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Some kinds of idealistic soft Γ-rings induced by fuzzy ideals

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Abstract. Based on "∈-*soft set*" and "q-*soft set*", some kinds of idealistic soft Γ-rings are given. Then, an idealistic soft quotient Γ-ring induced by a fuzzy ideal is introduced.

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1. Introduction

In dealing with the complicated problems in economics, engineering and environmental sciences, we are usually unable to apply the classical methods because there are various uncertainties in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets, rough sets, soft sets, i.e., which can be used as the fundamental tools for dealing with uncertainties.

Fuzzy sets, introduced by Zadeh [28], has been extensively applied to many scientific fields. In 1971, Rosenfeld [27] applied the concept to the theory of groupoids and groups. In 1982, Liu [14] defined and studied fuzzy subrings as well as fuzzy ideals. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature. The various constructions of fuzzy quotients rings and fuzzy isomorphisms have been investigated respectively by several researchers (see [5, 17, 18]).

After the concept of soft sets has been introduced by Molodtsov [24] in 1999, soft sets theory has been extensively studied by many authors. Maji et al. [21, 22] pointed out several directions for the applications of soft sets, they also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Aktas et al. [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences. In 2001, Maji et al. [23] defined fuzzy soft sets based on fuzzy sets and soft sets. In 2010, Feng et al. [6] introduced an adjustable approach to fuzzy soft set based decision making. In 2011, Feng et al. [8] introduced the notions of soft rough sets based on rough sets and soft sets.

The algebraic structure of soft set theory and fuzzy soft set theory dealing with uncertainties has been studied by some authors. Aktas et al. [1] applied the notion of soft sets to the theory of groups. Jun [10] introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties. We also noticed that Feng et al. [7] have already investigated the structure of soft semirings. In [15], we have proposed the definition of soft rings, given some properties of soft rings, and established three isomorphism theorems. Furthermore, we gave three fuzzy isomorphism theorems of soft rings in [16]. Recently, İnan et al. [9] introduced fuzzy soft rings and $(\in, \in \vee q)$-fuzzy soft subrings that was generalization of fuzzy soft rings.

Γ-rings are more general than rings, many mathematicians made good works on Γ-rings. In [11], Jun and Lee defined fuzzy Γ-rings. Dutta and Chanda studied the structures of fuzzy ideals of Γ-rings in [3] and introduced the notions of fuzzy prime ideals of Γ-rings in [4]. Recent research on fuzzy Γ-rings can be found in [19, 20, 26]. In [25] Öztürk et al. introduced soft Γ-ring, soft ideals and idealistic soft Γ-rings based on soft set theory.

In this paper, we attempt to study Γ-ring theory by using both soft sets and fuzzy sets. We first give some idealistic soft Γ-rings induced by $(\in, \in \vee q)$-fuzzy ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals. Next, we introduce an idealistic soft quotient Γ-ring induced by a fuzzy ideal.

2. Preliminaries

We let U be an initial universe and E a set of parameters. Denote the power set of U by $P(U)$ and consider $A \subset E$. Then we formulate the following definition.

Definition 2.1 ([24])**.** A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.2 ([3])**.** Let S and Γ be two additive abelian groups. S is called a Γ-ring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ by $(a, \alpha, b) \mapsto a\alpha b$ satisfying the following conditions:

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(2) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(3) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
If $a\alpha b = b\alpha a$ for all $a, b \in S$ and for all $\alpha \in \Gamma$, then S is said to be commutative.

Definition 2.3 ([3])**.** A left (resp., right) ideal of of a Γ-ring S is a subset A of S which is an additive subgroup of S and $S\Gamma A \subset A$ (resp.,$A\Gamma S \subset A$), where $S\Gamma A = \{x\alpha y \mid x \in S, y \in A, \alpha \in \Gamma\}$. If A is both a left and a right ideal, then A is called an ideal of S.

Let S be a Γ-ring and I an ideal of S, $x + I = \{y \in S \mid x - y \in I\}$ and $S/I = \{x + I \mid x \in S\}$, define two operations by

$$(x + I) + (y + I) = (x + y) + I,$$

$$(x + I)\alpha(y + I) = (x\alpha y) + I$$

for all $x, y \in S$ and for all $\alpha \in \Gamma$.

