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T-syntopogenous spaces

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Abstract. In this paper, we introduce the concepts of T-syntopogenous spaces and investigate some of their properties, where T stands for any continuous triangular norm. Their definitions subsumes that of fuzzy syntopogenous spaces due to A. K. Katsaras (Fuzzy Sets and Systems 36 (1990)), as our Min-syntopogenous spaces. In particular, we study the continuity of functions between T-syntopogenous spaces and the I-topological space associated with a T-syntopogenous space. Moreover, we describe the T-syntopogenous structures as fuzzy relations in (ordinary) power sets.

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1. Introduction

Katsaras and Petalas [5, 6, 7] introduced the fuzzy syntopogenous structures and studied the unified theory of Chang I-topologies [2] and Lowen Fuzzy uniformities [10]. In this manuscript, we introduce, for each continuous triangular norm T, a new structure of T-syntopogenous spaces that conforms well with Lowen I-topological spaces [9]. Our concept of T-syntopogenous structure generalizes, to arbitrary continuous triangular norm T, the fuzzy syntopogenous structure of A. K. Katsaras [7], now becoming the special case corresponding to $T = \text{Min}$. Also we deduce the notion of syntopogenous maps and here we show that the class of all T-syntopogenous spaces together with syntopogenous maps as arrows forms a concrete category. The basic idea is to introduce a degree of divergence between fuzzy subsets, which is a real number in the unit interval $I = [0, 1]$.

We proceed as follows: In Section 2, we state and supply some basic ideas and lemmas on the T-residuated implication and on the α-cuts of fuzzy subsets, which will be needed in the sequel. In the third section, we introduce our definition of T-topogenous order, and hence T-syntopogenous structure on a set, also we define the I-topology associated with a T-syntopogenous structure. We deduce the notions of image and inverse image of T-topogenous orders. Moreover, we give the examples of T-topogenous orders and T-syntopogenous spaces together with the I-topology generated by them. In Section 4, we deduce the notion of syntopogenous maps (syntopogenously continuous functions) and we define a functor from category of T-syntopogenous spaces into category of I-topological spaces. The fifth section characterizes both T-topogenous orders and syntopogenous maps, uniquely, in terms of fuzzy binary relations in power sets.

2. Preliminaries

A triangular norm T (cf. [12]) is a binary operation on the unit interval $I = [0, 1]$ that is associative, symmetric, monotone in each argument and has neutral element 1. The basic two (continuous) triangular norms are their simplest, namely Min (also denoted by $\wedge$) and product.

The triangular conorm of a triangular norm T is the binary operation T^* on the unit interval I given by : $\alpha T^* \beta = 1 - [(1-\alpha)T(1-\beta)]$, $\alpha,\ \beta \in I$. A continuous triangular norm T is uniformly continuous, that is for every $\epsilon > 0$ there is $\theta = \theta_{T,\epsilon}$ such that for every $(\alpha, \beta) \in I \times I$, we have

$$(\alpha T \beta) - \epsilon \leq (\alpha - \theta)T(\beta - \theta) \leq \alpha T \beta \leq (\alpha + \theta)T(\beta + \theta) \leq (\alpha T \beta) + \epsilon. \tag{2.1}$$

For a continuous triangular norm T, the following binary operation on I

$$\mathscr{I}(\alpha, \gamma) = \sup\{\epsilon \in I : \alpha T \epsilon \leq \gamma\}, \quad \alpha,\ \gamma \in I,$$

is called the residuation implication of T [11]. For this implication, we shall use the following properties, $\forall\ \alpha, \epsilon, \gamma \in I$:
By continuity of T,

$$\alpha\ T\ \mathscr{I}(\alpha, \epsilon) = \alpha \wedge \epsilon. \tag{2.2}$$

By the definition of $\mathscr{I}$, we have :

$$\alpha \leq \epsilon \text{ iff } \mathscr{I}(\alpha, \epsilon) = 1, \text{ and } \alpha T \gamma \leq \epsilon \text{ iff } \alpha \leq \mathscr{I}(\gamma, \epsilon). \tag{2.3}$$

A fuzzy set λ in a universe set X, introduced by Zadeh in [13], is a function $\lambda : X \to I = [0, 1]$. The height of a fuzzy set $\lambda \in I^X$ is the following real number :

$$\text{hgt } \lambda = \sup\{\lambda(x) : x \in X\}.$$

We shall often need to consider a subset $H \subseteq X$ as a fuzzy subset of X, said to be a crisp fuzzy subset of X, which we shall denote by the symbol $\mathbf{1}_H$. We do this by identifying $\mathbf{1}_H$ with its characteristic function. We also denote the constant fuzzy set of X with value $\alpha \in I$ by $\underline{\underline{\alpha}}$.

Given a fuzzy set $\lambda \in I^X$ and a real number $\alpha \in I_1 = [0, 1[$, the strong α-cut of λ is the following subset of X :

$$\lambda^\alpha = \{x \in X : \lambda(x) > \alpha\},$$

and for a real number $\alpha \in I$, the weak α-cut of λ is the subset of X:

$$\lambda_{\alpha^*} = \{x \in X : \lambda(x) \geq \alpha\}.$$

It is directly verified that every $\lambda \in I^X$ has the following formulations

$$\lambda = \bigvee_{\alpha \in I} [\underline{\underline{\alpha}} \wedge \mathbf{1}_{\lambda_{\alpha^*}}] = \bigvee_{\alpha \in I} [\underline{\underline{\alpha}} T \mathbf{1}_{\lambda_{\alpha^*}}]. \tag{2.4}$$

$$\lambda = \bigwedge_{\alpha \in I} \left[\underline{\underline{\alpha}} \vee \mathbf{1}_{\lambda_{\alpha^*}}\right] = \bigwedge_{\alpha \in I} \left[\underline{\underline{\alpha}} T^* \mathbf{1}_{\lambda_{\alpha^*}}\right]. \tag{2.5}$$

Given two fuzzy sets $\mu,\, \lambda \in I^X$, we denote by $\mu T \lambda$ the fuzzy subset of X given by : $(\mu T \lambda)(x) = \mu(x) T \lambda(x)$, $x \in X$. Hence for all $\epsilon > 0$ and the above $\theta = \theta_{T,\epsilon}$, we find that for all $\mu,\, \lambda,\, \nu,\, \rho \in I^X$:

$$\|\mu - \nu\| \vee \|\lambda - \rho\| \leq \theta \;\Rightarrow\; \|\mu T \lambda - \nu T \rho\| \leq \epsilon,$$

where $\| - \|$ denotes the L_∞-distance on I^X, given by

$$\|\mu - \nu\| = \sup\{|\mu(x) - \nu(x)| : x \in X\},\ \mu, \nu \in I^X,$$

that is, the function $T : I^X \times I^X \to I^X$ is also uniformly continuous with respect to the L_∞-distance on I^X.

Lemma 2.1 ([4])**.** *For every $\mu,\, \lambda \in I^X$ and $\epsilon \in I$, we have*

(i) $(\mu T \lambda)_{\epsilon^*} = \bigcup_{\theta T \gamma \geq \epsilon} (\mu_{\theta^*} \cap \lambda_{\gamma^*})$;
(ii) $(\mu T \underline{\underline{\alpha}})_{\epsilon^*} = \bigcup_{\theta T \gamma \geq \epsilon} (\mu_{\theta^*})$;
(iii) $\mathrm{hgt}\ (\lambda T \mathbf{1}_H) \leq \epsilon \;\Rightarrow\; \lambda^\epsilon \cap H = \varnothing$;
(iv) $\mathrm{hgt}\ f(\mu) = \mathrm{hgt}\ \mu$ *and* $f(\mu T \lambda) \leq f(\mu) T f(\lambda)$;
(v) *For all* $\alpha \in [0, \epsilon]$, *let* $\gamma_{\alpha,\epsilon} = \inf\{\theta \in I : \theta T \epsilon \geq \alpha\}$. *Then* $\gamma_{\alpha,\epsilon} T \epsilon = \alpha$ *and consequently,* $(\mu T \underline{\underline{\epsilon}})_{\alpha^*} = \mu_{\gamma^*_{\alpha,\epsilon}}$.

Lemma 2.2. *For every $\mu,\, \lambda \in I^X$ and $\alpha,\, \epsilon \in I$, we have*

(i) $(\mu T \lambda)^\epsilon = \bigcup_{\theta T \gamma \geq \epsilon} \left(\mu^\theta \cap \lambda^\gamma\right)$;
(ii) $\left((\underline{\underline{1 - \alpha}}) T^* \lambda\right)_{(1-\epsilon)^*} = \lambda_{(1 - \mathscr{I}(\alpha,\epsilon))^*}$.

Proof. (i) Let $\mu,\, \lambda \in I^X$ and $\epsilon \in I$. Then for every $x \in X$, we get the equivalences:

$$\begin{aligned} x \in (\mu\ T \lambda)^\epsilon \text{ iff }\ & \mu(x) T \lambda(x) > \epsilon \\ \text{iff }\ & \exists\ \theta, \gamma \in I \text{ such that } \mu(x) > \theta \text{ and } \lambda(x) > \gamma \text{ with } \theta T \gamma \geq \epsilon \\ \text{iff }\ & \exists\ \theta, \gamma \in I \text{ such that } x \in \mu^\theta \cap \lambda^\gamma \text{ with } \theta T \gamma \geq \epsilon \\ \text{iff }\ & x \in \bigcup_{\theta T \gamma \geq \epsilon} \left(\mu^\theta \cap \lambda^\gamma\right). \end{aligned}$$

This proves (i).

(ii) Let $\lambda \in I^X$ and α, $\epsilon \in I$. Then

$$\begin{aligned}\left((\underline{\underline{1-\alpha}})T^*\lambda\right)_{(1-\epsilon)^*} &= \{x \in X : (1-\alpha)T^*\lambda(x) \geq 1-\epsilon\} \\ &= \{x \in X : 1-[\alpha T(\underline{\underline{1}}-\lambda)(x)] \geq 1-\epsilon\} \\ &= \{x \in X : \alpha T(\underline{\underline{1}}-\lambda)(x) \leq \epsilon\} \\ &= \{x \in X : (\underline{\underline{1}}-\lambda)(x) \leq \mathscr{J}(\alpha,\epsilon)\}, \quad \text{by (2.3)} \\ &= \{x \in X : \lambda(x) \geq 1-\mathscr{J}(\alpha,\epsilon)\} \\ &= \lambda_{(1-\mathscr{J}(\alpha,\epsilon))^*}.\end{aligned}$$

Rendering (ii). □

Lemma 2.3 ([4])**.** *Suppose γ, α, α', $\epsilon \in I$ are such that $\alpha \leq \gamma$ and $\epsilon T\alpha' < \epsilon T\alpha$. Then $\mathscr{J}(\gamma, \epsilon T\alpha') < \epsilon T\mathscr{J}(\gamma,\alpha)$.*

3. T-syntopogenous spaces

T-topogenous orders and T-syntopogenous spaces are introduced in this section, and some of their properties are given. We generate an I-topological space from a T-syntopogenous space. We introduce the discrete and indiscrete T-topogenous orders, as special examples. Also, the image and inverse image of T-topogenous orders are define.

