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On interval valued fuzzy ideals in hemirings

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Abstract. In this paper we define and discuss interval valued fuzzy prime ideals, interval valued fuzzy irreducible ideals, interval valued fuzzy normal ideals of hemirings and some properties of the associated topology.

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1. Introduction

There are many generalizations of associative rings. Some of them, in particular, nearrings and semirings are very useful for solving problems in different areas of applied mathematics and information sciences. Semirings which are common generalization of associative ring and distributive lattices were introduced by H. S. Vandiver in 1934 [13]. In more recent times semirings have been deeply studied, especially in relations with applications [5]. Semirings have also been used for studying optimization, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes [2, 3, 5, 6]. An additively commutative semiring with zero element is called hemiring. Hemirings, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in information sciences [8, 11]. In 1965 Zadeh [16] introduced the concept of fuzzy sets. Since then fuzzy set has been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [12] and he introduced the notion of fuzzy subgroups. In [1] J. Ahsan initiated the study of fuzzy semirings. The fuzzy algebraic structures play an important role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [7, 14]. The concept of interval valued fuzzy set in algebra was initiated in [4] by Biswas and further this concept was investigated in [9]. As Ideals of hemirings and semirings play a vital role in the structure theory and are very useful for many purposes in the study of hemirings. So keeping in view the importance of ideals in semirings and hemirings, in this paper we define interval valued fuzzy prime ideals, interval valued fuzzy irreducible ideals, interval valued fuzzy normal ideals of hemirings and some properties of the associated topology.

2. Preliminaries

A set $R \neq \phi$ together with two associative binary operations addition " + " and multiplication " $\cdot$" is called semiring if "+" and "$\cdot$" are connected by the distributive laws, that is for all $a, b, c \in R$,

$$a\,(b+c) = ab + ac \text{ and } (a+b)\,c = ac + bc$$

An element $0 \in R$ satisfying the conditions, $0x = x0 = 0$ and $0 + x = x + 0 = x$, for all $x \in R$, is called zero of the semiring R. Further an element $1 \in R$ satisfying the condition, $1 \cdot x = x \cdot 1 = x$ for all $x \in R$, is called multiplicative identity or simply identity of the semiring R. A semiring R is called commutative if and only if, for all $a, b \in R$, $a.b = b.a$. A semiring with commutative addition and zero element is called a hemiring. A non-empty $A \subseteq R$ is called a subhemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. An element $a \in R$ is called multiplicatively idempotent if $a^2 = a$. A hemiring R is called multiplicatively idempotent if each element of R is multiplicatively idempotent. A non-empty $I \subseteq R$ is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I$ ($IR \subseteq I$). If I and J are left (respectively right) ideals of a hemiring R then $I \cap J$ is a left (respectively right) ideal of R. If I is a subset of R, then intersection of all left (right) ideals of R which contain I is a left (right) ideal of R containing I . Of course this is the smallest left (right) ideal of R containing I and is called the left (right) ideal of R generated by I. If I and J are left (respectively right) ideals of a hemiring R then $I + J$ is the smallest left (respectively right) ideal of R containing both I and J. If I and J are ideals of a hemiring R then IJ is an ideal of R contained in $I \cap J$. An ideal P of a hemiring R is called prime ideal of R if for any ideals I and J of R if $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. An ideal I of a hemiring R is called irreducible ideal of R if for any ideals A and B of R, if $A \cap B = I \Rightarrow A = I$ or $B = I$. An ideal I of a hemiring R is called idempotent if $I^2 = I$. A hemiring R is called fully idempotent if each two-sided ideal of R is idempotent, that is $I^2 = I$. If $A \subseteq R$, then characteristic function C_A of A is a function from X into $\{0, 1\}$, defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

A fuzzy subset λ of a universe X is a function $\lambda : X \longrightarrow [0, 1]$. The symbols $\lambda \wedge \mu$ and $\lambda \vee \mu$ will mean the following fuzzy subsets of X

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$$

for all $x \in X$, respectively. More generally, if $\{\lambda_i : i \in \Lambda\}$ is a family of fuzzy subsets of X, then $\wedge_{i\in\Lambda}\lambda_i$ and $\vee_{i\in\Lambda}\lambda_i$ are defined by

$$(\wedge_{i\in\Lambda}\lambda_i)(x) = \wedge_{i\in\Lambda}(\lambda_i(x))$$
$$(\vee_{i\in\Lambda}\lambda_i)(x) = \vee_{i\in\Lambda}(\lambda_i(x))$$

and are called the intersection and the union of the family $\{\lambda_i : i \in \Lambda\}$ of fuzzy subsets of X, respectively.

Definition 2.1. Let λ and μ be any two fuzzy subsets of a hemiring R. Then the sum of λ and μ is defined as

$$(\lambda + \mu)(x) = \vee_{x=y+z} [\lambda(y) \wedge \mu(z)] \qquad \text{for all } x \in R.$$

Definition 2.2 ([15])**.** Let λ and μ be any two fuzzy subsets of a hemiring R. Then the product of λ and μ is defined as

$$(\lambda\mu)(x) = \vee_{x=\Sigma_{i=1}^{n} y_i z_i} [\wedge_i \{\lambda(y_i) \wedge \mu(z_i)\}] \qquad \text{for all } x \in R.$$

Definition 2.3. A hemiring R is called von Neumann regular if for any $x \in R$ there exists $a \in R$ such that $x = xax$ or we have $x \in xRx$, for all $x \in R$.

Theorem 2.4 ([1])**.** *A hemiring R is von Neumann regular if and only if for any right ideal A and any left ideal B of R, $A \cap B = AB$.*

Theorem 2.5. *If R is commutative hemiring, then R is fully idempotent if and only if R is von Neumann regular.*

Definition 2.6 ([1])**.** A fuzzy subset λ of a hemiring R is said to be a fuzzy left (respectively right) ideal of the hemiring R if for all $x, y \in R$
(i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$
(ii) $\lambda(xy) \geq \lambda(y)$ (respectively $\lambda(xy) \geq \lambda(x)$).

A fuzzy subset λ of a hemiring R is called a fuzzy ideal of R if it is both fuzzy left and right ideal of R

Theorem 2.7. *If λ and μ are fuzzy left (respectively right) ideals of a hemiring R then so are $\lambda \cap \mu, \lambda + \mu$ and $\lambda\mu$.*

Let $\pounds$ denotes the family of all closed subintervals of $[0, 1]$ with minimal element $\tilde{O}$ $= [0, 0]$ and maximal element $\tilde{I} = [1, 1]$ according to the partial order $[\alpha, \alpha'] \leq [\beta, \beta']$ if and only if $\alpha \leq \beta, \alpha' \leq \beta'$ defined on $\pounds$ for all $[\alpha, \alpha'], [\beta, \beta'] \in \pounds$.