The following propositions are similar to rings, and we omit the proofs.

Proposition 2.4. *Let S be a Γ-ring and I an ideal of S, then S/I is also a Γ-ring.*

Proposition 2.5. *Let S be a Γ-ring, I and J two ideals of S with $I \subset J$, then J/I is an ideal of S/I.*

Definition 2.6 ([4]). Let S and K be two Γ-rings, and f a mapping of S into K. Then f is called a Γ-homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in S, \alpha \in \Gamma$. If f is surjective(resp., bijective), then f is called a Γ-epimorphism(resp., a Γ-isomorphism).

Definition 2.7 ([3]). A fuzzy set μ in a Γ-ring S is called a fuzzy ideal of S if, for all $x, y \in S$ and $\alpha \in \Gamma$, the following requirements are met:

(5) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and

(6) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

If $\mu(a\alpha b) = \mu(b\gamma a)$ for all $a, b \in S$ and for all $\alpha \in \Gamma$, then μ is said to be commutative.

Definition 2.8 ([19]). A fuzzy subset μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support x and value t, denoted by x_t.

Definition 2.9 ([19]). A fuzzy point x_t is said to "belong to" (resp., be quasico-incident with) a fuzzy set μ, written by $x_t \in \mu$ (resp., $x_t q\mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$).

If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we write $x_t \overline{\in} \mu$ (resp., $x_t \overline{q}\mu$). The symbol $\overline{\in \vee q}$ is to mean that $\in \vee q$ does not hold.

3. Idealistic soft Γ-rings induced by $(\in, \in \vee q)$-fuzzy ideals

Definition 3.1 ([25]). Let (F, A) be a soft set over Γ-ring S, then (F, A) is called a soft Γ-ring over S if $F(x)$ is a subΓ-ring of S for all $x \in A$.

Definition 3.2 ([25]). Let (F, A) be a soft Γ-ring over Γ-ring S, then (F, A) is said to be an idealistic soft Γ-ring over S if $F(x)$ is an ideal of S for all $x \in A$.

Example 3.3. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and (F, A) be a soft set over S, where $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \iff \overline{xy} \in \{\overline{0}, \overline{2}, \overline{4}\}$ for all $\overline{x} \in A$. Then $F(\overline{0}) = Z_6, F(\overline{1}) = \{\overline{0}, \overline{2}, \overline{4}\}, F(\overline{2}) = Z_6$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. It is clear that S is a Γ-ring and (F, A) is an idealistic soft Γ-ring over S.

Given a fuzzy set μ in any Γ-ring S and $A \subset [0, 1]$, consider the following two set-valued functions

$$\mathscr{F} : A \to \mathcal{P}(S), \quad t \mapsto \{x \in S \mid x_t \in \mu\}$$

and

$$\mathscr{F}_{\mathrm{q}} : A \to \mathcal{P}(S), \quad t \mapsto \{x \in R \mid x_t \,\mathrm{q}\, \mu\}.$$

Then $(\mathscr{F}, A)$ and $(\mathscr{F}_{\mathrm{q}}, A)$ are called an "$\in$-*soft set*" and "q-*soft set*" over S, respectively, see [12, 13, 29].

Proposition 3.4 ([11])**.** *A fuzzy set μ in a Γ-ring S is a fuzzy ideal of S if and only if $U(\mu, \alpha) = \{x \in S \mid \mu(x) \geq \alpha)$ is an ideal of S.*

Theorem 3.5. *Let μ be a fuzzy set in a Γ-ring S and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is an idealistic soft Γ-ring over S if and only if μ is a fuzzy ideal of S.*

Proof. It follows from Proposition 3.4 and thus its proof is omitted. □

Theorem 3.6. *Let μ be a fuzzy set in a Γ-ring S and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S if and only if μ is a fuzzy ideal of S.*