Definition 3.1. A T-topogenous order on a set X is a function $\zeta : I^X \times I^X \to I$, that satisfies, for any μ, λ, $\nu \in I^X$ and $\alpha \in I$, the following:

(TT1) $\zeta(\underline{\underline{1}}, \underline{\underline{\alpha}}) = \alpha$ and $\zeta(\underline{\underline{\alpha}}, \underline{\underline{0}}) = 1-\alpha$;
(TT2) $\zeta(\mu \vee \lambda, \nu) = \zeta(\mu,\nu) \wedge \zeta(\lambda,\nu)$ and $\zeta(\mu, \lambda \wedge \nu) = \zeta(\mu,\lambda) \wedge \zeta(\mu,\nu)$;
(TT3) If $\zeta(\mu,\lambda) > 1-(\theta\ T\ \beta)$ for some θ, $\beta \in I_0 =]0,1]$, there is $C \subseteq X$ such that $\zeta(\mu, \mathbf{1}_C) \geq 1-\theta$ and $\zeta(\mathbf{1}_C, \lambda) \geq 1-\beta$;
(TT4) $\zeta(\mu,\lambda) \leq 1-\text{hgt}[\mu T(\underline{\underline{1}}-\lambda)]$;
(TT5) $\zeta(\underline{\underline{\alpha}} T\mu, \lambda) = (1-\alpha)T^*\zeta(\mu,\lambda) = \zeta(\mu, (\underline{\underline{1-\alpha}})T^*\lambda)$.

The real number $\zeta(\mu,\lambda)$ can be interpreted as the degree of farness (divergence) of the fuzzy sets μ and $(\underline{\underline{1}}-\lambda)$.

Definition 3.2 ([7])**.** A T-topogenous order ζ on a set X is said to be :

(i) perfect if $\zeta\left(\bigvee_{j\in J}\mu_j, \lambda\right) = \bigwedge_{j\in J}\zeta(\mu_j,\lambda)$, $\quad \mu_j$, $\lambda \in I^X$;
(ii) biperfect if it is perfect and $\zeta\left(\mu, \bigwedge_{j\in J}\lambda_j\right) = \bigwedge_{j\in J}\zeta(\mu,\lambda_j)$, $\quad \mu$, $\lambda_j \in I^X$;
(iii) symmetrical if $\zeta(\mu,\lambda) = \zeta(\underline{\underline{1}}-\lambda, \underline{\underline{1}}-\mu)$, $\quad \mu$, $\lambda \in I^X$.

Definition 3.3. For T-topogenous orders ζ, η on X, we define the T-composition of ζ and η by:

$$(\zeta \circ_T \eta)(\mu,\lambda) = \sup_{C\subseteq X}[\eta(\mu,\mathbf{1}_C)T\zeta(\mathbf{1}_C,\lambda)], \quad \mu,\lambda \in I^X.$$

Definition 3.4. (i) A T-syntopogenous structure on a set X is a family $\mathscr{P}$ of T-topogenous orders on X satisfying the following conditions

(TS1) $\mathscr{P}$ is directed in the sense that, given ζ, $\eta \in \mathscr{P}$ there is $\xi \in \mathscr{P}$ such that

$$\xi \geq \zeta \vee \eta;$$

(TS2) for every $\zeta \in \mathscr{P}$ and $\epsilon \in I_0$, there is $\zeta_\epsilon \in \mathscr{P}$ such that

$$(\zeta_\epsilon \circ_T \zeta_\epsilon) + \underline{\underline{\epsilon}} \geq \zeta.$$

(ii) A T-syntopogenous structure $\mathscr{P}$ is called perfect (resp. biperfect, resp. symmetrical) if every member of $\mathscr{P}$ is perfect (resp. biperfect, resp. symmetrical). The pair $(X, \mathscr{P})$ is said to be a T-syntopogenous space.

First, we see how a T-syntopogenous structure can generate an I-topology. Take a T-syntopogenous space $(X, \mathscr{P})$, and for any $\mu \in I^X$ and $x \in X$, we define a map ${}^o : I^X \to I^X$ by:

$$\mu^o(x) = \sup_{\zeta \in \mathscr{P}} \zeta(\mathbf{1}_x, \mu). \tag{3.1}$$

The following lemma and two propositions lead to the proof of Theorem 3.8, below, which states that the mapping $\mu \to \mu^o$ is a fuzzy interior operator.

Given a real number $0 < \gamma < 1$, we denote by $n(\gamma)$ the unique positive integer n satisfying $n(\gamma) < 1 \leq (n+1)\gamma$.

Lemma 3.5 (El-Rayes and Morsi [3])**.** *For every $\mu \in I^X$ and $0 < \gamma < 1$,*

$$\mu \leq \bigvee_{j=0}^{n(\gamma)} \left[\underline{\underline{j\gamma}} T \mathbf{1}_{(\mu^{j\gamma})}\right] \leq \mu + \underline{\underline{\gamma}}.$$

We use this lemma to establish :

Proposition 3.6. *If $\zeta : I^X \times I^X \to I$ satisfies* (TT2)*, then the following are equivalent statements:*

(i) $\zeta(\underline{\underline{\alpha}} T \mu, \lambda) = (1-\alpha) T^* \zeta(\mu, \lambda) = \zeta(\mu, (\underline{\underline{1-\alpha}}) T^* \lambda),\ \forall\, \alpha \in I,\ \mu, \lambda \in I^X$;

(ii) $\zeta(\mu, \lambda) = \bigwedge_{\theta \in I} \left[\theta T^* \zeta(\mathbf{1}_{\mu_{(1-\theta)^*}}, \lambda)\right]$
$\qquad = \bigwedge_{\beta \in I} \left[\beta T^* \zeta(\mu, \mathbf{1}_{\lambda_{\beta^*}}\right], \qquad \forall\, \mu, \lambda \in I^X$;

(iii) $\zeta(\mu, \lambda) = \bigwedge_{\theta, \beta \in I} \left[\theta T^* \beta T^* \zeta(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{\lambda_{\beta^*}}\right], \qquad \forall\, \mu, \lambda \in I^X$.

Proof. (i) $\Rightarrow$ (ii) : Suppose that (i) holds. Then for all μ, $\lambda \in I^X$,

$$\begin{aligned} \zeta(\mu, \lambda) &\leq \zeta\left((\underline{\underline{1-\theta}}) T \mathbf{1}_{\mu_{(1-\theta)^*}}, \lambda\right), \quad \forall\, \theta \in I, \text{ by (2.4) and (TT2)} \\ &= \theta T^* \zeta\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \lambda\right), \qquad \forall\, \theta \in I, \text{ by (i)} \end{aligned}$$

hence, $\zeta(\mu, \lambda) \leq \bigwedge_{\theta \in I} \left[\theta T^* \zeta(\mathbf{1}_{\mu_{(1-\theta)^*}}, \lambda\right]$.

For the converse inequality, since

$$\mu \leq \bigvee_{j=0}^{n(\gamma)} \left[\underline{\underline{j\gamma}} T \mathbf{1}_{(\mu^{j\gamma})}\right] \leq \bigvee_{j=0}^{n(\gamma)} \left[\underline{\underline{j\gamma}} T \mathbf{1}_{\mu_{j\gamma^*}}\right],$$

we have

$$\begin{aligned}\zeta(\mu,\lambda) &\geq \zeta\left(\bigvee_{j=0}^{n(\gamma)}\left[\underline{\underline{j\gamma}}T\mathbf{1}_{\mu_{j\gamma^*}}\right],\lambda\right)\\ &= \bigwedge_{j=0}^{n(\gamma)}\left[(1-j\gamma)T^*\zeta\left(\mathbf{1}_{\mu_{j\gamma^*}},\lambda\right)\right], \qquad \text{by (TT2) and (i)}\\ &\geq \bigwedge_{\theta\in I}\left[\theta T^*\zeta\left(\mathbf{1}_{\mu_{(1-\theta)^*}},\lambda\right)\right]\end{aligned}$$

Hence, we get the conclusion, which proves the first equality of (ii).

For the second equality, we have as above, by (2.5),

$$\zeta(\mu,\lambda)\leq\bigwedge_{\beta\in I}\left[\beta T^*\zeta\left(\mu,\mathbf{1}_{\lambda_{\beta^*}}\right)\right].$$

For the converse inequality, putting $\lambda=\underline{\underline{1}}-\nu$, for some $\nu\in I^X$, we have

$$\begin{aligned}\zeta(\mu,\lambda) &= \zeta(\mu,\underline{\underline{1}}-\nu)\\ &\geq \zeta\left(\mu,\underline{\underline{1}}-\left[\bigvee_{j=0}^{n(\gamma)}\left(\underline{\underline{j\gamma}}T\mathbf{1}_{(\nu^{j\gamma})}\right)\right]\right), \qquad \text{by (TT2)}\\ &= \zeta\left(\mu,\bigwedge_{j=0}^{n(\gamma)}\left[(\underline{\underline{1-j\gamma}})T^*(\underline{\underline{1}}-\mathbf{1}_{(\nu^{j\gamma})})\right]\right)\\ &= \bigwedge_{j=0}^{n(\gamma)}\left[(1-j\gamma)T^*\zeta(\mu,\mathbf{1}_{(X-\nu^{j\gamma})})\right], \qquad \text{by (TT2) and (i)}\\ &= \bigwedge_{j=0}^{n(\gamma)}\left[(1-j\gamma)T^*\zeta(\mu,\mathbf{1}_{(\underline{\underline{1}}-\nu)_{(1-j\gamma)^*}})\right], \qquad \text{clear}\\ &\geq \bigwedge_{\beta\in I}\left[\beta T^*\zeta(\mu,\mathbf{1}_{\lambda_{\beta^*}})\right].\end{aligned}$$

Hence, we get the conclusion, which proves the second equality of (ii).

(ii)$\Rightarrow$(iii): Direct.

(iii)$\Rightarrow$(i): Suppose (iii) holds. Then for all μ, $\lambda\in I^X$ and all $\alpha\in I$,

$$\begin{aligned}\zeta(\underline{\underline{\alpha}}T\mu,\lambda) &= \bigwedge_{\theta,\beta\in I}\left[\theta T^*\beta T^*\zeta(\mathbf{1}_{(\underline{\underline{\alpha}}T\mu)_{(1-\theta)^*}},\mathbf{1}_{\lambda_{\beta^*}})\right], \qquad \text{by (iii)}\\ &= \bigwedge_{\substack{\beta\in I\\ \theta\in[1-\alpha,1]}}\left[\theta T^*\beta T^*\zeta\left(\mathbf{1}_{\mu_{(\gamma_{(1-\theta),\alpha})^*}},\mathbf{1}_{\lambda_{\beta^*}}\right)\right], \qquad \text{by Lemma 2.1 (v)}\\ &= \bigwedge_{\epsilon,\beta\in I}\left[(1-\alpha)T^*\epsilon T^*\beta T^*\zeta(\mathbf{1}_{\mu_{(1-\epsilon)^*}},\mathbf{1}_{\lambda_{\beta^*}}\right],\end{aligned}$$

because $\theta \geq 1-\alpha$ if and only if $\theta = (1-\alpha)T^*\epsilon$ for some $\epsilon \in I$, equivalently $1-\theta = \alpha T(1-\epsilon)$ (due to the continuity of T), and such $(1-\epsilon)$ is greater or equal to $\gamma_{(1-\theta),\alpha}$, hence $\mu_{(1-\epsilon)^*} \subseteq \mu_{(\gamma_{(1-\theta),\alpha})^*}$. Therefore,

$$\begin{aligned}\zeta(\underline{\underline{\alpha}}T\mu,\lambda) &= (1-\alpha)T^* \wedge_{\epsilon,\beta\in I}\left[\epsilon T^*\beta T^*\zeta\left(\mathbf{1}_{\mu_{(1-\epsilon)^*}},\mathbf{1}_{\lambda_{\beta^*}}\right)\right]\\ &= (1-\alpha)T^*\zeta(\mu,\lambda), \qquad \text{by (iii)}.\end{aligned}$$

Similarly, by using Lemma 2.2 (ii), we can show

$$\zeta(\mu,(\underline{\underline{1-\alpha}})T^*\lambda) = (1-\alpha)T^*\zeta(\mu,\lambda),$$

which winds up the proof. □

Proposition 3.7. *Let ζ be a T-topogenous order on a set X. Then for all $\mu,\lambda \in I^X$, we have*

(i) $\lambda^o \leq \lambda$;
(ii) $\zeta(\mu,\lambda^o) = \zeta(\mu,\lambda)$.