Definition 2.8. An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is a function $\tilde{\lambda} : R \to \pounds$.

We write $\tilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0, 1]$ for all $x \in R$ where $\lambda^-, \lambda^+ : R \to [0, 1]$ are lower and upper fuzzy sets of R, giving lower and upper limits of the image interval for each $x \in R$. Note that we have $0 \leq \lambda^-(x) \leq \lambda^+(x) \leq 1$ for all $x \in R$. For simplicity we write $\tilde{\lambda} = [\lambda^-, \lambda^+]$.

Definition 2.9. Let A be a subset of a hemiring R. Then the interval valued characteristic function $\tilde{C}_A$ of A is defined to be a function $\tilde{C}_A : R \to \pounds$ such that for all $x \in R$

$$\tilde{C}_A(x) = \begin{cases} \tilde{I} = [1, 1] & \text{if } x \in A, \\ \tilde{O} = [0, 0] & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of R is also an interval valued fuzzy subset of R. The interval valued characteristic function can be used to indicate either membership or non-membership of any member of R in a subset A of R. Note that $\tilde{C}_R(x) = \tilde{I}$ for all $x \in R$.

Lemma 2.10. *Let $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\nu}$ be the interval valued fuzzy subsets of a hemiring R. If $\tilde{\lambda} \subseteq \tilde{\mu}$ then $\tilde{\lambda}\tilde{\nu} \subseteq \tilde{\mu}\tilde{\nu}$ and $\tilde{\nu}\tilde{\lambda} \subseteq \tilde{\nu}\tilde{\mu}$.*

Proof. Proof is straightforward. □

Definition 2.11 ([10])**.** Let $\tilde{\lambda}$ be an interval valued fuzzy subset of a hemiring R. Then $\tilde{\lambda}$ is said to be an interval valued fuzzy left (resp. right) ideal of R if and only if for all $x, y \in R$

(i) $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$

(ii) $\tilde{\lambda}(xy) \geq \tilde{\lambda}(y)$ $\Big($resp. $\tilde{\lambda}(xy) \geq \tilde{\lambda}(x)\Big)$.

An interval valued fuzzy subset $\tilde{\lambda} : R \to £$ is called an interval valued fuzzy ideal of R if it is both, interval valued fuzzy left and right ideal of R.

Lemma 2.12. *An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is an interval valued fuzzy left (respectively right) ideal of R if and only if $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$ and $\tilde{C}_R\tilde{\lambda} \subseteq \tilde{\lambda}$ (respectively $\tilde{\lambda}\tilde{C}_R \subseteq \tilde{\lambda}$).*

Proof. Proof is straightforward. □

Theorem 2.13 ([10])**.** *A subset A of a hemiring R is a left (respectively right) ideal of R if and only if the interval valued characteristic function $\tilde{C}_A$ is an interval valued fuzzy left (respectively right) ideals of R.*

Theorem 2.14. *If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of R then $\tilde{\lambda} + \tilde{\mu}$ and $\tilde{\lambda}\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of R.*

Proof. Proof is straightforward. □

Remark 2.15. If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of R then $\tilde{\lambda} \cap \tilde{\mu}$ is an interval valued fuzzy left (respectively right) ideal of R. In general, $\tilde{\lambda} \cap \tilde{\mu} \neq \tilde{\lambda}\tilde{\mu}$.

Theorem 2.16. *For a hemiring R with identity 1, the following assertions are equivalent:*

(i) *R is fully idempotent.*
(ii) *Each interval valued fuzzy ideal of R is idempotent.*
(iii) *If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy ideals of R then $\tilde{\lambda} \cap \tilde{\mu} = \tilde{\lambda}\tilde{\mu}$.*

If R is commutative then the above conditions are equivalent to

(iv) *R is regular.*

Proof. (i) $\Rightarrow$ (ii) Let λ be an interval valued fuzzy ideal of R and $x \in R$. Then

$$\begin{aligned}
\tilde{\lambda}^2(x) &= \left(\tilde{\lambda}\tilde{\lambda}\right)(x) \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \lambda^-(y_i z_i), \lambda^+(y_i z_i) \wedge \lambda^+(y_i z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\wedge_i \lambda^-(y_i z_i), \wedge_i \lambda^+(y_i z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-\left(\sum_{i=1}^n y_i z_i\right), \lambda^+\left(\sum_{i=1}^n y_i z_i\right)\right] \\
&= \tilde{\lambda}(x).
\end{aligned}$$

Thus $\tilde{\lambda}^2 \subseteq \tilde{\lambda}$. For the reverse inclusion, since R is fully idempotent, so

$$x \in < x > = < x >^2 = RxRRxR.$$

Thus $x = \sum_{i=1}^n a_i x a_i' b_i x b_i'$ for some $a_i, a_i', b_i, b_i' \in R$. Hence

$$\tilde{\lambda}(x) = \tilde{\lambda}(x) \wedge \tilde{\lambda}(x) \leq \tilde{\lambda}(a_i x a_i') \wedge \tilde{\lambda}(b_i x b_i')$$

for all $a_i, a_i', b_i, b_i' \in R$, which implies

$$\begin{aligned}
\tilde{\lambda}(x) &\leq \wedge_i \left[\lambda^-(a_i x a_i') \wedge \lambda^-(b_i x b_i'), \lambda^+(a_i x a_i') \wedge \mu^+(b_i x b_i')\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n a_i x a_i' b_i x b_i'} \left\{\wedge_i \begin{bmatrix} \lambda^-(a_i x a_i') \wedge \lambda^-(b_i x b_i'), \\ \lambda^+(a_i x a_i') \wedge \lambda^+(b_i x b_i') \end{bmatrix}\right\} \\
&\leq \vee_{x=\Sigma_{j=1}^m y_j z_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \lambda^-(z_j), \lambda^+(y_j) \wedge \lambda^+(z_j)\right]\right\} \\
&= (\tilde{\lambda}\tilde{\lambda})(x) = \tilde{\lambda}^2(x).
\end{aligned}$$

This shows $\tilde{\lambda} \subseteq \tilde{\lambda}^2$. Hence $\tilde{\lambda} = \tilde{\lambda}^2$.