Proof. Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S, then for all $t \in A$, $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S, we must prove (5) and (6) hold. If (5) is not true, then there exist $a, b \in S$ such that $\mu(a - b) < \min\{\mu(a), \mu(b)\}$, then we can choose $t \in A$ such that $\mu(a - b) + t \leq 1 < \min\{\mu(a), \mu(b)\} + t$. Hence , $\mu(a) + t > 1$ and $\mu(b) + t > 1$, but $\mu(a - b) + t \leq 1$, i.e., $a, b \in F_{\mathrm{q}}(t)$, $a - b \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is a contradiction. If (6) is not true, then for a fixed $\alpha \in \Gamma$, there exist $a, b \in S$ such that $\mu(a\alpha b) < \max\{\mu(a), \mu(b)\}$. If $\mu(a) \geq \mu(b)$, then $\mu(a\alpha b) < \mu(a)$, we can choose $t \in A$ such that $\mu(a\alpha b) + t \leq 1 < \mu(a) + t$. Hence , $\mu(a) + t > 1$, but $\mu(a\alpha b) + t \leq 1$, i.e., $a \in F_{\mathrm{q}}(t)$, $b \in S$, $a\alpha b \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is a contradiction. If $\mu(a) < \mu(b)$, we can also get a contradiction. Therefore, μ is a fuzzy ideal of S.

Conversely, suppose that μ is a fuzzy ideal of S. Let $t \in A$, $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, $z \in S$ and $\alpha \in \Gamma$, then $\mu(x-y)+t \geq \min\{\mu(x), \mu(y)\}+t > 1$, $\mu(x\alpha z)+t \geq \max\{\mu(x), \mu(z)\}+t > 1$ and $\mu(z\alpha x) + t \geq \max\{\mu(x), \mu(z)\} + t > 1$, so $x - y \in \mathscr{F}_{\mathrm{q}}(t)$, $x\alpha z, z\alpha x \in \mathscr{F}_{\mathrm{q}}(t)$. This proves that $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S and hence $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S. □

In the following theorems, we characterize the soft Γ-rings over S by $(\in, \in \vee q)$-fuzzy ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals of S.

Definition 3.7 ([19])**.** A fuzzy set μ is called an $(\in, \in \vee q)$-fuzzy ideal of S if for all $t, r \in (0, 1]$, $x, y \in S$ and $\alpha \in \Gamma$, the following conditions are satisfied:
(7) $x_t \in \mu$ and $y_r \in \mu$ imply $(x - y)_{\min(t,r)} \in \vee q\mu$,
(8) $x_t \in \mu$, imply $(x\alpha y)_t \in \vee q\mu$ and $(y\alpha x)_t \in \vee q\mu$.

Lemma 3.8 ([19])**.** *A fuzzy set μ in a Γ-ring S is an $(\in, \in \vee q)$-fuzzy ideal of S if and only if for $x, y \in S$ and $\alpha \in \Gamma$, the following conditions are satisfied:*
(9) $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(10) $\mu(x\alpha y) \geq \min\{\mu(x), 0.5\}$ *and* $\mu(x\alpha y) \geq \min\{\mu(y), 0.5\}$.

Theorem 3.9. Let μ be a fuzzy set in a Γ-ring S and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is an idealistic soft Γ-ring over S if and only if μ is an $(\in, \in \vee q)$-fuzzy ideals of S.

Proof. Assume that $(\mathscr{F}, A)$ is an idealistic soft Γ-ring over S, then $\mathscr{F}(t)$ is an ideal of S for all $t \in A$, we must prove (9) and (10) hold. If (9) is not true, then there exist $x, y \in S$ such that $\mu(x - y) < \min\{\mu(x), \mu(y), 0.5\}$, then we can choose $t \in (0, 0.5]$ such that $\mu(x - y) < t \leq \min\{\mu(x), \mu(y), 0.5\}$. Thus $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(x - y) < t$, that is, $x, y \in \mathscr{F}(t)$, but $x - y \overline{\in} \mathscr{F}(t)$, which is a contradiction. If (10) is not true, then there exist $x, y \in S$ and $\alpha \in \Gamma$ such that $\mu(x\alpha y) < \min\{\mu(x), 0.5\}$ or $\mu(x\alpha y) < \min\{\mu(y), 0.5\}$. If $\mu(x\alpha y) < \min\{\mu(x), 0.5\}$, we can choose $t \in A$ such that $\mu(x\alpha y) < t \leq \min\{\mu(x), 0.5\}$, that is, $x \in \mathscr{F}(t)$, but $x\alpha y \overline{\in} \mathscr{F}(t)$, which is a contradiction. If $\mu(x\alpha y) < \min\{\mu(y), 0.5\}$, we can also get a contradiction. By Lemma 3.8, μ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy ideal of S. Let $t \in A$, if $x, y \in \mathscr{F}(t)$, $z \in S$ and $\alpha \in \Gamma$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. These imply that $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t$ and so $x - y \in \mathscr{F}(t)$. We can also show that $x\alpha z \in \mathscr{F}(t)$ and $z\alpha x \in \mathscr{F}(t)$. Thus $\mathscr{F}(t)$ is an ideal of S and $(\mathscr{F}, A)$ is indeed an idealistic soft Γ-ring over S. □