Proof. (i) By (TT4), we have for all $x \in X$,

$$\lambda^o(x) = \zeta(\mathbf{1}_x,\lambda) \leq 1 - \text{hgt}\left[\mathbf{1}_x T(\underline{\underline{1}}-\lambda)\right] = \lambda(x).$$

(ii) By the continuity of T, for every real number $\theta > 1-\zeta(\mu,\lambda)$ and every $\epsilon > 0$, there is $\beta_\theta > 0$ such that

$$[1-\zeta(\mu,\lambda)] + \epsilon > \theta T\beta_\theta > 1-\zeta(\mu,\lambda).$$

Hence, by (TT3) and the continuity of T, there is $C_\theta \subseteq X$ such that

$$\zeta(\mu,\mathbf{1}_{C_\theta}) \geq 1-\beta_\theta \text{ and } \zeta(\mathbf{1}_{C_\theta},\lambda) \geq 1-\theta. \tag{3.2}$$

Consequently, for every $z \in C_\theta$, we have

$$\begin{aligned}\lambda^o(z) = \zeta(\mathbf{1}_z,\lambda) &\geq \zeta(\mathbf{1}_{C_\theta},\lambda), && \text{by (TT2)}\\ &\geq 1-\theta, && \text{by (3.2)}\\ &= \alpha, && \text{by putting } 1-\theta=\alpha.\end{aligned}$$

Therefore, $z \in (\lambda^o)_{\alpha^*}$, that is

$$C_\theta \subseteq (\lambda^o)_{\alpha^*}. \tag{3.3}$$

Hence,

$$\begin{aligned}
\zeta(\mu,\lambda^o) &= \bigwedge_{\alpha\in I}\left[\alpha T^*\zeta\mu, \mathbf{1}_{(\lambda^o)_{\alpha^*}}\right], && \text{by Proposition 3.6}\\
&= \left\{\bigwedge_{\alpha<\zeta(\mu,\lambda)}\left[\alpha T^*\zeta(\mu, \mathbf{1}_{(\lambda^o)_{\alpha^*}}\right]\right\} \wedge \left\{\bigwedge_{\alpha\geq\zeta(\mu,\lambda)}\left[\alpha T^*\zeta(\mu, \mathbf{1}_{(\lambda^o)_{\alpha^*}}\right]\right\}, && \text{clear}\\
&\geq \left\{\bigwedge_{\alpha<\zeta(\mu,\lambda)}\left[\alpha T^*\zeta(\mu, \mathbf{1}_{(\lambda^o)_{\alpha^*}}\right]\right\} \wedge \left\{\bigwedge_{\alpha\geq\zeta(\mu,\lambda)}\alpha\right\}\\
&\geq \left\{\bigwedge_{\theta>1-\zeta(\mu,\lambda)}[(1-\theta)T^*\zeta(\mu,\mathbf{1}_{C_\theta})]\right\}\wedge\zeta(\mu,\lambda), && \text{by (3.3) and (TT2)}\\
&\geq \left\{\bigwedge_{\theta>1-\zeta(\mu,\lambda)}[(1-\theta)T^*(1-\beta_\theta)]\right\}\wedge\zeta(\mu,\lambda), && \text{by (3.2)}\\
&= \left\{\bigwedge_{\theta>1-\zeta(\mu,\lambda)}[1-(\theta T\beta_\theta)]\right\}\wedge\zeta(\mu,\lambda)\\
&\geq [\zeta(\mu,\lambda)-\epsilon]\wedge\zeta(\mu,\lambda)\\
&= \zeta(\mu,\lambda)-\epsilon.
\end{aligned}$$

By the arbitrariness of ϵ, we get the inequality $\zeta(\mu,\lambda^o)\geq\zeta(\mu,\lambda)$. The opposite inequality follows from (i) and (TT2). □

Theorem 3.8. *The above mapping $\mu\to\mu^o$ is a fuzzy interior operator.*

Proof. We have shown that ^{o} is a lower operator. Also, for every μ, $\lambda\in I^X$, $\alpha\in I$ and $x\in X$, we have

$$\begin{aligned}
\underline{\underline{\alpha}}^o(x) &= \sup_{\zeta\in\mathscr{P}}\zeta(\mathbf{1}_x, \underline{\underline{\alpha}}T^*\underline{\underline{0}})\\
&= \alpha T^*\sup_{\zeta\in\mathscr{P}}\zeta(\mathbf{1}_x,\underline{\underline{0}}), && \text{by (TT5)}\\
&= \alpha, && \text{by (TT4)}\\
&= \underline{\underline{\alpha}}(x).
\end{aligned}$$

By monotonicity of ^{o}, we have

$$(\mu^o\wedge\lambda^o)\geq(\mu\wedge\lambda)^o.$$

On the other hand, if $\epsilon>0$, then there are ζ, $\eta\in\mathscr{P}$ such that

$$\mu^o(x)-\epsilon<\zeta(\mathbf{1}_x,\mu) \text{ and } \lambda^o(x)-\epsilon<\eta(\mathbf{1}_x,\lambda).$$

By (TS1), we can get $\zeta \in \mathscr{P}$ such that $\xi \geq \zeta,\, \eta$, thus

$$\begin{aligned}(\mu^o \wedge \lambda^o)(x) - \epsilon &= [\mu^o(x) - \epsilon] \wedge [\lambda^o(x) - \epsilon] \\ &< [\xi(\mathbf{1}_x, \mu) \wedge \xi(\mathbf{1}_x, \lambda)] \\ &= [\xi(\mathbf{1}_x, \mu \wedge \lambda)]\,, \qquad \text{by (TT2)} \\ &\leq (\mu \wedge \lambda)^o(x).\end{aligned}$$

This proves that $(\mu^o \wedge \lambda^o) \leq (\mu \wedge \lambda)^o$ and so $(\mu \wedge \lambda)^o = \mu^o \wedge \lambda^o$. Moreover, by (ii) of the preceding proposition, we get

$$(\mu^o)^o(x) = \sup_{\zeta \in \mathscr{P}} \zeta(\mathbf{1}_x, \mu^o) = \sup_{\zeta \in \mathscr{P}} \zeta(\mathbf{1}_x, \mu) = \mu^o(x);$$

that is, ^{o} is idempotent. This proves that ^{o} is a fuzzy interior operator. □

As a consequence of this theorem we may define an I-topology in the usual way, namely assuming a fuzzy set μ to be open if and only if $\mu = \mu^o$. We shall denote this I-topology by $\tau(\mathscr{P})$, and we shall refer to it as the I-topology generated by $\mathscr{P}$.

Obviously one can equip the set of all T-topogenous orders on a set X, with a partial order by defining ζ_1 is coarser than ζ_2 (and ζ_2 is finer than ζ_1) if $\zeta_1(\mu, \lambda) \leq \zeta_2(\mu, \lambda)$ for every pair of fuzzy sets $\mu,\, \lambda \in I^X$. Consequently, the T-syntopogenous structure $\mathscr{P}_1$ on X is said to be coarser than another one $\mathscr{P}_2$ (and $\mathscr{P}_2$ is finer than $\mathscr{P}_1$) if for every $\zeta \in \mathscr{P}_1$, there is $\zeta' \in \mathscr{P}_2$ such that $\zeta \leq \zeta'$.

It clearly follows that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are T-syntopogenous structures on a set X, and $\mathscr{P}_1$ is coarser than $\mathscr{P}_2$, then $\tau(\mathscr{P}_1) \subseteq \tau(\mathscr{P}_2)$.

Proposition 3.9. *Let $\mathscr{P}$ be a T-syntopogenous structure on a set X and define $\zeta_s : I^X \times I^X \to I$, by :*

$$\zeta_s(\mu, \lambda) = \sup_{\zeta \in \mathscr{P}} \zeta(\mu, \lambda), \qquad \mu, \lambda \in I^X.$$

Then ζ_s is a T-topogenous order on X, with $\tau(\{\zeta_s\}) = \tau(\mathscr{P})$.

Proof. It is easy to see that ζ_s satisfies (TT1), (TT3),(TT4) and (TT5). To prove (TT2), let $\mu,\, \lambda,\, \nu \in I^X$. Then

$$\begin{aligned}\zeta_s(\mu \vee \lambda, \nu) &= \sup_{\zeta \in \mathscr{P}} \zeta(\mu \vee \lambda, \nu) \\ &= \sup_{\zeta \in \mathscr{P}} [\zeta(\mu, \nu) \wedge \zeta(\lambda, \nu)]\,, \qquad \text{by (TT2)} \\ &\leq \left[\sup_{\zeta \in \mathscr{P}} \zeta(\mu, \nu)\right] \wedge \left[\sup_{\zeta \in \mathscr{P}} \zeta(\lambda, \nu)\right] \\ &= \zeta_s(\mu, \nu) \wedge \zeta_s(\lambda, \nu).\end{aligned}$$

For the opposite inequality, let $\epsilon \in I_0$ be such that

$$\epsilon < \zeta_s(\mu, \nu) \wedge \zeta_s(\lambda, \nu).$$

Then there are $\zeta_1,\, \zeta_2 \in \mathscr{P}$ such that

$$\epsilon \leq \zeta_1(\mu, \nu) \wedge \zeta_2(\lambda, \nu).$$

Since $\mathscr{P}$ is directed, then there is $\eta \in \mathscr{P}$ such that $\eta \geq \zeta_1 \vee \zeta_2$. Hence

$$\begin{aligned}\epsilon &\leq \eta(\mu,\nu) \wedge \eta(\lambda,\nu)\\ &= \eta(\mu \vee \lambda, \nu), \qquad\qquad \text{by (TT2)}\\ &\leq \sup_{\zeta\in\mathscr{P}} \zeta(\mu\vee\lambda,\nu)\\ &= \zeta_s(\mu\vee\lambda,\nu).\end{aligned}$$

This proves that

$$\zeta_s(\mu,\nu) \wedge \zeta_s(\lambda,\nu) \leq \zeta_s(\mu\vee\lambda,\nu).$$

So, $\zeta_s(\mu\vee\lambda,\nu) = \zeta_s(\mu,\nu) \wedge \zeta_s(\lambda,\nu)$. Analogously, we show that

$$\zeta_s(\mu,\lambda\wedge\nu) = \zeta_s(\mu,\lambda) \wedge \zeta_s(\mu,\nu).$$

Finally, we denote the fuzzy interior operators associated with $\tau(\{\zeta_s\})$ and $\tau(\mathscr{P})$, respectively by ^{o_1} and ^{o_2}. Let $\lambda \in I^X$ and $x \in X$. Then we have

$$\lambda^{o_1}(x) = \zeta_s(\mathbf{1}_x,\lambda) = \sup_{\zeta\in\mathscr{P}} \zeta(\mathbf{1}_x,\lambda) = \lambda^{o_2}(x).$$

That is, $\lambda^{o_1} = \lambda^{o_2}$, which implies that $\tau(\{\zeta_s\}) = \tau(\mathscr{P})$. Hence, the result follows. $\square$

We call ζ_s is the supremum of the T-syntopogeneous structure $\mathscr{P}$.