(ii) $\Rightarrow$ (i) Let I be an ideal of R and let $\tilde{C}_I$ be the interval valued characteristic function of I. Then $\tilde{C}_I$ is the interval valued fuzzy ideal of R which is, by hypothesis, idempotent, that is,

$$\left(\tilde{C}_I\right)^2 = \tilde{C}_I \tilde{C}_I = \tilde{C}_I.$$

This implies $\tilde{C}_{I^2} = \tilde{C}_I$. Hence $I^2 = I$. Thus R is fully idempotent.

(i) $\Rightarrow$ (iii) Let $\tilde{\lambda}$ and $\tilde{\mu}$ be interval valued fuzzy ideals of R. Then for any $x \in R$

$$\begin{aligned}
\left(\tilde{\lambda}\tilde{\mu}\right)(x) &= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \begin{bmatrix} \lambda^-(y_i z_i) \wedge \mu^-(y_i z_i), \\ \lambda^+(y_i z_i) \wedge \mu^+(y_i z_i) \end{bmatrix}\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\begin{array}{c} [\wedge_i \lambda^-(y_i z_i), \wedge_i \lambda^+(y_i z_i)] \wedge \\ [\wedge_i \mu^-(y_i z_i), \wedge_i \mu^+(y_i z_i)] \end{array}\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\begin{array}{c} [\lambda^-(\sum_{i=1}^n y_i z_i), \lambda^+(\sum_{i=1}^n y_i z_i)] \wedge \\ [\mu^-(\sum_{i=1}^n y_i z_i), \mu^+(\sum_{i=1}^n y_i z_i)] \end{array}\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\lambda(x) \wedge \mu(x)\right\} \\
&= \left(\tilde{\lambda} \cap \tilde{\mu}\right)(x).
\end{aligned}$$

This implies $\tilde{\lambda}\tilde{\mu} \subseteq \tilde{\lambda} \cap \tilde{\mu}$. For reverse containment, since R is fully idempotent so $x \in < x >=< x >^2$. Thus as we argued in (i) $\Rightarrow$ (ii)

$$\begin{aligned} \left(\tilde{\lambda} \cap \tilde{\mu}\right)(x) &= \tilde{\lambda}(x) \wedge \tilde{\mu}(x) \\ &\leq \vee_{x=\Sigma_{j=1}^m y_j z_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \mu^-(z_j), \lambda^+(y_j) \wedge \mu^+(z_j)\right]\right\} \\ &= \left(\tilde{\lambda}\tilde{\mu}\right)(x). \end{aligned}$$

This shows $\tilde{\lambda} \cap \tilde{\mu} \subseteq \tilde{\lambda}\tilde{\mu}$. Hence $\tilde{\lambda} \cap \tilde{\mu} = \tilde{\lambda}\tilde{\mu}$.

(iii) $\Rightarrow$ (ii) Let $\tilde{\lambda}$ be an interval valued fuzzy ideal of R . Then

$$\tilde{\lambda}^2 = \tilde{\lambda}\tilde{\lambda} = \tilde{\lambda} \cap \tilde{\lambda} = \tilde{\lambda}.$$

Thus $\tilde{\lambda}$ is idempotent.

If R is commutative then (i)$\Leftrightarrow$ (iv). □

Definition 2.17. An interval valued fuzzy ideal $\tilde{\xi}$ of a hemiring R is called interval valued fuzzy prime ideal of R if for any interval valued fuzzy ideals $\tilde{\lambda}$ and $\tilde{\mu}$ of R $\tilde{\lambda}\tilde{\mu} \subseteq \tilde{\xi} \Rightarrow \tilde{\lambda} \subseteq \tilde{\xi}$ or $\tilde{\mu} \subseteq \tilde{\xi}$.

Definition 2.18. An interval valued fuzzy ideal $\tilde{\xi}$ of a hemiring R is called interval valued fuzzy irreducible ideal of R if for any interval valued fuzzy ideals $\tilde{\lambda}$ and $\tilde{\mu}$ of R, $\tilde{\lambda} \cap \tilde{\mu} = \tilde{\xi} \Rightarrow \tilde{\lambda} = \tilde{\xi}$ or $\tilde{\mu} = \tilde{\xi}$.

Remark 2.19. (i) $\tilde{\lambda} = [\lambda^-, \lambda^+]$ is an interval valued fuzzy prime ideal of R if and only if λ^- and λ^+ are fuzzy prime ideals of R.

(ii) $\tilde{\lambda} = [\lambda^-, \lambda^+]$ is an interval valued fuzzy irreducible ideal of R if and only if λ^- and λ^+ are fuzzy irreducible ideals of R.

Lemma 2.20. *Let R be a fully idempotent hemiring. If $\tilde{\lambda}$ is an interval valued fuzzy ideal of R with $\tilde{\lambda}(x) = [\alpha, \beta] \in \pounds$ where $x \in R$, then there exists an interval valued fuzzy prime ideal $\tilde{\xi}$ of R such that $\tilde{\lambda} \subseteq \tilde{\xi}$ and $\tilde{\xi}(x) = [\alpha, \beta]$.*

Proof. Let

$$X = \left\{\tilde{\mu} : \tilde{\mu} \text{ is an interval valued fuzzy ideal of } R \text{ and } \tilde{\mu}(x) = [\alpha, \beta] \text{ and } \tilde{\lambda} \subseteq \tilde{\mu}\right\}.$$

Note that $X \neq \phi$ as $\tilde{\lambda} \in X$. Let $\pounds = \left\{\tilde{\lambda}_i : i \in I\right\}$ be a totally ordered subset of X, then we claim that $\cup_i \tilde{\lambda}_i$ is an interval valued fuzzy ideal of R. For this, let $x, y \in R$. Then $\left(\cup_i \tilde{\lambda}_i\right)(x) = \vee_i \left(\tilde{\lambda}_i(x)\right) \leq \vee_i \left(\tilde{\lambda}_i(xy)\right) = \left(\cup_i \tilde{\lambda}_i\right)(xy)$ and

$$\left(\cup_i \tilde{\lambda}_i\right)(x) = \vee_i \left(\tilde{\lambda}_i(x)\right) \leq \vee_i \left(\tilde{\lambda}_i(yx)\right) = \left(\cup_i \tilde{\lambda}_i\right)(yx).$$