Definition 3.10 ([19])**.** A fuzzy set μ is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S if for all $t, r \in (0, 1]$, $x, y \in S$ and $\alpha \in \Gamma$, the following conditions are satisfied:

(11) $(x - y)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$,

(12) $(x\alpha y)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee \overline{q} \mu$ and $x_t \overline{\in} \vee \overline{q} \mu$.

Lemma 3.11 ([19])**.** *A fuzzy set μ in a Γ-ring S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S if and only if for $x, y \in S$ and $\alpha \in \Gamma$, the following conditions are satisfied:*

(13) $\max\{\mu(x - y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,

(14) $\max\{\mu(x\alpha y), 0.5\} \geq \max\{\mu(x), \mu(y)\}$.

Theorem 3.12. *Let μ be a fuzzy set in a Γ-ring S and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is an idealistic soft Γ-ring over S if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.*

Proof. Let $(\mathscr{F}, A)$ be an idealistic soft Γ-ring over S. Then $\mathscr{F}(t)$ is an ideal of S for all $t \in A$. If there exist $x, y \in S$ such that $\max\{\mu(x - y), 0.5\} < \min\{\mu(x), \mu(y)\}$, then we can choose $t \in A$ such that $\max\{\mu(x - y), 0.5\} < t \leq \min\{\mu(x), \mu(y)\}$, then, $x, y \in \mathscr{F}(t)$, but $x - y \overline{\in} \mathscr{F}(t)$, which is a contradiction, and hence, $\max\{\mu(x - y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$. If there exist $x, y \in S$ and $\alpha \in \Gamma$ such that $\max\{\mu(x\alpha y), 0.5\} < \max\{\mu(x), \mu(y)\}$, then $\max\{\mu(x\alpha y), 0.5\} < \mu(x)$ or $\max\{\mu(x\alpha y), 0.5\} < \mu(y)$. If $\max\{\mu(x\alpha y), 0.5\} < \mu(x)$, then we can choose $t \in A$ such that $\max\{\mu(x\alpha y), 0.5\} < t \leq \mu(x)$, then $x \in \mathscr{F}(t)$, but $x\alpha y \overline{\in} \mathscr{F}(t)$, which is a contradiction. We have proved that $\max\{\mu(x\alpha y), 0.5\} \geq \mu(x)$ for all $x, y \in S$ and $\alpha \in \Gamma$. By the same arguments, we can also prove that $\max\{\mu(x\alpha y), 0.5\} \geq \mu(y)$ for all $x, y \in S$ and $\alpha \in \Gamma$. Thus, by Lemma 3.11, μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S. If we let $t \in A$ with $x, y \in \mathscr{F}(t)$, $\alpha \in \Gamma$, then $\mu(x) \geq t > 0.5$, $\mu(y) \geq t > 0.5$, and hence $\max\{\mu(x - y), 0.5\} \geq \min\{\mu(x), \mu(y)\} \geq t, \max\{\mu(x\alpha y), 0.5\} \geq \mu(y) \geq t$ and $\max\{\mu(x\alpha y), 0.5\} \geq \mu(x) \geq t$. Thus, $\mu(x - y) \geq t$ and $\mu(x\alpha y) \geq t$ and $\mu(x\alpha z) \geq t$, that is, $x - y \in \mathscr{F}(t)$, $z\alpha x \in \mathscr{F}(t)$ and $x\alpha z \in \mathscr{F}(t)$. These show that $\mathscr{F}(t)$ is an ideal of S and $(\mathscr{F}, A)$ is an idealistic soft Γ-ring over S. □

Theorem 3.13. Let μ be a fuzzy set in a Γ-ring S and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.