Example 3.10. Let X be a set and define $\zeta_1, \zeta_2 : I^X \times I^X \to I$, by for every μ, $\lambda \in I^X$:

$$\begin{aligned}\zeta_1(\mu,\lambda) &= 1 - \mathrm{hgt}[\mu T(\underline{\underline{1}} - \lambda)],\\ \zeta_2(\mu,\lambda) &= 1 - [(\mathrm{hgt}\mu) T \mathrm{hgt}(\underline{\underline{1}} - \lambda)].\end{aligned}$$

We verify that the function ζ_1 is a biperfect symmetrical T-topogenous order. It suffices to check (TT3), since the other axioms trivially hold. Let $\zeta_1(\mu,\lambda) > 1-(\theta T\beta)$ for some θ, $\beta \in I$. So for every $x \in X$,

$$[\mu T(\underline{\underline{1}} - \lambda)](x) \leq \mathrm{hgt}[\mu T(\underline{\underline{1}} - \lambda)] < \theta T \beta,$$

hence

$$\begin{aligned}\varnothing &= [\mu T(\underline{\underline{1}} - \lambda)]_{(\theta T\beta)^*}\\ &= \bigcup_{\epsilon T\gamma \geq \theta T\beta} \left[\mu_{\epsilon^*} \cap (\underline{\underline{1}} - \lambda)_{\gamma^*}\right], \qquad \text{by Lemma 2.1(i)}\\ &\supseteq \mu_{\theta^*} \cap (\underline{\underline{1}} - \lambda)_{\beta^*}.\end{aligned}$$

By taking $C = \mu_{\theta^*} \subseteq X$, we have

$$\begin{aligned}\zeta_1(\mu,\mathbf{1}_C) &= 1 - \mathrm{hgt}[\mu T(\underline{\underline{1}} - \mathbf{1}_C)] = 1 - \mathrm{hgt}[\mu T(\underline{\underline{1}} - \mathbf{1}_{\mu_{\theta^*}})] \geq 1-\theta,\\ \zeta_1(\mathbf{1}_C,\lambda) &= 1 - \mathrm{hgt}[\mathbf{1}_C T(\underline{\underline{1}} - \lambda)] = 1 - \mathrm{hgt}[\mathbf{1}_{\mu_{\theta^*}} T(\underline{\underline{1}} - \lambda)]\\ &\geq 1 - \mathrm{hgt}[(\underline{\underline{1}} - \mathbf{1}_{(\underline{\underline{1}}-\lambda)_{\beta^*}}) T(\underline{\underline{1}} - \lambda)] \geq 1-\beta.\end{aligned}$$

This yields (TT3). Moreover, it follows immediately that ζ_1 is a biperfect symmetrical.

The I-topology generated by ζ_1 is the discrete one (i.e. every fuzzy set is open), since for every $x \in X$ and $\mu \in I^X$, we have

$$\begin{aligned}\mu^o(x) &= \zeta_1(\mathbf{1}_x, \mu)\\ &= 1 - \text{hgt}[(\mathbf{1}_x)T(\underline{\underline{1}} - \mu)]\\ &= 1 - \sup_{z\in X}[(\mathbf{1}_x)T(\underline{\underline{1}} - \mu)](z)\\ &= 1 - (\underline{\underline{1}} - \mu)(x)\\ &= \mu(x).\end{aligned}$$

Also, the function ζ_1 is the finest (discrete) T-topogenous order on X, because for every T-topogenous order ζ on X, we have by (TT4),

$$\zeta(\mu, \lambda) \leq 1 - \text{hgt}[\mu T(\underline{\underline{1}} - \lambda)] = \zeta_1(\mu, \lambda).$$

To see that ζ_2 is a T-topogenous order, we need only check (TT3). Let $\zeta_2(\mu, \lambda) > 1 - (\theta T \beta)$ for some θ, $\beta \in I$, therefore

$$(\text{hgt}\mu)T(\text{hgt}(\underline{\underline{1}} - \lambda)) < \theta T \beta.$$

Hence, if $(\text{hgt}\mu) < \theta$, then $C = \varnothing$ yields;

$$\begin{aligned}\zeta_2(\mu, \mathbf{1}_C) &= \zeta_2(\mu, \underline{\underline{0}}) = 1 - [(\text{hgt}\mu)T\text{hgt}(\underline{\underline{1}} - \underline{\underline{0}})] = 1 - (\text{hgt}\mu) > 1 - \theta,\\ \zeta_2(\mathbf{1}_C, \lambda) &= \zeta_2(\underline{\underline{0}}, \lambda) = 1 - [(\text{hgt}\underline{\underline{0}})T\text{hgt}(\underline{\underline{1}} - \lambda)] = 1 > 1 - \beta.\end{aligned}$$

Whereas if $(\text{hgt}\mu) \geq \theta$, then $\text{hgt}(\underline{\underline{1}} - \lambda) < \beta$, and hence $C = X$ similarly yields

$$\zeta_2(\mu, \mathbf{1}_C) > 1 - \theta \text{ and } \zeta_2(\mathbf{1}_C, \lambda) > 1 - \beta.$$

This establishes (TT3). Moreover, it is easy to see that ζ_2 is a biperfect symmetrical. The I-topology generated by ζ_2 is the indiscrete one (exactly the constant fuzzy sets are open) because, for every $x \in X$ and $\mu \in I^X$, we have

$$\mu^o(x) = \zeta_2(\mathbf{1}_x, \mu) = 1 - [\text{hgt}(\mathbf{1}_x)T\text{hgt}(\underline{\underline{1}} - \mu)] = 1 - \text{hgt}(\underline{\underline{1}} - \mu).$$

Also, the function ζ_2 is the coarsest (indiscrete) T-topogenous order on X, because if $\text{hgt}\mu = \alpha$ and $\text{hgt}(\underline{\underline{1}} - \lambda) = \gamma$, then for every T-topogenous order ζ on X, we have

$$\begin{aligned}\zeta(\mu, \lambda) &\geq \zeta(\underline{\underline{\alpha}}, \underline{\underline{1 - \gamma}}), && \text{clearly by (TT2)}\\ &= \zeta(\underline{\underline{\alpha}} T \underline{\underline{1}}, (\underline{\underline{1 - \gamma}})T^* \underline{\underline{0}})\\ &= (1 - \alpha)T^*(1 - \gamma)T^* \zeta(\underline{\underline{1}}, \underline{\underline{0}}), && \text{by (TT5)}\\ &= (1 - \alpha)T^*(1 - \gamma), && \text{by (TT1)}\\ &= 1 - (\alpha T \gamma)\\ &= 1 - [(\text{hgt}\mu)T\text{hgt}(\underline{\underline{1}} - \lambda)]\\ &= \zeta_2(\mu, \lambda).\end{aligned}$$

Example 3.11. Let X be a nonempty set and let T =Min, take $\mathscr{P} = \{\zeta_1, \zeta_2\}$, where ζ_1, ζ_2 as in Example 3.10. We verify that $(X, \mathscr{P})$ is a biperfect symmetrical Min-syntopogenous space.

(TS1) It obviously holds because $\zeta_1 \geq \zeta_2$.

(TS2) Let $\mu,\ \lambda \in I^X$ and $\epsilon \in I_0$. Then

$$\begin{aligned}\left[(\zeta_1 \circ_T \zeta_1) + \underline{\epsilon}\right](\mu,\lambda) &= \sup_{C\subseteq X}\left[\zeta_1(\mu,\mathbf{1}_C)\wedge\zeta_1(\mathbf{1}_C,\lambda)\right]+\epsilon\\ &\geq \left[\zeta_1(\mu,\mathbf{1}_X)\wedge\zeta_1(\mathbf{1}_X,\lambda)\right]+\epsilon\\ &= \left\{[1-\mathrm{hgt}(\mu\wedge(\underline{\underline{1}}-\underline{\underline{1}}))]\wedge[1-\mathrm{hgt}(\underline{\underline{1}}\wedge(\underline{\underline{1}}-\lambda))]\right\}+\epsilon\\ &= [1-\mathrm{hgt}(\underline{\underline{1}}-\lambda)]+\epsilon\\ &= \left[\inf_{x\in X}\lambda(x)\right]+\epsilon\\ &\geq \lambda(x_0), \qquad \text{for some } x_0\in X.\end{aligned}$$

Also,

$$\begin{aligned}\left[(\zeta_1 \circ_T \zeta_1) + \underline{\epsilon}\right](\mu,\lambda) &= \sup_{C\subseteq X}\left[\zeta_1(\mu,\mathbf{1}_C)\wedge\zeta_1(\mathbf{1}_C,\lambda)\right]+\epsilon\\ &\geq \left[\zeta_1(\mu,\mathbf{1}_\varnothing)\wedge\zeta_1(\mathbf{1}_\varnothing,\lambda)\right]+\epsilon\\ &= \left\{[1-\mathrm{hgt}(\mu\wedge(\underline{\underline{1}}-\underline{\underline{0}})]\wedge\left[1-\mathrm{hgt}(\underline{\underline{0}}\wedge(\underline{\underline{1}}-\lambda))\right]\right\}+\epsilon\\ &= [1-(\mathrm{hgt}\mu)]+\epsilon\\ &= \left[\inf_{x\in X}(\underline{\underline{1}}-\mu)(x)\right]+\epsilon\\ &\geq (\underline{\underline{1}}-\mu)(y_0), \qquad \text{for some } y_0\in X.\end{aligned}$$

On the other hand,

$$\begin{aligned}\zeta_1(\mu,\lambda) &= 1-\mathrm{hgt}[\mu\wedge(\underline{\underline{1}}-\lambda)]\\ &= \inf_{x\in X}\left[(\underline{\underline{1}}-\mu)\vee\lambda\right](x)\\ &\leq (\underline{\underline{1}}-\mu)(x)\vee\lambda(x), \qquad \forall x\in X.\end{aligned}$$

So, for every $x\in X$, we have

$$\zeta_1(\mu,\lambda)\leq(\underline{\underline{1}}-\mu)(x) \text{ or } \zeta_1(\mu,\lambda)\leq\lambda(x).$$

If $\zeta_1(\mu,\lambda)\leq(\underline{\underline{1}}-\mu)(x)$, then $\zeta_1(\mu,\lambda)\leq(\underline{\underline{1}}-\mu)(y_0)\leq[(\zeta_1\circ_T\zeta_1)+\underline{\epsilon}](\mu,\lambda)$, and if $\zeta_1(\mu,\lambda)\leq\lambda(x)$, then $\zeta_1(\mu,\lambda)\leq\lambda(x_0)\leq[(\zeta_1\circ_T\zeta_1)+\underline{\epsilon}](\mu,\lambda)$. That is $(\zeta_1\circ_T\zeta_1)+\underline{\epsilon}\geq\zeta_1$. Also, $(\zeta_1\circ_T\zeta_1)+\underline{\epsilon}\geq\zeta_2$ since $\zeta_1\geq\zeta_2$. This renders (TS2) and shows that $\mathscr{P}$ is a Min-syntopogenous structure on X. Moreover, the members $\zeta_1,\ \zeta_2$ of $\mathscr{P}$ are biperfect symmetrical from Example 3.10. Also, obviously $\tau(\mathscr{P})=\tau(\{\zeta_1\})$, where ζ_1 is the supremum of $\mathscr{P}$.

Now, we clarify the relation between our T-syntopogenous structures and Katsaras' fuzzy syntopogenous structures.