Also

$$\begin{aligned}
\left(\cup_i\tilde{\lambda}_i\right)(x)\wedge\left(\cup_j\tilde{\lambda}_j\right)(y) &= \left(\cup_i\tilde{\lambda}_i\right)(x)\wedge\left(\vee_j\tilde{\lambda}_j(y)\right)\\
&= \vee_j\left[\left[\vee_i\tilde{\lambda}_i(x)\right]\wedge\tilde{\lambda}_j(y)\right]\\
&= \vee_j\left[\vee_i\left[\tilde{\lambda}_i(x)\wedge\tilde{\lambda}_j(y)\right]\right]\\
&\leq \vee_j\left[\vee_i\left[\tilde{\lambda}_i^j(x)\wedge\tilde{\lambda}_i^j(y)\right]\right] \text{ where } \lambda_i\vee\lambda_j=\tilde{\lambda}_i^j\in \pounds\\
&\leq \vee_j\left[\vee_i\left[\tilde{\lambda}_i^j(x+y)\right]\right]\\
&= \vee_{i,j}\left[\tilde{\lambda}_i^j(x+y)\right]\\
&\leq \vee_i\left[\tilde{\lambda}_i(x+y)\right]\\
&= \left(\cup_i\tilde{\lambda}_i\right)(x+y).
\end{aligned}$$

Thus $\cup_i\tilde{\lambda}_i$ is an interval valued fuzzy ideal of R. Clearly $\lambda\leq\cup_i\tilde{\lambda}_i$ and

$$\left(\cup_i\tilde{\lambda}_i\right)(x)=\vee_i\left(\tilde{\lambda}_i(x)\right)=\vee_i[\alpha,\beta]=[\alpha,\beta].$$

Thus $\cup_i\tilde{\lambda}_i$ is least upper bound of $\pounds$. Hence by Zorn's lemma, there exists an interval valued fuzzy ideal $\tilde{\xi}$ of R which is maximal with respect to the property that $\tilde{\lambda}\subseteq\tilde{\xi}$ and $\tilde{\xi}(x)=[\alpha,\beta]$. Let $\tilde{\Psi}$ and $\tilde{\theta}$ be any interval valued fuzzy ideals of R such that $\tilde{\theta}\cap\tilde{\Psi}=\tilde{\xi}$. Then $\tilde{\xi}\subseteq\tilde{\theta}$ and $\tilde{\xi}\subseteq\tilde{\Psi}$ and if $\tilde{\xi}\neq\tilde{\theta}$ and $\tilde{\xi}\neq\tilde{\Psi}$. Then, since $\tilde{\xi}$ is maximal with respect to the property that $\tilde{\xi}(x)=[\alpha,\beta]$ for $x\in R$. So $\tilde{\theta},\tilde{\Psi}\notin X$ and

$$\tilde{\theta}(x)\neq[\alpha,\beta]\neq\tilde{\Psi}(x).$$

Hence $[\alpha,\beta]=\tilde{\xi}(x)=\left(\tilde{\theta}\cap\tilde{\Psi}\right)(x)=\tilde{\theta}(x)\wedge\tilde{\Psi}(x)\neq[\alpha,\beta]$, which is impossible. Thus $\tilde{\xi}=\tilde{\theta}$ or $\tilde{\xi}=\tilde{\Psi}$. And since every interval valued fuzzy irreducible ideal of fully idempotent hemiring R is also interval valued fuzzy prime ideal, so $\tilde{\xi}$ is the required interval valued fuzzy prime ideal of R. □

Theorem 2.21. *Let R be a hemiring, Then the following assertions are equivalent:*

(i) *R is fully idempotent.*

(ii) *The set $\pounds_R$ of all interval valued fuzzy ideals of R (ordered by $\subseteq$) is distributive lattice under the sum and intersection of interval valued fuzzy ideals with $\tilde{\lambda}\cap\tilde{\mu}=\tilde{\lambda}\tilde{\mu}$ for each pair of interval valued fuzzy ideals $\tilde{\lambda},\tilde{\mu}$ of R.*

(iii) *Each interval valued fuzzy ideal is intersection of all those interval valued fuzzy prime ideals of R which contain it.*

If R is commutative then the above three assertions are equivalent to

(iv) *R is Von Neumann regular.*

Proof. (i) $\Rightarrow$ (ii) The set $\pounds_R$ of all interval valued fuzzy ideals of R (ordered by "$\subseteq$" i.e. $\tilde{\lambda}\subseteq\tilde{\mu}$ iff $\lambda^-(x)\leq\mu^-(x)$ and $\lambda^+(x)\leq\mu^+(x)$ for all $x\in R$) is clearly a lattice under the sum and intersection of interval valued fuzzy ideals. Moreover, since R is fully idempotent so by previous theorem $\tilde{\lambda}\cap\tilde{\mu}=\tilde{\lambda}\tilde{\mu}$ for each pair $\tilde{\lambda}$ and $\tilde{\mu}$ of interval valued fuzzy ideals of R. For distributive lattice, we have to show that for $\tilde{\lambda}, \tilde{\mu}, \tilde{\xi} \in \mathcal{L}_R$ $\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi} = \left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right)$. Let $x \in R$. Then

$$\begin{aligned}
\left[\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi}\right](x) &= \vee_{x=y+z} \left[\begin{array}{c} \left(\lambda^- \wedge \mu^-\right)(y) \wedge \xi^-(z), \\ \left(\lambda^+ \wedge \mu^+\right)(y) \wedge \xi^+(z) \end{array}\right] \\
&= \vee_{x=y+z} \left[\begin{array}{c} \lambda^-(y) \wedge \xi^-(z) \wedge \mu^-(y) \wedge \xi^-(z), \\ \lambda^+(y) \wedge \xi^+(z) \wedge \mu^+(y) \xi^+(z) \end{array}\right] \\
&= \vee_{x=y+z} \left[\lambda^-(y) \wedge \xi^-(z), \lambda^+(y) \wedge \xi^+(z)\right] \wedge \\
&\quad \left[\mu^-(y) \wedge \xi^-(z), \mu^+(y) \wedge \xi^+(z)\right] \\
&\leq \vee_{x=y+z} \left[\left(\lambda^- + \xi^-\right)(x), \left(\lambda^+ + \xi^+\right)(x)\right] \wedge \\
&\quad \left[\left(\mu^- + \xi^-\right)(x), \left(\mu^+ + \xi^+\right)(x)\right] \\
&= \vee_{x=y+z} \left[\left(\tilde{\lambda} + \tilde{\xi}\right)(x) \wedge \left(\tilde{\mu} + \tilde{\xi}\right)(x)\right] \\
&= \left[\left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right)\right](x).
\end{aligned}$$