Proof. Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S. Then $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S for all $t \in A$. If there exist $x, y \in S$ such that $\max\{\mu(x - y), 0.5\} < \min\{\mu(x), \mu(y)\}$, then we can select $t \in A$ such that $\max\{\mu(x - y), 0.5\} + t \leq 1 < \min\{\mu(x), \mu(y)\} + t$, then $x, y \in F_{\mathrm{q}}(t)$, but $x - y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is a contradiction.

Hence, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$. In the same way, we can also prove that $\max\{\mu(x\alpha y), 0.5\} \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$. It hence follows from Lemma 3.11 that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S. Then for all $x, y \in S$ and $\alpha \in \Gamma$, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ and $\max\{\mu(x\alpha y), 0.5\} \geq \max\{\mu(x), \mu(y)\}$. Let $t \in A$, $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, $\alpha \in \Gamma$ and $z \in S$. Then $\mu(x) + t > 1$, $\mu(y) + t > 1$, hence, $\max\{\mu(x-y), 0.5\} + t \geq \min\{\mu(x), \mu(y)\} + t$, $\max\{\mu(z\alpha x), 0.5\} + t \geq \max\{\mu(x), \mu(z)\} + t$ and $\max\{\mu(x\alpha z), 0.5\} + t \geq \max\{\mu(x), \mu(z)\} + t$. Thus, $\mu(x-y) + t > 1$, $\mu(z\alpha x) + t > 1$, and $\mu(x\alpha z) + t > 1$, i.e., $x - y \in \mathscr{F}_{\mathrm{q}}(t)$, $z\alpha x \in \mathscr{F}_{\mathrm{q}}(t)$ and $x\alpha z \in \mathscr{F}_{\mathrm{q}}(t)$, so $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S and $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S. □

Theorem 3.14. Let μ be a fuzzy set in a Γ-ring S and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S if and only if μ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Proof. Let $(\mathscr{F}_{\mathrm{q}}, A)$ be an idealistic soft Γ-ring over S. Then $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S, for all $t \in A$. If there exist $x, y \in S$ such that $\mu(x-y) < \min\{\mu(x), \mu(y), 0.5\}$, then we can choose $t \in (0.5, 1]$ such that $\mu(x-y) + t \leq 1 < \min\{\mu(x), \mu(y), 0.5\} + t$. Thus $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, but $x - y \overline{\in} F_{\mathrm{q}}(t)$. This is a contradiction. Hence, $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in S$. By using the same arguments, we can prove that $\mu(z\alpha x) \geq \min\{\mu(x), \mu(z), 0.5\}$ and $\mu(x\alpha z) \geq \min\{\mu(x), \mu(z), 0.5\}$ for all $x, z \in S$ and $\alpha \in \Gamma$. It follows from Lemma 3.8 that μ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy ideal of S. Using Lemma 3.8, we have $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$,$\mu(x\alpha y) \geq \min\{\mu(y), 0.5\}$ and $\mu(x\alpha y) \geq \min\{\mu(x), 0.5\}$ for all $x, y \in S$ and $\alpha \in \Gamma$. Let $t \in A$, $x, y \in F_{\mathrm{q}}(t)$, $z \in S$ and $\alpha \in \Gamma$, then $\mu(x) + t > 1$ and $\mu(y) + t > 1$. These imply that $\mu(x-y) + t \geq \min\{\mu(x), \mu(y), 0.5\} + t > 1$, and so $x - y \in \mathscr{F}_{\mathrm{q}}(t)$. We can also similarly prove that $z\alpha x, x\alpha z \in F_{\mathrm{q}}(t)$. Thus $\mathscr{F}_{\mathrm{q}}(t)$ is an ideal of S and $(\mathscr{F}_{\mathrm{q}}, A)$ is an idealistic soft Γ-ring over S. □

4. An idealistic soft quotient Γ-ring induced by a fuzzy ideal

Let μ be a fuzzy ideal of a Γ-ring S. For any $x, y \in S$, define a binary relation $\sim$ on S by $x \sim y$ if and only if

$$\mu(x-y) = \mu(0),$$

where 0 is the zero element of S.