Proposition 3.12 ([3]). *Let T be a continuous triangular norm. If $\boldsymbol{\Omega}: I^X\to I$ satisfies for all $\mu,\ \lambda\in I^X$, $H\subseteq X$ and $\alpha\in I$: $\boldsymbol{\Omega}(\mu\vee\lambda)=\boldsymbol{\Omega}(\mu)\vee\boldsymbol{\Omega}(\lambda)$ and $\boldsymbol{\Omega}(\underline{\underline{\alpha}}\wedge\mathbf{1}_H)=\alpha T\boldsymbol{\Omega}(\mathbf{1}_H)$, then $\boldsymbol{\Omega}(\underline{\underline{\alpha}})=\boldsymbol{\Omega}(\underline{\underline{1}})T\alpha$ and $\boldsymbol{\Omega}$ is uniformly continuous with respect to the L_∞-distance on I^X. Specifically, for given $\epsilon>0$, let $\theta=\theta_{T,\epsilon}$ be as in (2.1). Then for all $\mu,\ \lambda\in I^X$:*

$$\|\mu-\lambda\|\leq\theta\ \Rightarrow\ |\boldsymbol{\Omega}(\mu)-\boldsymbol{\Omega}(\lambda)|\leq\epsilon.$$

Proposition 3.13. *If $\zeta : I^X \times I^X \to I$ satisfies* (TT2) *and* (TT5)*, then ζ is uniformly continuous with respect to the L_∞-distance on I^X.*

Proof. Let μ, λ, ν, $\rho \in I^X$. For a given $\epsilon > 0$, let $\theta_{T,\epsilon}$ be as in (2.1). Put $\theta = \frac{1}{2}(\frac{\epsilon}{2} \wedge \theta_{T,\epsilon})$. Suppose that $\|(\mu, \underline{\underline{1}} - \lambda) - (\nu, \underline{\underline{1}} - \rho)\| = \|\mu - \nu\| \vee \|\rho - \lambda\| < \theta$. Then

$$\begin{aligned}|\zeta(\mu, \underline{\underline{1}} - \lambda) - \zeta(\nu, \underline{\underline{1}} - \rho)| &\leq |\zeta(\mu, \underline{\underline{1}} - \lambda) - \zeta(\mu, \underline{\underline{1}} - \rho)| + |\zeta(\mu, \underline{\underline{1}} - \rho) - \zeta(\nu, \underline{\underline{1}} - \rho))| \\ &\leq \epsilon + \epsilon = 2\epsilon\end{aligned}$$

because, for fixed fuzzy sets μ, ρ; $\boldsymbol{\Omega}_1(\lambda) = 1 - \zeta(\mu, \underline{\underline{1}} - \lambda)$, satisfies

$$\begin{aligned}\boldsymbol{\Omega}_1(\lambda \vee \lambda') &= 1 - \zeta(\mu, \underline{\underline{1}} - (\lambda \vee \lambda')) \\ &= 1 - \zeta(\mu, (\underline{\underline{1}} - \lambda) \wedge (\underline{\underline{1}} - \lambda')), \quad \lambda, \lambda' \in I^X \\ &= 1 - \left[\zeta(\mu, (\underline{\underline{1}} - \lambda)) \wedge \zeta(\mu, (\underline{\underline{1}} - \lambda'))\right], \qquad \text{by (TT2)} \\ &= \left[1 - \zeta(\mu, (\underline{\underline{1}} - \lambda))\right] \vee \left[1 - \zeta(\mu, (\underline{\underline{1}} - \lambda'))\right] \\ &= \boldsymbol{\Omega}_1(\lambda) \vee \boldsymbol{\Omega}_1(\lambda'),\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\Omega}_1(\underline{\underline{\alpha}} \wedge \mathbf{1}_H) &= 1 - \zeta(\mu, \underline{\underline{1}} - (\underline{\underline{\alpha}} \wedge \mathbf{1}_H)), \qquad \alpha \in I,\ H \in 2^X \\ &= 1 - \zeta(\mu, (\underline{\underline{1 - \alpha}}) \vee (\underline{\underline{1}} - \mathbf{1}_H)) \\ &= 1 - \zeta(\mu, (\underline{\underline{1 - \alpha}}) T^* (\underline{\underline{1}} - \mathbf{1}_H)) \\ &= 1 - \left[(1 - \alpha) T^* \zeta(\mu, (\underline{\underline{1}} - \mathbf{1}_H))\right], \qquad \text{by (TT5)} \\ &= \alpha T \left[1 - \zeta(\mu, (\underline{\underline{1}} - \mathbf{1}_H))\right] \\ &= \alpha T \boldsymbol{\Omega}_1(\mathbf{1}_H).\end{aligned}$$

In an analogous way, we can show that $\boldsymbol{\Omega}_2(\nu) = 1 - \zeta(\nu, \underline{\underline{1}} - \rho)$ also satisfies the above two conditions in Proposition 3.12. This establishes the uniform continuity of ζ. □

Remark 3.14. It follows from (TT1), (TT4) and Proposition 3.13 that ζ satisfies the axioms (i)-(v) of Definition 3.1 in [7], when T =Min. That is, the T-topogenous order (T-syntopogenous structure) is a generalization of Katsaras' fuzzy topogenous order (Katsaras' fuzzy syntopogenous structure).

In the following, we deduce the notion of image and inverse image of T-topogenous orders and T-syntopogenous structures.

Let $f : X \to Y$ be a function and η be a T-topogenous order on Y, we define the mapping $f^{\leftarrow}(\eta) : I^X \times I^X \to I$, by:

$$(f^{\leftarrow}(\eta))(\mu, \lambda) = \eta(f(\mu), \underline{\underline{1}} - f(\underline{\underline{1}} - \lambda)), \qquad \mu,\ \lambda \in I^X.$$

We call $f^{\leftarrow}(\eta)$ the inverse image of η under the function f.

Proposition 3.15. *For the mapping $f^{\leftarrow}(\eta)$ defined above, one has the following:*

(i) *$f^{\leftarrow}(\eta)$ is a T-topogenous order on X;*
(ii) *If η is a perfect (resp. biperfect, resp. symmetrical), then $f^{\leftarrow}(\eta)$ is a perfect (resp. biperfect, resp. symmetrical).*

Proof. (i) Let η be a T-topogenous order on Y. We verify that $f^{\leftarrow}(\eta)$ is a T-topogenous order on X. Let μ, λ, $\nu \in I^X$. Clearly, (TT1) holds.
(TT2)

$$\begin{aligned}(f^{\leftarrow}(\eta)(\mu\vee\lambda,\nu) &= \eta(f(\mu\vee\lambda),\underline{\underline{1}}-f(\underline{\underline{1}}-\nu))\\ &= \eta(f(\mu)\vee f(\lambda),\underline{\underline{1}}-f(\underline{\underline{1}}-\nu))\\ &= \eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\nu))\wedge\eta(f(\lambda),\underline{\underline{1}}-f(\underline{\underline{1}}-\nu))\\ &= (f^{\leftarrow}(\eta))(\mu,\nu)\wedge(f^{\leftarrow}(\eta))(\lambda,\nu).\end{aligned}$$

Similarly, we can show

$$(f^{\leftarrow}(\eta))(\mu,\lambda\wedge\nu) = (f^{\leftarrow}(\eta))(\mu,\lambda)\wedge(f^{\leftarrow}(\eta))(\mu,\nu).$$

(TT3) Obviously, $f(\underline{\underline{1}}-f^{\leftarrow}(\mathbf{1}_H)) = \underline{\underline{1}}-\mathbf{1}_H$, for all $H\subseteq Y$. Now, let $(f^{\leftarrow}(\eta))(\mu,\lambda) > 1-(\theta T\beta)$ for some θ, $\beta\in I_0$. Then $\eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda)) > 1-(\theta T\beta)$, so there is $H\subseteq Y$ such that $\eta(f(\mu),\mathbf{1}_H)\geq 1-\theta$ and $\eta(\mathbf{1}_H,\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\geq 1-\beta$, which implies by taking $C=f^{-1}(H)\subseteq X$, that

$$\begin{aligned}(f^{\leftarrow}(\eta))(\mu,\mathbf{1}_C) &= \eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\mathbf{1}_C))\\ &= \eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-f^{-1}(\mathbf{1}_H)))\\ &= \eta(f(\mu),\mathbf{1}_H)\\ &\geq 1-\theta,\end{aligned}$$

and

$$\begin{aligned}(f^{\leftarrow}(\eta))(\mathbf{1}_C,\lambda) &= \eta(f(\mathbf{1}_C),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &= \eta(f(f^{\leftarrow}(\mathbf{1}_H)),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &\geq \eta(\mathbf{1}_H,\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda)), \qquad \text{by (TT2)}\\ &\geq 1-\beta.\end{aligned}$$

(TT4)

$$\begin{aligned}(f^{\leftarrow}(\eta))(\mu,\lambda) &= \eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &\leq 1-\mathrm{hgt}[f(\mu)Tf(\underline{\underline{1}}-\lambda)]\\ &\leq 1-\mathrm{hgt}[f(\mu T(\underline{\underline{1}}-\lambda))], \quad \text{by Lemma 2.1(iv)}\\ &= 1-\mathrm{hgt}[\mu T(\underline{\underline{1}}-\lambda)]. \qquad \text{by Lemma 2.1(iv) again}\end{aligned}$$

(TT5)

$$\begin{aligned}(f^{\leftarrow}(\eta))(\underline{\underline{\alpha}}T\mu,\lambda) &= \eta(f(\underline{\underline{\alpha}}T\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &= \eta(\underline{\underline{\alpha}}Tf(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &= (1-\alpha)T^*\eta(f(\mu),\underline{\underline{1}}-f(\underline{\underline{1}}-\lambda))\\ &= (1-\alpha)T^*(f^{\leftarrow}(\eta))(\mu,\lambda).\end{aligned}$$

Analogously we can show

$$(f^{\leftarrow}(\eta))(\mu,(\underline{\underline{1-\alpha}})T^*\lambda) = (1-\alpha)T^*(f^{\leftarrow}(\eta))(\mu,\lambda).$$

This proves that $f^{\leftarrow}(\eta)$ is a T-topogenous order on X.

(ii) The perfect (resp. biperfect) of $f^{\leftarrow}(\eta)$ is immediately follows from the obviously fact that $f\left(\bigvee_{j\in J}\mu_j\right) = \bigvee_{j\in J} f(\mu_j)$ for any nonempty index set J. Now, we show that $f^{\leftarrow}(\eta)$ is symmetrical.

$$\begin{aligned}(f^{\leftarrow}(\eta))(\mu,\lambda) &= \eta(f(\mu), \underline{\underline{1}} - f(\underline{\underline{1}} - \lambda))\\ &= \eta(f(\underline{\underline{1}} - \lambda), \underline{\underline{1}} - f(\mu)), \qquad \text{by Definition 3.2}\\ &= (f^{\leftarrow}(\eta))(\underline{\underline{1}} - \lambda, \underline{\underline{1}} - \mu)\end{aligned}$$

This completes the proof. □

From the above proposition, we arrive

Proposition 3.16. *Let $f : X \to Y$ be a function and $\mathscr{H}$ be a T-syntopogenous structure on Y. Then $f^{\leftarrow}(\mathscr{H}) = \{f^{\leftarrow}(\eta) : \eta \in \mathscr{H}\}$ is a T-syntopogenous structure on X.*

Proof. Let $\mathscr{H}$ be a T-syntopogenous structure on Y. We verify that $f^{\leftarrow}(\mathscr{H})$ is a T-syntopogenous structure on X as:
(TS1) To show that $f^{\leftarrow}(\mathscr{H})$ is directed, given $f^{\leftarrow}(\eta)$, $f^{\leftarrow}(\zeta) \in f^{\leftarrow}(\mathscr{H})$, that is η, $\zeta \in \mathscr{H}$. Since $\mathscr{H}$ is directed, then there is $\xi \in \mathscr{H}$ such that $\xi \geq \eta \vee \zeta$. This meaning that, there is $f^{\leftarrow}(\xi) \in f^{\leftarrow}(\mathscr{H})$, which satisfies $f^{\leftarrow}(\xi) \geq f^{\leftarrow}(\eta \vee \zeta) = f^{\leftarrow}(\eta) \vee f^{\leftarrow}(\zeta)$.
(TS2) Let $f^{\leftarrow}(\eta) \in f^{\leftarrow}(\mathscr{H})$ and $\epsilon \in I_0$. Then there is $\eta_\epsilon \in \mathscr{H}$ such that

$$\eta \leq (\eta_\epsilon \circ_T \eta_\epsilon) + \underline{\underline{\epsilon}}.$$