This implies $\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi} \subseteq \left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right)$. For reverse containment,

$$\begin{aligned}
&\left[\left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right)\right](x) = \left[\left(\tilde{\lambda} + \tilde{\xi}\right)\left(\tilde{\mu} + \tilde{\xi}\right)\right](x) \\
&= \vee_{x=\ \Sigma_{i=1}^n y_i z_i} \{\wedge_i [\left(\lambda^- + \xi^-\right)(y_i) \wedge \left(\mu^- + \xi^-\right)(z_i), \\
&\qquad \left(\lambda^+ + \xi^+\right)(y_j) \wedge \left(\mu^+ + \xi^+\right)(z_i)]\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{(\vee_{y_i=r_i+s_i} \left[\lambda^-(r_i) \wedge \xi^-(s_i), \lambda^+(r_i) \wedge \xi^+(s_i)\right]) \\
&\qquad \wedge \left(\vee_{z_i=t_i+\mu_i} \left[\mu^-(t_i) \wedge \xi^-(\mu_i), \mu^+(t_i) \wedge \xi^+(\mu_i)\right]\right)\}] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}} [\lambda^-(r_i) \wedge \xi^-(s_i) \wedge \mu^-(t_i) \wedge \xi^-(\mu_i), \\
&\qquad \lambda^+(r_i) \wedge \xi^+(s_i) \wedge \mu^+(t_i) \wedge \xi^+(\mu_i)]\}] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}} [\lambda^-(r_i) \wedge \xi^-(s_i) \wedge \xi^-(s_i) \wedge \mu^-(t_i) \wedge \xi^-(\mu_i), \\
&\qquad \lambda^+(r_i) \wedge \xi^+(s_i) \wedge \xi^+(s_i) \wedge \mu^+(t_i) \wedge \xi^+(\mu_i)]\}] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{\substack{y_i=r_i+s_i \\ z_i=t_i+\mu_i}} [\lambda^-(r_i t_i) \wedge \mu^-(r_i t_i) \wedge \xi^-(s_i t_i) \wedge \xi^-(s_i u_i) \wedge \xi^-(r_i \mu_i), \\
&\qquad \lambda^+(r_i t_i) \wedge \mu^+(r_i t_i) \wedge \xi^+(s_i t_i) \wedge \xi^+(s_i u_i) \wedge \xi^+(r_i \mu_i)]\}] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{y_i z_i=r_i t_i+(s_i t_i+r_i u_i+s_i u_i)} [\left(\lambda^- \wedge \mu^-\right)(r_i t_i) \wedge \xi^-(s_i t_i + s_i u_i + r_i u_i), \\
&\qquad \left(\lambda^+ \wedge \mu^+\right)(r_i t_i) \wedge \xi^+(s_i t_i + s_i u_i + r_i u_i)]\}] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} [\wedge_i \{\vee_{y_i z_i=r_i t_i+(s_i t_i+r_i u_i+s_i u_i)} [\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(y_i z_i), \\
&\qquad \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(y_i z_i)]\}] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(y_i z_i), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(y_i z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(\Sigma_i y_i z_i), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(\Sigma_i y_i z_i)\right] \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\left(\left(\lambda^- \wedge \mu^-\right) + \xi^-\right)(x), \left(\left(\lambda^+ \wedge \mu^+\right) + \xi^+\right)(x)\right]
\end{aligned}$$

$$= \left(\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi}\right)(x).$$

This shows $\left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right) \subseteq \left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi}$, and so $\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi} = \left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right)$. Hence $\pounds_R$ is a distributive lattice.

(ii) $\Rightarrow$ (i) Let $\pounds_R$ be a distributive lattice under the sum and intersection of interval valued fuzzy ideals of R and let $\tilde{\lambda} \cap \tilde{\mu} = \tilde{\lambda}\tilde{\mu}$ for each pair of interval valued fuzzy ideals $\tilde{\lambda}$ and $\tilde{\mu}$ of R. Then for any interval valued fuzzy ideal $\tilde{\lambda}$ of R, we have

$$\tilde{\lambda}^2 = \tilde{\lambda}\tilde{\lambda} = \tilde{\lambda} \cap \tilde{\lambda} = \tilde{\lambda}.$$

Thus R is fully idempotent.

(i) $\Rightarrow$ (iii) Suppose that R is fully idempotent hemiring. Let $\tilde{\lambda}$ be an interval valued fuzzy ideal of R and $\left\{\tilde{\lambda}_i : i \in I\right\}$ be the family of all interval valued fuzzy prime ideals of R which contain $\tilde{\lambda}$. Then obviously $\tilde{\lambda} \subseteq \cap_{i\in I}\tilde{\lambda}_i$. For reverse containment, let $x \in R$ then by the above lemma, there exists an interval valued fuzzy prime ideal ξ of R such that $\tilde{\lambda} \subseteq \xi$ and $\tilde{\lambda}(x) = \tilde{\xi}(x)$. Then $\tilde{\xi} \in \left\{\tilde{\lambda}_i : i \in I\right\}$. Hence $\cap_{i\in I}\tilde{\lambda}_i \subseteq \tilde{\xi}$. Thus $\cap_{i\in I}\tilde{\lambda}_i(x) \leq \tilde{\xi}(x) = \tilde{\lambda}(x)$. This shows $\cap_{i\in I}\tilde{\lambda}_i \subseteq \tilde{\lambda}$. Thus $\tilde{\lambda} = \cap_{i\in I}\tilde{\lambda}_i$.

(iii) $\Rightarrow$ (i) Let $\tilde{\lambda}$ be an interval valued fuzzy ideal of R then $\tilde{\lambda}^2$ is also an interval valued fuzzy ideal of R and by hypothesis $\tilde{\lambda}^2$ can be written as $\tilde{\lambda}^2 = \cap_{i\in I}\tilde{\lambda}_i$ where $\left\{\tilde{\lambda}_i : i \in I\right\}$ is a family of interval valued fuzzy prime ideals of R which contain $\tilde{\lambda}^2$. Now since $\tilde{\lambda}^2 \subseteq \tilde{\lambda}_i$ for all i and since $\tilde{\lambda}_i$ are interval valued fuzzy prime ideals so $\tilde{\lambda} \subseteq \tilde{\lambda}_i$ for all i. Hence $\tilde{\lambda} \subseteq \cap_{i\in I}\tilde{\lambda}_i = \tilde{\lambda}^2 \subseteq \tilde{\lambda}$. Thus $\tilde{\lambda} = \tilde{\lambda}^2$. Hence R is fully idempotent. □

Theorem 2.22. *Let R be a fully idempotent hemiring. An interval valued fuzzy ideal $\tilde{\xi}$ of R is interval valued fuzzy prime if and only if it is interval valued fuzzy irreducible.*

Proof. Assume that $\tilde{\xi}$ is an interval valued fuzzy prime ideal of R. Then $\tilde{\xi}$ is also interval valued fuzzy irreducible.

