F$. This completes the proof. □

Let $\mu$ be a fuzzy field of a field $F$, $\mu_t$ be a $t-$level congruence relation of $\mu$ on $F$, and $X$ be a nonempty subset of $F$. The congruence class of $\mu_t$ containing the element $[(x,y)]_R$ of $F$ will be denoted by $[(x,y)]_R^{\mu}$. Then the sets

$$\underline{\mu_t}(X)=\{[(x,y)]_R\in F:[(x,y)]_R^{\mu}\subseteq X\}$$

and

$$\overline{\mu_t}(X)=\{[(x,y)]_R\in F:[(x,y)]_R^{\mu}\cap X\neq\varnothing\}$$

are called, respectively, the lower and upper approximations of the sets $X$ with respect to $\mu_t$.

**Proposition 3.16.** *Let $\mu$ be a fuzzy field of a field $F$, and $t\in[0,1]$. If $X$ is a subfield of $F$, then $\overline{\mu_t}(X)$ is a subfield of $F$.*

*Proof.* Let $X$ be a subfield of $F$, and $[(x,y)]_R$, $[(a,b)]_R$ be any elements of $\overline{\mu_t}(X)$. Then there exist elements $[(x_1,y_1)]_R$ and $[(a_1,b_1)]_R$ in $F$ such that

$$[(x_1,y_1)]_R\in[(x,y)]_R^{\mu}\cap X$$

and

$$[(a_1,b_1)]_R\in[(a,b)]_R^{\mu}\cap X.$$

Thus, $[(x_1,y_1)]_R \in [(x,y)]_R^\mu$, $[(x_1,y_1)]_R \in X$, $[(a_1,b_1)]_R \in [(a,b)]_R^\mu$ and $[(a_1,b_1)]_R \in X$. Since $\mu_t$ is a congruence relation on $F$,

$$[(x_1,y_1)]_R - [(a_1,b_1)]_R \in ([(x,y)]_R^\mu - [(a,b)]_R^\mu) = ([(x,y)]_R - [(a,b)]_R)^\mu.$$

Since $X$ is a subfield of $F$, $[(x_1,y_1)]_R - [(a_1,b_1)]_R \in X$. Thus, $[(x_1,y_1)]_R - [(a_1,b_1)]_R \in ([(x,y)]_R - [(a,b)]_R)^\mu \cap X$, and so $([(x,y)]_R - [(a,b)]_R) \in \overline{\mu_t}(X)$. If $[(x,y)]_R$ and $[(a,b)]_R$ are any elements of $\overline{\mu_t}(X)$, then there exist elements $[(x_1,y_1)]_R$ and $[(a_1,b_1)]_R$ in $F$ such that

$$[(x_1,y_1)]_R \in [(x,y)]_R^\mu \cap X$$

and

$$[(a_1,b_1)]_R \in [(a,b)]_R^\mu \cap X.$$

Thus, $[(x_1,y_1)]_R \in [(x,y)]_R^\mu$, $[(x_1,y_1)]_R \in X$, $[(a_1,b_1)]_R \in [(a,b)]_R^\mu$ and $[(a_1,b_1)]_R \in X$. Since $\mu_t$ is a congruence relation on $F$,

$$[(x_1,y_1)]_R[(b_1,a_1)]_R \in ([(x,y)]_R^\mu[(b,a)]_R^\mu) = ([(x,y)]_R[(b,a)]_R)^\mu.$$

Since $X$ is a subfield of $F$, $[(x_1,y_1)]_R[(b_1,a_1)]_R \in X$. Thus, $[(x_1,y_1)]_R[(b_1,a_1)]_R \in ([(x,y)]_R[(b,a)]_R)^\mu \cap X$, and so $[(x,y)]_R[(b,a)]_R \in \overline{\mu_t}(X)$. □

**Proposition 3.17.** *Let $\mu$ be a fuzzy field of a field $F$, and $t \in [0,1]$. If $X$ is a subfield of $F$, then $\underline{\mu_t}(X)$ is a subfield of $F$.*

*Proof.* Let $X$ be a subfield of $F$, and $[(x,y)]_R$ and $[(a,b)]_R$ be any elements of $\underline{\mu_t}(X)$. Then

$$[(x,y)]_R^\mu \subseteq X$$

and

$$[(a,b)]_R^\mu \subseteq X.$$

Since $\mu_t$ is a congruence relation on $F$,

$$([(x,y)]_R^\mu - [(a,b)]_R^\mu) = ([(x,y)]_R - [(a,b)]_R)^\mu \subseteq X$$

and so $([(x,y)]_R - [(a,b)]_R) \in \overline{\mu_t}(X)$. If $[(x,y)]_R$ and $[(a,b)]_R$ are any elements of $\overline{\mu_t}(X)$, then

$$[(x,y)]_R^\mu \subseteq X$$

and

$$[(a,b)]_R^\mu \subseteq X.$$

Since $\mu_t$ is a congruence relation on $F$,

$$([(x,y)]_R^\mu[(b,a)]_R^\mu) = ([(x,y)]_R[(b,a)]_R)^\mu \subseteq X.$$

and so $[(x,y)]_R[(b,a)]_R \in \underline{\mu_t}(X)$. □

## References


[1] H. Aktaş and N. Çağman, Some structure properties of the cyclic fuzzy group family, Algebra Colloq. 12(3) (2005) 471-476.
[2] B. Bismas and S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42 (1994) 251-254.
[3] R. Bismas, On rough sets and fuzzy rough sets, Bull. Polish Acad. Sci. Math. 42 (1994) 345-349.
[4] Z. Bonikowaski, Algebric structures of rough sets, Fuzzy Sets and Knowledge Discovery (1995) 242-247.
[5] B. Davvaz, Roughness in rings, Inform. Sci. 164 (2004) 147-163.
[6] V. N. Dixit, R. Kumar and N. Ajmal, On fuzzy rings, Fuzzy Sets and Systems 49 (1992) 205-213.
[7] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17(2-3) (1990) 191-209.
[8] J. B. Fraleigh, A First Course in Abstract Algebra, New York, 1994.
[9] T. Iwinski, Algebric approach to rough sets, Bull. Polish Acad. Sci. Math. 35 (1987) 673-683.
[10] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100(1997) 139-163.
[11] N. Kuroki and P. P. Wang, The lower and upper approximations in fuzzy group, Inform. Sci. 90 (1996) 203-220.
[12] S. Nanda, Fuzzy fields and fuzzy linear spaces, Fuzzy Sets and Systems 19 (1986) 89-94.
[13] S. Nanda and S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems 45 (1992) 157-160.
[14] Z. Pawlak, Rough Sets, Internat. J. Comput. Inform. Sci. 11 (1982) 341-356.
[15] Z. Pawlak, Rough sets: basic notions, ICS PAS Rep. 436, 1981.
[16] Z. Pawlak, Some remarks on rough sets, Bull. Polish Acad. Sci. Tech. Sci. 33 (1985) 567-572.
[17] Z. Pawlak, Rough sets and fuzzy sets, Fuzzy Sets and Systems 17 (1985) 99-102.
[18] J. Pomykala and J. A. Pomykala, The stone algebra of rough sets, Bull. Polish Acad. Sci. Math. 36 (1988) 495-508.
[19] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512-517.



Hacı Aktaş (haktas@nevsehir.edu.tr)
Department of mathematics, faculty of art and science, university of Nevşehir, postal code 50350, Nevşehir, Turkey

Filiz Çitak (filiz.citak@gop.edu.tr)
Department of mathemathics, faculty of art and science, university of Gaziosmanpaşa, postal code 60100, Tokat, Turkey