Hence for every μ, $\lambda \in I^X$, we have

$$\begin{aligned}&(f^{\leftarrow}(\eta))(\mu,\lambda)\\ &= \eta(f(\mu), \underline{\underline{1}} - f(\underline{\underline{1}} - \lambda))\\ &\leq (\eta_\epsilon \circ_T \eta_\epsilon)(f(\mu), \underline{\underline{1}} - f(\underline{\underline{1}} - \lambda)) + \epsilon\\ &= \sup_{H\subseteq Y} [\eta_\epsilon(f(\mu), \mathbf{1}_H) T \eta_\epsilon(\mathbf{1}_H, \underline{\underline{1}} - f(\underline{\underline{1}} = \lambda))] + \epsilon\\ &\leq \sup_{H\subseteq Y} \{[\eta_\epsilon(f(\mu), \underline{\underline{1}} - f(\underline{\underline{1}} - f^{\leftarrow}(\mathbf{1}_H)))] T [\eta_\epsilon(f(f^{\leftarrow}(\mathbf{1}_H)), \underline{\underline{1}} - f(\underline{\underline{1}} - \lambda))]\} + \epsilon\\ &= \sup_{H\subseteq Y} \{[(f^{\leftarrow}(\eta_\epsilon))(\mu, f^{\leftarrow}(\mathbf{1}_H))] T [(f^{\leftarrow}(\eta_\epsilon))(f^{\leftarrow}(\mathbf{1}_H), \lambda)]\} + \epsilon\\ &\leq \sup_{C\subseteq Y} \{[(f^{\leftarrow}(\eta_\epsilon))(\mu, \mathbf{1}_C)] T [(f^{\leftarrow}(\eta_\epsilon))(\mathbf{1}_C, \lambda)]\} + \epsilon\\ &= [f^{\leftarrow}(\eta_\epsilon) \circ_T f^{\leftarrow}(\eta_\epsilon)](\mu,\lambda) + \underline{\underline{\epsilon}}.\end{aligned}$$

Thus, there is $f^{\leftarrow}(\eta_\epsilon)$ an element in $f^{\leftarrow}(\mathscr{H})$ satisfies

$$f^{\leftarrow}(\eta) \leq [f^{\leftarrow}(\eta_\epsilon) \circ_T f^{\leftarrow}(\eta_\epsilon)] + \underline{\underline{\epsilon}}.$$

This completes the proof. □

Let $f : X \to Y$ be a function and ζ be a T-topogenous order on X, we define the mapping $f(\zeta) : I^Y \times I^Y \to I$ by:

$$(f(\zeta))(\nu,\rho) = \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)), \qquad \nu,\ \rho \in I^Y.$$

We call $f(\zeta)$ the image of ζ under the function f.

Proposition 3.17. *For the mapping $f(\zeta)$ defined above, one has the following:*

(i) *If f is a bijective, then $f(\zeta)$ is a T-topogenous order on Y;*
(ii) *If ζ is a perfect (resp. biperfect, resp. symmetrical) and f is a bijective, then $f(\zeta)$ is a perfect (resp. biperfect, resp. symmetrical).*

Proof. (i) Let ζ be a T-topogenous order on X. Then obviously $f(\zeta)$ satisfies (TT1), (TT2) and (TT5).
(TT3) Let $\nu,\ \rho \in I^Y$ and $\theta,\ \beta \in I_0$, be such that $(f(\zeta))(\nu,\rho) > 1-(\theta T\beta)$. Then $\zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)) \geq 1-(\theta T\beta)$, so there is $C \subseteq X$ such that $\zeta(f^{\leftarrow}(\nu), \mathbf{1}_C) \geq 1-\theta$ and $\zeta(\mathbf{1}_C, f^{\leftarrow}(\rho)) \geq 1-\beta$, which implies by putting $H = f(C) \subseteq Y(i.e, f^{\leftarrow}(H) = C$ because f is injective), that $(f(\zeta))(\nu,\mathbf{1}_H) = \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\mathbf{1}_H)) = \zeta(f^{\leftarrow}(\nu),\mathbf{1}_C) \geq 1-\theta$ and $(f(\zeta))(\mathbf{1}_H,\rho)) = \zeta(f^{\leftarrow}(\mathbf{1}_H), f^{\leftarrow}(\rho)) = \zeta(\mathbf{1}_C, f^{\leftarrow}(\rho)) \geq 1-\beta$.
(TT4)

$$\begin{aligned}
(f(\zeta))(\nu,\rho) &= \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)) \\
&\leq 1-\mathrm{hgt}[f^{\leftarrow}(\nu)T(\underline{\underline{1}}-f^{\leftarrow}(\rho))] \\
&= 1-\mathrm{hgt}[f^{\leftarrow}(\nu)Tf^{\leftarrow}(\underline{\underline{1}}-\rho)] \\
&= 1-\mathrm{hgt}f^{\leftarrow}(\nu T(\underline{\underline{1}}-\rho)) \\
&= 1-\sup_{x\in X}[f^{\leftarrow}(\nu T(\underline{\underline{1}}-\rho))](x) \\
&= 1\sup_{x\in X}[\nu T(\underline{\underline{1}}-\rho)](f(x)) \\
&= 1-\sup_{y\in Y}[\nu T(\underline{\underline{1}}-\rho)](y), \qquad \text{because } f \text{ is surjective} \\
&= 1-\mathrm{hgt}[\nu T(\underline{\underline{1}}-\rho)].
\end{aligned}$$

This proves that $f(\zeta)$ is a T-topogenous order on Y.

(ii) The perfect(resp. biperfect) of $f(\zeta)$ is immediately follows from the obviously facts that $f^{\leftarrow}\left(\bigvee_{j\in J}\nu_j\right) = \bigvee_{j\in J} f^{\leftarrow}(\nu_j)$ and $f^{\leftarrow}\left(\bigwedge_{j\in J}\nu_j\right) = \bigwedge_{j\in J} f^{\leftarrow}(\nu_j)$ for any index set J. Also, the symmetrical of $f(\zeta)$ is trivially hold. □

Proposition 3.18. *Let $f : X \to Y$ be a bijective function and $\mathscr{P}$ a T-syntopogenous structure on X. Then $f(\mathscr{P}) = \{f(\zeta) : \zeta \in \mathscr{P}\}$ is a T-syntopogenous structure on Y.*

Proof. The proof can be along similar lines of Proposition 3.16. □

4. Syntopogenously continuous functions

The aim of this section is to study the continuity of functions between T-syntopogenous spaces.

Definition 4.1. Let $(X,\mathscr{P})$ and $(Y,\mathscr{H})$ be T-syntopogenous spaces. A function $f : X \to Y$ is called syntopogenous map, or syntopogenously continuous, if for every $\eta \in \mathscr{H}$ there is $\zeta \in \mathscr{P}$ such that

$$\eta(\nu,\rho) \leq \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)), \qquad \nu,\ \rho \in I^Y. \tag{4.1}$$

Equivalently, if for every $\eta \in \mathscr{H}$ there is $\zeta \in \mathscr{P}$ such that

$$\eta(f(\mu),\underline{\underline{1}}-f(\lambda)) \leq \zeta(\mu,\underline{\underline{1}}-\lambda), \qquad \mu,\ \lambda \in I^X. \tag{4.2}$$

We can see that the conditions (4.1) and (4.2) above are equivalent as follows: Suppose (4.1) holds. Then for every μ, $\lambda \in I^X$,

$$\begin{aligned}\eta(f(\mu), \underline{\underline{1}} - f(\lambda)) &\leq \zeta(f^{\leftarrow}(f(\mu)), f^{\leftarrow}(\underline{\underline{1}} - f(\lambda))) \\ &\leq \zeta(\mu, \underline{\underline{1}} - f^{\leftarrow}(f(\lambda))), && \text{by (TT2), for } f^{\leftarrow}(f(\mu)) \geq \mu \\ &\leq \zeta(\mu, \underline{\underline{1}} - \lambda), && \text{by (TT2) again}\end{aligned}$$

Rendering (4.2).
Suppose (4.2) holds. Then for every ν, $\rho \in I^Y$,

$$\begin{aligned}\eta(\nu, \rho) &\leq \eta(f(f^{\leftarrow}(\nu)), \underline{\underline{1}} - f(f^{\leftarrow}(\underline{\underline{1}} - \rho))), && \text{by (TT2), for } f(f^{\leftarrow}(\nu) \leq \nu \\ &= \eta(f(f^{\leftarrow}(\nu)), \underline{\underline{1}} - f(\underline{\underline{1}} - f^{\leftarrow}(\rho))) \\ &\leq \zeta(f^{\leftarrow}(\nu), \underline{\underline{1}} - (\underline{\underline{1}} - f^{\leftarrow}(\rho))), && \text{by (4.2)} \\ &= \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)),\end{aligned}$$

which yields (4.1).

The next two theorems follow immediately from definitions (cf. [8]).

Theorem 4.2. *Let $f : (X, \mathscr{P}) \to (Y, \mathscr{H})$ and $g : (Y, \mathscr{H}) \to (Z, \mathscr{C})$ be syntopogenous maps. Then the composition $g \circ f$ is also a syntopogenous map.*

Theorem 4.3. *If $f : (X, \mathscr{P}) \to (Y, \mathscr{H})$ is a syntopogenous map, then $f : (X, \tau(\mathscr{P})) \to (Y, \tau(\mathscr{H}))$ is continuous.*

The above shows that the class of all T-syntopogenous spaces, forms a concrete category; together with syntopogenous maps as arrows [1]. We denote this category by T-SS. Also, a functor $\digamma_T$ is defined from this category to the category FTS of Lowen I-topological spaces by $\digamma_T(X, \mathscr{P}) = (X, \tau(\mathscr{P}))$, on objects, and by leaving arrows unchanged.

5. Characterization of a T-topogenous order in terms of crisp fuzzy subsets

We provide axioms for a function $\zeta : 2^X \times 2^X \to I$ to be a restriction of a (unique) T-topogenous order on X.

Theorem 5.1. *A function $\zeta : I^X \times I^X \to I$ is a T-topogenous order on X if and only if it satisfies the following five axioms, the first four of which are properties of its restriction $\zeta : 2^X \times 2^X \to I$. For all H, M, $N \in 2^X$:*

(TT1') $\zeta(\mathbf{1}_X, \mathbf{1}_X) = \zeta(\mathbf{1}_\varnothing, \mathbf{1}_\varnothing) = 1$ *and* $\zeta(\mathbf{1}_X, \mathbf{1}_\varnothing) = 0$*;*
(TT2') $\zeta(\mathbf{1}_{(H \cup M)}, \mathbf{1}_N) = \zeta(\mathbf{1}_H, \mathbf{1}_N) \wedge \zeta(\mathbf{1}_M, \mathbf{1}_N)$*, and* $\zeta(\mathbf{1}_H, \mathbf{1}_{(M \cap N)}) = \zeta(\mathbf{1}_H, \mathbf{1}_M) \wedge \zeta(\mathbf{1}_H, \mathbf{1}_N)$*;*
(TT3') *If* $\zeta(\mathbf{1}_H, \mathbf{1}_M) > 1 - (\theta T \beta)$ *for some* θ, $\beta \in I_0$*, there is* $C \subseteq X$ *such that* $\zeta(\mathbf{1}_H, \mathbf{1}_C) \geq 1 - \theta$ *and* $\zeta(\mathbf{1}_C, \mathbf{1}_M) \geq 1 - \beta$*;*
(TT4') *If* $H \not\subset M$*, then* $\zeta(\mathbf{1}_H, \mathbf{1}_M) = 0$*;*
(TT5') $\zeta(\mu, \lambda) = \bigwedge_{\theta, \beta \in I} [\theta T^* \beta T^* \zeta(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{\lambda_{\beta^*}})]$.