Conversely, let $\tilde{\xi}$ be an interval valued fuzzy irreducible ideal of R and $\tilde{\lambda}$, $\tilde{\mu}$ be any two interval valued fuzzy ideals of R such that $\tilde{\lambda}\tilde{\mu} \subseteq \tilde{\xi}$. Then since R is fully idempotent so $\tilde{\lambda}\tilde{\mu} = \tilde{\lambda} \cap \tilde{\mu}$. Thus we have $\tilde{\lambda} \cap \tilde{\mu} \subseteq \tilde{\xi}$. This implies $\left(\tilde{\lambda} \cap \tilde{\mu}\right) + \tilde{\xi} = \tilde{\xi}$. Again since R is fully idempotent so the set of all interval valued fuzzy ideals of R is a distributive lattice under sum and intersection of interval valued fuzzy ideals, hence $\left(\tilde{\lambda} + \tilde{\xi}\right) \cap \left(\tilde{\mu} + \tilde{\xi}\right) = \tilde{\xi}$. Thus $\tilde{\lambda} + \tilde{\xi} = \tilde{\xi}$ or $\tilde{\mu} + \tilde{\xi} = \tilde{\xi}$ since $\tilde{\xi}$ is irreducible. Hence $\tilde{\lambda} \subseteq \tilde{\xi}$ or $\tilde{\mu} \subseteq \tilde{\xi}$. Thus $\tilde{\xi}$ is an interval valued fuzzy prime ideal of R. □

Example 2.23. Let S be a non-empty set. Define a binary operation "$*$"on S by $x * y = y$ for all $x, y \in S$. Then $(S, *)$ is a semigroup. Now let $R = S \cup \{\infty\} \cup \{0\}$ where $\{\infty\}$ is a ring with a single element "∞", and "0" is absorbing zero i.e., $x * 0 = 0 * x = \infty * 0 = 0 * \infty = 0$ for all $x \in S \cup \{0\}$ and $x * \infty = \infty * x = \infty$ for all $x \in S$. Now define another binary operation "$+$" on R as $0 + 0 = 0$ and $x + y = \infty$ for all $x, y \in S$ and $x + \infty = \infty + x = \infty$ for all $x \in R$. Then $(R, +, *)$ is a hemiring.

FACT 1. Every element of R is multiplicatively idempotent.

FACT 2. R is a regular hemiring.

FACT 3. An interval valued fuzzy subset $\tilde{\lambda}$ of R is interval valued fuzzy right ideal of R if and only if

(i) $\tilde{\lambda}(0) \geq \tilde{\lambda}(x)$ for all $x \in R$,
(ii) $\tilde{\lambda}(\infty) \geq \tilde{\lambda}(x)$ for all $x \in S \cup \{\infty\}$,
(iii) $\tilde{\lambda}(x) = \tilde{\lambda}(y)$ for all $x, y \in S$.

Proof. Suppose that (i), (ii) and (iii) hold. If $x, y \in S \cup \{\infty\}$, then $x \neq 0 \neq y$, and so $\tilde{\lambda}(x+y) = \tilde{\lambda}(\infty) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ by (i). If any one of x and y, say $x = 0$ and $y \in S \cup \{\infty\}$, then $x+y = 0+y = y$ and so $\tilde{\lambda}(x+y) = \tilde{\lambda}(y) = \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ because $\tilde{\lambda}(x) = \tilde{\lambda}(0) \geq \tilde{\lambda}(a)$ for all $a \in R$. When $x = y = 0$, we have

$$\tilde{\lambda}(x+y) = \tilde{\lambda}(0+0) = \tilde{\lambda}(0) = \tilde{\lambda}(0) \wedge \tilde{\lambda}(0).$$

Thus in each case, $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ for all $x, y \in R$. If $x, y \in S$, then $\tilde{\lambda}(x * y) = \tilde{\lambda}(y) = \tilde{\lambda}(x)$ by (ii). If $x = 0$ and $y \in R$, then $\tilde{\lambda}(x * y) = \tilde{\lambda}(0 * y) = \tilde{\lambda}(0) = \tilde{\lambda}(x)$ and $\tilde{\lambda}(y * x) = \tilde{\lambda}(y * 0) = \tilde{\lambda}(0) \geq \tilde{\lambda}(y)$. If $x = \infty$ and $y \in R$, then $\tilde{\lambda}(x * y) = \tilde{\lambda}(\infty * y) = \tilde{\lambda}(\infty) = \tilde{\lambda}(x)$ and $\tilde{\lambda}(y * x) = \tilde{\lambda}(y * \infty) = \tilde{\lambda}(\infty) \geq \tilde{\lambda}(y)$ by (i). Thus in any case $\tilde{\lambda}(x * y) \geq \tilde{\lambda}(x)$ for all $x, y \in R$. Hence $\tilde{\lambda}$ is an interval valued fuzzy right ideal of R.

Conversely, suppose that $\tilde{\lambda}$ is an interval valued fuzzy right ideal of R and let $x \in S \cup \{\infty\}$ then $\tilde{\lambda}(\infty) = \tilde{\lambda}(x * \infty) \geq \tilde{\lambda}(x)$. Also for any $x, y \in S$

$$\begin{aligned}
\tilde{\lambda}(x) &= \tilde{\lambda}(y * x) \geq \tilde{\lambda}(y), \\
\tilde{\lambda}(y) &= \tilde{\lambda}(x * y) \geq \tilde{\lambda}(x), \\
\tilde{\lambda}(0) &= \tilde{\lambda}(x * 0) \geq \tilde{\lambda}(x).
\end{aligned}$$

Thus $\tilde{\lambda}(x) = \tilde{\lambda}(y)$. □

FACT 4. The (crisp) right ideals of R are $\{0\}, \{0, \infty\}$ and R itself which are all idempotent.

FACT 5. All interval valued fuzzy right ideals of R are idempotent.

Proof. Let $\tilde{\lambda} : R \rightarrow \pounds$ be an interval valued fuzzy right ideal of $R = S \cup \{0, \infty\}$. Then

$$\begin{aligned}
\tilde{\lambda}^2(0) &= \vee_{0=\Sigma_{i=1}^{n} y_i z_i} \left[\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right] \\
&\geq \left[\lambda^-(0) \wedge \lambda^-(0), \lambda^+(0) \wedge \lambda^+(0)\right] \\
&= \tilde{\lambda}(0) \geq \tilde{\lambda}^2(0).
\end{aligned}$$

Thus $\tilde{\lambda}^2(0) = \tilde{\lambda}(0)$. Now for $x \in S$, no expression of the form $x = \sum_{i=1}^{n} y_i z_i$ involves only 0 and ∞. Thus $\wedge_i [\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)] \neq \lambda(0)$ and $\wedge_i [\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)] \neq \lambda(\infty)$ Note that $x = x \cdot x$ is among the possible expressions of x and since $\tilde{\lambda}(x) = \tilde{\lambda}(y)$ for all $x, y \in S$. So

$$\begin{aligned}
\tilde{\lambda}^2(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{ \wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \right\} \\
&\geq \vee_{x=x.x} \left[\lambda^-(x) \wedge \lambda^-(x), \lambda^+(x) \wedge \lambda^+(x) \right] \\
&= \tilde{\lambda}(x) \geq \tilde{\lambda}^2(x)
\end{aligned}$$

for all $x \in S$. Hence $\tilde{\lambda}^2(x) = \tilde{\lambda}(x)$ for all $x \in S$. Now we calculate $\tilde{\lambda}^2(\infty)$. Clearly, no expression for ∞ contains only 0 and one expression for ∞ is $\infty = \infty \cdot \infty$. Thus for $\infty = \sum_{i=1}^{n} y_i z_i$

$$\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] \neq \tilde{\lambda}(0)$$

and

$$\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i) \right] = \tilde{\lambda}(\infty).$$

Thus $\vee_{0=\Sigma_{i=1}^{n} y_i z_i} \{ \wedge_i [\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)] \} = \tilde{\lambda}(\infty)$. Hence $\tilde{\lambda}^2(\infty) = \tilde{\lambda}(\infty)$. Thus the interval valued fuzzy right ideal $\tilde{\lambda}$ is idempotent. □

Example 2.24. Consider the hemiring $R = \{0, a, b, c, d\}$ defined by the following operations

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	c	a	b	a
b	b	a	b	c	b
c	c	b	c	a	c
d	d	a	b	c	d

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	b	b	b	d
c	0	c	b	a	d
d	0	d	d	d	0

FACT 1. The (crisp) ideals of R are $\{0\}, \{0, d\}, \{0, b, d\}$ and R itself. Now

$$\{0\}^2 = \{0\}, \ \{0, d\}^2 = \{0\} \neq \{0, d\}, \ \{0, b, d\}^2 = \{0, b, d\}.$$

Thus the ideal $\{0, d\}$ is not idempotent.

FACT 2. If we define two interval valued fuzzy subsets $\tilde{\mu}$ and $\tilde{\lambda}$ of R as

$$\tilde{\mu}(x) = \begin{cases} \tilde{O} & \text{if } x = a, b, c, \\ \tilde{I} & \text{if } x = 0, d, \end{cases} \qquad \tilde{\lambda}(x) = \begin{cases} \tilde{O} & \text{if } x = a, c, \\ \tilde{I} & \text{if } x = 0, b, d, \end{cases}$$

Then both $\tilde{\mu}$ and $\tilde{\lambda}$ are interval valued fuzzy ideals of R but $\tilde{\lambda}$ is idempotent while $\tilde{\mu}$ is not as $\tilde{\mu}^2(d) = [0, 0] = \tilde{O} \neq \tilde{\mu}(d)$.

Example 2.25. Consider the set $R = \{0, x\}$ with binary operations defined as

$+$	0	x
0	0	x
x	x	x

$\cdot$	0	x
0	0	0
x	0	x

Then R is clearly a hemiring with an absorbing element '0'. Its only proper (crisp) ideal is zero ideal $\{0\}$. Since each ideal $\{0\}$and R are idempotent, so R is fully idempotent hemiring. Since R is also commutative so it is von Neumann regular.

Then the lattice of all interval valued fuzzy ideals of R (ordered by $\subseteq$) is distributive under the sum and intersection of interval valued fuzzy ideals.

Definition 2.26. An interval valued fuzzy ideal $\tilde{\lambda}$ of R is called normal if $\tilde{\lambda}(0) = [1, 1]$.

Let R be a fully idempotent hemiring, $\pounds_R$ is the lattice of all normal interval valued fuzzy ideals of R and $\pounds_P$ is the set of all proper normal interval valued fuzzy prime ideals of R. For any interval valued fuzzy ideal $\tilde{\lambda}$ of R, we define

$$\theta_{\tilde{\lambda}} = \left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda} \nsubseteq \tilde{\mu}\right\} \text{ and } \Im = \left\{\theta_{\tilde{\lambda}} : \tilde{\lambda} \in \pounds_R\right\}.$$

Theorem 2.27. *The set $\Im$ forms a topology on the set $\pounds_P$. The assignment $\tilde{\lambda} \to \theta_{\tilde{\lambda}}$ is an isomorphism between the lattice $\pounds_R$ and the lattice of open subsets of $\pounds_P$.*

Proof. First we show that $\Im$ forms a topology on the set $\pounds_P$. Let $\tilde{\psi}$ be the interval valued fuzzy ideal of R defined by

$$\tilde{\psi}(x) = \begin{cases} \tilde{O} & \text{if } x \neq 0, \\ \tilde{I} & \text{if } x = 0. \end{cases}$$

Then $\theta_{\tilde{\psi}} = \left\{\tilde{\mu} \in \pounds_P : \tilde{\psi} \nsubseteq \tilde{\mu}\right\} = \varphi$. If $\tilde{C}_R$ is the interval valued characteristic function of R then

$$\tilde{C}_R\left(x\right) = [1, 1] = \tilde{I}, \ \forall x \in R$$

and by definition of $\pounds_P$, we have $\tilde{\mu} \subset \tilde{C}_R \ \forall \tilde{\mu} \in \pounds_P$. Thus

$$\theta_{\tilde{C}_R} = \left\{\tilde{\mu} \in \pounds_P : \tilde{C}_R \nsubseteq \tilde{\mu}\right\} = \pounds_P$$

and hence $\theta_{\tilde{C}_R} = \pounds_P$ is an element of $\Im$.