We need the next two propositions and two definitions, in the course of proving this theorem.

Proposition 5.2. *Let $\zeta : 2^X \times 2^X \to I$ be a function satisfies the five conditions* (TT1')-(TT5'). *Let H, M_1, M_2, ..., $M_n \subseteq X$, γ_1, ..., $\gamma_n \in I$ and put*

$$\lambda = \bigwedge_{k=1}^{n} \left(\underline{\underline{\gamma_k}} \vee \mathbf{1}_{M_k}\right) \in I^X.$$

If $\zeta(\mathbf{1}_H, \lambda) > 1 - (\theta T \beta)$ for some θ, $\beta \in I_0$, then there is $C \subseteq X$ such that $\zeta(\mathbf{1}_H, \mathbf{1}_C) \geq 1 - \theta$ and $\zeta(\mathbf{1}_C, \lambda) \geq 1 - \beta$.

Proof. First, we notice that ζ satisfies (TT2), by repeated application of (TT2') and (TT5'). Also, by continuity of T, there is $\beta' \in I$ such that

$$\begin{aligned}
1 - (\theta T \beta) &< 1 - (\theta T \beta') \\
&< \zeta(\mathbf{1}_H, \lambda) \\
&= \zeta\left(\mathbf{1}_H, \bigwedge_{k=1}^{n} (\underline{\underline{\gamma_k}} \vee \mathbf{1}_{M_k})\right) \\
&= \bigwedge_{k=1}^{n} \zeta(\mathbf{1}_H, (\underline{\underline{\gamma_k}} T^* \mathbf{1}_{M_k})), && \text{by (TT2) which yields above} \\
&= \bigwedge_{k=1}^{n} [\gamma_k T^* \zeta(\mathbf{1}_H, \mathbf{1}_{M_k})], && \text{by Proposition 3.6}(i),
\end{aligned}$$

that is $\theta T \beta > \theta T \beta' > \bigvee_{k=1}^{n} \{(1 - \gamma_k) T [1 - \zeta(\mathbf{1}_H, \mathbf{1}_{M_k})]\}$. Hence, for every $k = 1, \ldots, n$ such that $1 - \gamma_k > \beta$,

$$\theta T \beta' > (1 - \gamma_k) T [1 - \zeta(\mathbf{1}_H, \mathbf{1}_{M_k})].$$

Thus, by (2.3) then Lemma 2.3,

$$1 - \zeta(\mathbf{1}_H, \mathbf{1}_{M_k}) \leq \mathscr{I}(1 - \gamma_k, \theta T \beta') < \theta T \mathscr{I}(1 - \gamma_k, \beta),$$

that is

$$\zeta(\mathbf{1}_H, \mathbf{1}_{M_k}) > 1 - [\theta T \mathscr{I}(1 - \gamma_k, \beta))].$$

Then from (TT3') there is $C_k \subseteq X$ such that

$$\zeta(\mathbf{1}_H, \mathbf{1}_{C_k}) > 1 - \theta \text{ and } \zeta(\mathbf{1}_{C_k}, \mathbf{1}_{M_k}) \geq 1 - \mathscr{I}(1 - \gamma_k, \beta). \tag{5.1}$$

For every $k = 1, \ldots, n$ such that $1 - \gamma_k \leq \beta$, we take $C_k = X$, Then

$$\begin{aligned}
\zeta(\mathbf{1}_H, \mathbf{1}_{C_k}) &= \zeta(\mathbf{1}_H, \mathbf{1}_X) \geq \zeta(\mathbf{1}_X, \mathbf{1}_X), && \text{by (TT2')} \\
&= 1 \geq 1 - \theta
\end{aligned}$$

and

$$\begin{aligned}
\zeta(\mathbf{1}_{C_k}, \mathbf{1}_{M_k}) &= \zeta(\mathbf{1}_X, \mathbf{1}_{M_k}) \geq \zeta(\mathbf{1}_X, \mathbf{1}_\varnothing), && \text{by (TT2')} \\
&= \underline{\underline{0}} = 1 - 1 = 1 - \mathscr{I}(1 - \gamma_k, \beta), && \text{by (2.3)}
\end{aligned}$$

which again yields (5.1). By taking $C = \bigcup_{i=1}^{n} C_i \subseteq X$, we get

$$\zeta(\mathbf{1}_H, \mathbf{1}_C) = \zeta\left(\mathbf{1}_H, \bigvee_{i=1}^{n} \mathbf{1}_{C_i}\right) = \bigvee_{i=1}^{n} \zeta(\mathbf{1}_H, \mathbf{1}_{C_i}) \geq 1-\theta, \quad \text{by (5.1)}$$

$$\begin{aligned}\zeta(\mathbf{1}_C, \lambda) &= \zeta\left(\bigvee_{i=1}^{n} \mathbf{1}_{C_i}, \bigwedge_{k=1}^{n} (\underline{\underline{\gamma_k}} T^* \mathbf{1}_{M_k})\right) \\ &= \bigwedge_{k=1}^{n} [\gamma_k T^* \zeta(\mathbf{1}_{C_k}, \mathbf{1}_{M_k})], \quad \text{by (TT2) and Proposition 3.6} \\ &\geq \bigwedge_{k=1}^{n} \{\gamma_k T^* [1 - \mathscr{J}(1-\gamma_k, \beta)]\}, \quad \text{by (5.1)} \\ &= 1 - \left\{\bigvee_{k=1}^{n} [(1-\gamma_k) T \mathscr{J}(1-\gamma_k, \beta)]\right\} \\ &= 1 - \left\{\bigvee_{k=1}^{n} [(1-\gamma_k) \wedge \beta]\right\}, \quad \text{by (2.2)} \\ &\geq 1-\beta,\end{aligned}$$

which proves our assertion. ∎

Proposition 5.3. *Let $\zeta : 2^X \times 2^X \to I$ be a function satisfies the five conditions* (TT1')-(TT5'). *Let $H_1, \ldots, H_r$, $M_1, \ldots, M_n \subseteq X$, $\alpha_1, \ldots, \alpha_r$, $\gamma_1, \ldots, \gamma_n \in I$, and write $\mu = \bigvee_{i=1}^{r} (\underline{\underline{\alpha_i}} \wedge \mathbf{1}_{H_i})$, $\lambda = \bigwedge_{k=1}^{n} (\underline{\underline{\gamma_k}} \vee \mathbf{1}_{M_k}) \in I^X$. If $\zeta(\mu, \lambda) > 1 - (\theta T \beta)$ for some $\theta, \beta \in I_0$, then there is $C \subseteq X$ such that $\zeta(\mu, \mathbf{1}_C) \geq 1-\beta$ and $\zeta(\mathbf{1}_C, \lambda) \geq 1-\theta$.*

Proof. By continuity of T, there is $\beta' \in I$ such that

$$\begin{aligned}1 - (\theta T \beta) < 1 - (\theta T \beta') < \zeta(\mu, \lambda) &= \zeta\left(\bigvee_{i=1}^{r} (\underline{\underline{\alpha_i}} \wedge \mathbf{1}_{H_i}), \lambda\right) \\ &= \bigwedge_{i=1}^{r} \zeta(\underline{\underline{\alpha_i}} T \mathbf{1}_{H_i}, \lambda) \\ &= \bigwedge_{i=1}^{r} [(1-\alpha_i) T^* \zeta(\mathbf{1}_{H_i}, \lambda)], \quad \text{by Proposition 3.6}\end{aligned}$$

That is, $\theta T \beta > \theta T \beta' > \bigvee_{i=1}^{r} \{\alpha_i T [1 - \zeta(\mathbf{1}_{H_i}, \lambda)]\}$. Therefore, by (2.3) then Lemma 2.3, we get for every $i = 1, \ldots, r$ such that $\alpha_i > \beta$:

$$1 - \zeta(\mathbf{1}_{H_i}, \lambda) \leq \mathscr{J}(\alpha_i, \theta T \beta') < \theta T \mathscr{J}(\alpha_i, \beta) = \mathscr{J}(\alpha_i, \beta) T \theta,$$

that is,

$$\zeta(\mathbf{1}_H, \lambda) > 1 - [\mathscr{J}(\alpha_i, \beta) T \theta].$$

Consequently, from Proposition 5.2, there is $C_i \subseteq X$ such that

$$\zeta(\mathbf{1}_{H_i}, \mathbf{1}_{C_i}) \geq 1 - \mathscr{J}(\alpha_i, \beta) \text{ and } \zeta(\mathbf{1}_{C_i}, \lambda) \geq 1-\theta. \tag{5.2}$$

For every $i = 1, \ldots, r$ with $\alpha_i \leq \beta$, we take $C_i = X$. Then (as shown for (5.1)) C_i will also satisfy (5.2). By taking $C = \bigcap_{t=1}^{r} C_t \subseteq X$, we get

$$\begin{aligned}
\zeta(\mu, \mathbf{1}_C) &= \zeta\left(\bigvee_{i=1}^{r}(\underline{\underline{\alpha_i}} \wedge \mathbf{1}_{H_i}), \bigwedge_{t=1}^{r} \mathbf{1}_{C_t}\right) \\
&= \bigwedge_{i=1}^{r} \zeta(\underline{\underline{\alpha_i}} T \mathbf{1}_{H_i}, \mathbf{1}_{C_i}), \qquad \text{by (TT2) which yields above} \\
&= \bigwedge_{i=1}^{r} [(1-\alpha_i) T^* \zeta(\mathbf{1}_{H_i}, \mathbf{1}_{C_i})], \qquad \text{by Proposition 3.6} \\
&\geq \bigwedge_{i=1}^{r} \{(1-\alpha_i) T^* [1 - \mathscr{J}(\alpha_i, \beta)]\}, \quad \text{by (5.2)} \\
&= 1 - \left\{\bigvee_{i=1}^{r} [\alpha_i T \mathscr{J}(\alpha_i, \beta)]\right\} \\
&= 1 - \left[\bigvee_{i=1}^{r} (\alpha_i \wedge \beta)\right], \qquad \text{by (2.2)} \\
&\geq 1 - \beta
\end{aligned}$$

and $\zeta(\mathbf{1}_C, \lambda) = \zeta\left(\bigwedge_{t=1}^{r} \mathbf{1}_{C_t}, \lambda\right) = \bigvee_{t=1}^{r} \zeta(\mathbf{1}_{C_t}, \lambda) \geq 1 - \theta$ by (TT2') and (5.2), which completes the proof. □

A possibility distribution on a nonempty set X [12], is an assignment of possibility values in $[0, 1]$ to the elements of X, such that those values have supremum 1. Such a function is numerically equal to a normalized fuzzy subset of X (i.e. one with height 1).

Definition 5.4 ([3])**.** A generalized possibility measure, GPM, on a set X is a function $\boldsymbol{f} : 2^X \to I$ which satisfies the following three axioms:
(GPM1) $\boldsymbol{f}(X) = 1$;
(GPM2) $\boldsymbol{f}(\varnothing) = 0$;
(GPM3) $\boldsymbol{f}\left(\bigcup_{i=1}^{n} H_i\right) = \max\{\boldsymbol{f}(H_i) : i = 1, 2, \ldots, n\}$ for every nonempty, finitely indexed family $\{H_i\}_{i=1}^{n}$ of subsets of X.