Now let $\theta_{\tilde{\lambda}_1}, \theta_{\tilde{\lambda}_2} \in \Im$ with $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \pounds_R$. Then

$$\theta_{\tilde{\lambda}_1} \cap \theta_{\tilde{\lambda}_2} = \left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda}_1 \nsubseteq \tilde{\mu} \text{ and } \tilde{\lambda}_2 \nsubseteq \tilde{\mu}\right\}.$$

Since R is fully idempotent hemiring so $\tilde{\lambda}_1 \cap \tilde{\lambda}_2 = \tilde{\lambda}_1\tilde{\lambda}_2$. Now if $\tilde{\lambda}_1 \cap \tilde{\lambda}_2 \subseteq \tilde{\mu}$ then $\tilde{\lambda}_1\tilde{\lambda}_2 \subseteq \tilde{\mu}$. But $\tilde{\mu}$ is an interval valued fuzzy prime ideal of R so $\tilde{\lambda}_1 \subseteq \tilde{\mu}$ or $\tilde{\lambda}_2 \subseteq \tilde{\mu}$, which is a contradiction. Therefore $\tilde{\lambda}_1 \cap \tilde{\lambda}_2 \nsubseteq \tilde{\mu}$.

Conversely, if $\tilde{\lambda}_1 \cap \tilde{\lambda}_2 \nsubseteq \tilde{\mu}$ then $\tilde{\lambda}_1 \nsubseteq \tilde{\mu}$ and $\tilde{\lambda}_2 \nsubseteq \tilde{\mu}$. Thus

$$\begin{aligned} \theta_{\tilde{\lambda}_1} \cap \theta_{\tilde{\lambda}_2} &= \left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda}_1 \nsubseteq \tilde{\mu} \text{ and } \tilde{\lambda}_2 \nsubseteq \tilde{\mu}\right\} \\ &= \left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda}_1 \cap \tilde{\lambda}_2 \nsubseteq \tilde{\mu}\right\} \\ &= \theta_{\tilde{\lambda}_1 \cap \tilde{\lambda}_2}. \end{aligned}$$

And surely $\tilde{\lambda}_1 \cap \tilde{\lambda}_2 \in \pounds_R$. Thus $\theta_{\tilde{\lambda}_1} \cap \theta_{\tilde{\lambda}_2} \in \Im$ for all $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \pounds_R$. Consider a family $\{\theta_{\tilde{\lambda}_i}\}_{i \in I}$ of elements of $\Im$. Then

$$\begin{aligned} \cup_{i \in I}\theta_{\tilde{\lambda}_i} &= \cup_{i \in I}\left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda}_i \nsubseteq \tilde{\mu}\right\} \\ &= \left\{\tilde{\mu} \in \pounds_P : \tilde{\lambda}_k \nsubseteq \tilde{\mu} \text{ for some } k \in I\right\} \\ &= \theta_{\Sigma_i\tilde{\lambda}_i} \in \Im \qquad \left(\because \Sigma_i\tilde{\lambda}_i \in \pounds_R\right). \end{aligned}$$

Thus $\Im$ is a topology on $\pounds_P$. Define a map $\Psi : \pounds_R \to \Im$ by

$$\Psi\left(\tilde{\lambda}\right) = \theta_{\tilde{\lambda}} \quad \forall \tilde{\lambda} \in \pounds_R$$

Then by definition of $\theta_{\tilde{\lambda}}$, we have

$$\tilde{\lambda}_1 = \tilde{\lambda}_2 \Rightarrow \theta_{\tilde{\lambda}_1} = \theta_{\tilde{\lambda}_2} \quad \forall \tilde{\lambda}_1, \tilde{\lambda}_2 \in \pounds_R.$$

And from (ii) and (iii) above, Ψ preserves the finite intersection and arbitrary union. Thus Ψ is a lattice homomorphism. Now for isomorphism, we will show that Ψ is a bijection. Ψ is clearly onto, and for $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \pounds_R$ let

$$\Psi\left(\tilde{\lambda}_1\right) = \Psi\left(\tilde{\lambda}_2\right).$$

Then $\theta_{\tilde{\lambda}_1} = \theta_{\tilde{\lambda}_2}$, and so $\tilde{\lambda}_1 = \tilde{\lambda}_2$ because if $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ then there exists $x \in R$ such that $\tilde{\lambda}_1(x) \neq \tilde{\lambda}_2(x)$. Therefore anyone, say, $\tilde{\lambda}_1(x)$ is greater than $\tilde{\lambda}_2(x)$. Then for $\tilde{\lambda}_2$, there exists an interval valued fuzzy prime ideal $\tilde{\mu}$ of R such that $\tilde{\lambda}_2(x) = \tilde{\mu}(x)$. Then $\tilde{\lambda}_1 \nleq \tilde{\mu}$ because $\tilde{\lambda}_1(x) > \tilde{\lambda}_2(x) = \tilde{\mu}(x)$. This implies $\tilde{\mu} \in \theta_{\tilde{\lambda}_1} = \theta_{\tilde{\lambda}_2}$, and thus $\tilde{\mu} \in \theta_{\tilde{\lambda}_2}$. Hence $\tilde{\lambda}_2 \nleq \tilde{\mu}$, which is a contradiction. Thus Ψ is an isomorphism. $\square$

References

[1] J. Ahsan, K. Saifullah and M. F. Khan, Fuzzy Semirings, Fuzzy Sets and Systems 60 (1993) 309–320.

[2] L. B. Beasley and N. G. Pullman, Operators that preserves semiring matrix functions, Linear Algebra Appl. 99 (1988) 199–216.

[3] L. B. Beasley and N. G. Pullman, Linear operators strongly preserving idempotent matrices over semirings, Linear Algebra Appl. 160 (1992) 217–229.

[4] R. Biswas, Rosenfeld's fuzzy subgroups with intervalued membership function, Fuzzy Sets and Systems 63 (1994) 87–90.

[5] S. Ghosh, Matrices over semirings, Inform. Sci. 90 (1996) 221–230.

[6] K. Glazek, A guide to litrature on semirings and their applications in mathematics and information sciences: with complete bibliography, Kluwer Acad. Publ. Nederland, 2002.

[7] J. S. Golan, Semirings and their applications, Kluwer Acad. Publ. 1999.

[8] U. Hebisch and H. J. Weinert, Semirings: Algebraic Theory and Applications in the Computer Science, World Scientific, 1998.

[9] X. P. Li and G. J. Wang, The S_H interval valued fuzzy subgroups, Fuzzy Sets and Systems 112 (2000) 319–325.

[10] X. Ma and J. Zhan, On fuzzy h-ideals of hemirings, J. Syst. Sci. Complex. 20 (2007) 470–478.

[11] J. N. Mordeson and D. S. Malik, Fuzzy Automata and Languages, Theory and Applications, Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton 2002.

[12] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.

[13] H. S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc. 40 (1934) 914–920.

[14] W. Wechler, The concept of fuzziness in automata and language theory, Akademic verlog, Berlin, 1978.

[15] Y. Yin and H. Li, The Characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform. Sci. 178 (2008) 3451–3464.

[16] L. A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965) 338–353.

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