Definition 5.5 ([3])**.** An extended generalized possibility measure, (extended GPM) on X is a function $\boldsymbol{\Omega} : I^X \to I$ that satisfies:
(E-GPM1) $\boldsymbol{\Omega}(X) = 1$;
(E-GPM2) $\boldsymbol{\Omega}(\varnothing) = 0$;
(E-GPM3) $\boldsymbol{\Omega}(\mu \vee \lambda) = \boldsymbol{\Omega}(\mu) \vee \boldsymbol{\Omega}(\lambda)$, for every μ, $\lambda \in I^X$.

It is evident that the restriction of an extended GPM to 2^X is a GPM on X.

Proof. (**Proof of Theorem 5.1**) Suppose that $\zeta : I^X \times I^X \to I$ is a T-topogenous order on X. Then (TT1')-(TT4') immediately follow from the corresponding axioms in Definition 3.1, and (TT5') follows from Proposition 3.6. Conversely, let $\zeta : I^X \times I^X \to I$ be satisfies (TT1')-(TT5'). Then (TT1), (TT2) are satisfied by (TT1'), (TT2') and repeated applications of (TT5'), and then (TT5) follows from

Proposition 3.6. To prove (TT3), let $\zeta(\mu,\lambda) > 1-(\theta T\beta)$ for some μ, $\lambda \in I^X$ and θ, $\beta \in I_0$. Then on one hand, by continuity of T, there is $\epsilon > 0$ in such a way that $\zeta(\mu,\lambda) > 1-[(\theta-\epsilon)T(\beta-\epsilon)]+\epsilon$.

On the other hand, (TT2) and (TT5) (proved above) imply, by Proposition 3.13, that such ζ is uniformly continuous on $I^X \times I^X$ with respect to the L_∞-distance, and so there is $\gamma = \gamma_{T,\epsilon} > 0$ such that for every λ, λ', ρ, $\rho' \in I^X$,

$$\|\lambda-\lambda'\| \vee \|\rho-\rho'\| < \gamma \;\Rightarrow\; |\zeta(\lambda,\rho)-\zeta(\lambda',\rho')| \le \epsilon. \tag{5.3}$$

But there exist μ_1, $\lambda_1 \in I^X$ with finite ranges such that

$$\|\mu_1-\mu\| \vee \|\lambda_1-\lambda\| < \gamma,$$

hence

$$|\zeta(\mu_1,\lambda_1)-\zeta(\mu,\lambda)| \le \epsilon,$$

i.e.,

$$\zeta(\mu_1,\lambda_1) \ge \zeta(\mu,\lambda)-\epsilon > 1-[(\theta-\epsilon)T(\beta-\epsilon)].$$

So, by Proposition 5.3, there is $C \subseteq X$ such that

$$\zeta(\mu_1,\mathbf{1}_C) \ge 1-(\theta-\epsilon) \text{ and } \zeta(\mathbf{1}_C,\lambda_1) \ge 1-(\beta-\epsilon).$$

Consequently, by (5.3),

$$\zeta(\mu,\mathbf{1}_C) \ge \zeta(\mu_1,\mathbf{1}_C)-\epsilon \ge 1-(\theta-\epsilon)-\epsilon = 1-\theta,$$

and also,

$$\zeta(\mathbf{1}_C,\lambda) \ge \zeta(\mathbf{1}_C,\lambda_1)-\epsilon \ge 1-(\beta-\epsilon)-\epsilon = 1-\beta$$

which renders (TT3).

We next prove (TT4). For every real number $\epsilon > 1-\mathrm{hgt}[\mu T(\underline{\underline{1}}-\lambda)]$, we have

$$\begin{aligned}
\varnothing &\ne [\mu T(\underline{\underline{1}}-\lambda)]^{(1-\epsilon)} \\
&= \bigcup_{\theta T\beta \ge 1-\epsilon} [\mu^\theta \cap (\underline{\underline{1}}-\lambda^\beta)], && \text{by Lemma 2.2 (i)} \\
&\subseteq \bigcup_{\theta T\beta \ge 1-\epsilon} [\mu_{\theta^*} \cap (X-\lambda_{(1-\beta)^*})], && \text{clear, since } \mu^\theta \subseteq \mu_{\theta^*}.
\end{aligned}$$

Consequently, there exist θ, $\beta \in I$ with $\epsilon \ge 1-(\theta T\beta) = (1-\theta)T^*(1-\beta)$ such that $\mu_{\theta^*} \cap (X-\lambda_{(1-\beta)^*}) \ne \varnothing$, that is $\mu_{\theta^*} \not\subseteq \lambda_{(1-\beta)^*}$ and so by (TT4'), we have $\zeta(\mathbf{1}_{\mu_{\theta^*}},\mathbf{1}_{\lambda_{(1-\beta)^*}}) = 0$. Hence,

$$\epsilon \ge (1-\theta)T^*(1-\beta) = [(1-\theta)T^*(1-\beta)T^*\zeta(\mathbf{1}_{\mu_{\theta^*}},\mathbf{1}_{\lambda_{(1-\beta)^*}})] \ge \zeta(\mu,\lambda).$$

This establishes that $\zeta(\mu,\lambda) \le 1-\mathrm{hgt}[\mu T(\underline{\underline{1}}-\lambda)]$, which completes the proof. □

Theorem 5.6. *For a T-topogenous order ζ on a set X, we have*

(i) *If ζ is a perfect, then $\zeta\left(\bigcup_{j\in J}\mathbf{1}_{H_j},\mathbf{1}_M\right) = \bigwedge_{j\in J}\zeta(\mathbf{1}_{H_j},\mathbf{1}_M)$, H_j , $M \in 2^X$;*

(ii) *ζ is a biperfect if and only if it is a perfect and*

$$\zeta\left(\mathbf{1}_H,\bigcap_{j\in J}\mathbf{1}_{M_j}\right) = \bigwedge_{j\in J}\zeta(\mathbf{1}_H,\mathbf{1}_{M_j}), \qquad H,\ M_j \in 2^X. \tag{5.4}$$

(iii) *ζ is a symmetrical if and only if $\zeta(\mathbf{1}_H, \mathbf{1}_M) = \zeta(\mathbf{1}_{(X-M)}, \mathbf{1}_{(X-H)})$, H, $M \in 2^X$. Also, the order relation on the set of T-topogenous orders on X, is completely determined by the order on their restrictions to pairs of crisp fuzzy subsets of X.*

Proof. (i) Obviously holds.

(ii) Suppose that ζ is a biperfect. Then it is a perfect and

$$\zeta\left(\mathbf{1}_H, \bigcap_{j\in J} \mathbf{1}_{M_j}\right) = \bigwedge_{j\in J} \zeta(\mathbf{1}_H, \mathbf{1}_{M_j}), \text{ for all } H,\ M_j \in 2^X.$$

Conversely, let ζ be a perfect and satisfies (5.4). Then by (TT5'), we have

$$\begin{aligned}
\zeta\left(\mu, \bigwedge_{j\in J} \lambda_j\right) &= \bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \zeta(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{(\bigwedge_{j\in J}\lambda_j)_{\beta^*}})\right] \\
&= \bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \zeta\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \bigcap_{j\in J} \mathbf{1}_{(\lambda_j)_{\beta^*}}\right)\right], \quad \text{clear} \\
&= \bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \bigwedge_{j\in J} \zeta\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{(\lambda_j)_{\beta^*}}\right)\right], \quad \text{by hypothesis} \\
&= \bigwedge_{j\in J} \left\{\bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \zeta\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{(\lambda_j)_{\beta^*}}\right))\right]\right\} \\
&= \bigwedge_{j\in J} \zeta(\mu, \lambda_j),
\end{aligned}$$

which proves that ζ is a biperfect.

(iii) can be proved analogously in similar lines. Now, let $\zeta_1(\mathbf{1}_H, \mathbf{1}_M) \leq \zeta_2(\mathbf{1}_H, \mathbf{1}_M)$ for every H, $M \in 2^X$, where ζ_1, ζ_2 are T-topogenous orders on X. Then for every μ, $\lambda \in I^X$ we have, by (TT5') that

$$\begin{aligned}
\zeta_1(\mu, \lambda) &= \bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \zeta_1\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{\lambda_{\beta^*}}\right)\right] \\
&\leq \bigwedge_{\theta,\beta\in I} \left[\theta T^* \beta T^* \zeta_2\left(\mathbf{1}_{\mu_{(1-\theta)^*}}, \mathbf{1}_{\lambda_{\beta^*}}\right)\right] \\
&= \zeta_2(\mu, \lambda),
\end{aligned}$$

which establishes that $\zeta_1 \leq \zeta_2$. The converse is immediate. □

Theorem 5.7. *Let $(X, \mathscr{P})$ and $(Y, \mathscr{H})$ be T-syntopogenous spaces. A function $f : X \to Y$ is a syntopogenous map, if and only if for every $\eta \in \mathscr{H}$ there is $\zeta \in \mathscr{P}$ such that*

$$\eta(\mathbf{1}_H, \mathbf{1}_M) \leq \zeta(\boldsymbol{f}^{\leftarrow}(\mathbf{1}_H), f^{\leftarrow}(\mathbf{1}_M)), \qquad H,\ M \in 2^Y.$$

Proof. The " only if " part, obviously follows. For the " if " part , suppose f satisfies the stated condition and $\eta \in \mathscr{H}$. Then by (TT5'), we have for every $\nu,\ \rho \in I^Y$,

$$\begin{aligned}
\eta(\nu,\rho) &= \bigwedge_{\theta,\beta\in I} \left[\theta T^*\beta T^*\eta(\mathbf{1}_{\nu_{(1-\theta)^*}}, \mathbf{1}_{\rho_{\beta^*}})\right] \\
&\le \bigwedge_{\theta,\beta\in I} \left[\theta T^*\beta T^*\zeta(f^{\leftarrow}(\mathbf{1}_{\nu_{(1-\theta)^*}}), f^{\leftarrow}(\mathbf{1}_{\rho_{\beta^*}}))\right], \quad \text{by hypothesis} \\
&= \bigwedge_{\theta,\beta\in I} \left[\theta T^*\beta T^*\zeta(\mathbf{1}_{f^{\leftarrow}(\nu_{(1-\theta)^*})}, \mathbf{1}_{f^{\leftarrow}(\rho_{\beta^*})})\right], \quad \text{obvious} \\
&= \bigwedge_{\theta,\beta\in I} \left[\theta T^*\beta T^*\zeta(\mathbf{1}_{[f^{\leftarrow}(\nu)]_{(1-\theta)^*}}, \mathbf{1}_{[f^{\leftarrow}(\rho)]_{\beta^*}})\right], \quad \text{clear} \\
&= \zeta(f^{\leftarrow}(\nu), f^{\leftarrow}(\rho)).
\end{aligned}$$

This proves that f is a syntopogenous map. □

Theorem 5.8. *Let $(X,\mathscr{P})$ and $(Y,\mathscr{H})$ be T-syntopogenous spaces. A function $f : X \to Y$ is a syntopogenous map, if and only if for every $\eta \in \mathscr{H}$ there is $\zeta \in \mathscr{P}$ such that*

$$\eta(f(\mathbf{1}_E), \underline{\underline{1}} - f(\mathbf{1}_G)) \le \zeta(\mathbf{1}_E, \mathbf{1}_{(X-G)}), \quad E,\ G \in 2^X.$$

Proof. The proof is analogous to the above one. □

6. Conclusion

This manuscript introduces a new structure of T-syntopogenous spaces which is interpreted as enlarge of fuzzy syntopogenous spaces introduced by A. K. Katsaras (1990). It gives express the concept of T-syntopogenous spaces in terms of fuzzy binary relations in power sets. The motivation of this study is to will lead us and contribute, in future research, to show that the T-syntopogenous structures compatible with fuzzy T-uniform structures (1998), T-proximity and T-neighbourhood structures (2002).

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