Theorem 4.1. *$\sim$ is a congruence relation of S.*

Proof. (a) For any $x \in S$, $\mu(x-x) = \mu(0)$, then $x \sim x$.

(b) For any $x, y \in S$, if $x \sim y$, then $\mu(x-y) = \mu(0)$, $\mu(y-x) = \mu(0)$, hence $y \sim x$.

(c) For any $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $\mu(x-y) = \mu(0)$, $\mu(y-z) = \mu(0)$, $\mu(x-z) = \mu((x-y) - (y-z)) = \mu(0)$, i.e., $x \sim z$.

(d) For any $x, y, z \in S$, if $x \sim y$, then $\mu(x-y) = \mu(0)$, $\mu((x+z) - (y+z)) = \mu(x-y) = \mu(0)$, it means $(x+z) \sim (y+z)$.

(e) For any $x, y, u, v \in S$, if $x \sim y$ and $u \sim v$, it follows from (4) that $(x+u) \sim (y+u)$ and $(y+u) \sim (y+v)$, by (3), we obtain $(x+u) \sim (y+v)$. Similarly, we have $(x+v) \sim (y+u)$. The proof is complete. □

Let $\mu_x = \{y \in S \mid y \sim x\}$ be the equivalence class containing x and $S/\mu = \{\mu_x \mid x \in S\}$ the set of all equivalence classes of S. Define two operations by

$$\mu_x + \mu_y = \mu_{x+y},$$

$$\mu_x \alpha \mu_y = \mu_{x\alpha y}$$

for $x, y \in S$, $\alpha \in \Gamma$. We can verify that the two operations are well defined and S/μ is a Γ-ring, the Γ-ring is called the quotient Γ-ring of S induced by fuzzy ideal μ.

Lemma 4.2. *Let I be an ideal of a Γ-ring S and μ a fuzzy ideal of S. We have*
(i) *if μ is restricted to I, written μ_I , then μ_I is fuzzy ideal of I;*
(ii) *I/μ_I is an ideal of S/μ.*

Proof. (i)It follows from the definition of fuzzy ideals of a Γ-ring that μ_I is fuzzy ideal of I.

(ii) Let μ_{I_x},$\mu_{I_y} \in I/\mu_I$, $\mu_z \in S/\mu$ and $\alpha \in \Gamma$, where $x, y \in I$ and $z \in S$, then $x-y \in I$, $x\alpha z \in I$ and $z\alpha x \in I$, hence $\mu_{I_x} - \mu_{I_y} = \mu_{I_{x-y}} \in I/\mu_I$, $\mu_{I_x}\alpha\mu_z = \mu_{x\alpha z} = \mu_{I_{x\alpha z}} \in I/\mu_I$ and $\mu_z\alpha\mu_{I_x} = \mu_{z\alpha x} = \mu_{I_{z\alpha x}} \in I/\mu_I$. Therefore, I/μ_I is an ideal of S/μ. □

Let μ be a fuzzy ideal of a Γ-ring S and (F, A) an idealistic soft Γ-ring over S. Now, we restrict μ to $F(x)$ for all $x \in A$, then for all $x \in A$, μ is fuzzy ideal of $F(x)$ and $F(x)/\mu$ is an ideal of S/μ. Thus, we can define a set-valued function $F/\mu : A \to P(S/\mu)$ by $(F/\mu)(x) = F(x)/\mu$.

Theorem 4.3. If μ is a fuzzy ideal of a Γ-ring S and (F, A) is an idealistic soft Γ-ring over S, then $(F/\mu, A)$ is an idealistic soft Γ-ring over S/μ.

Proof. For any $x \in A$, $F(x)$ is an ideal of S, it follows from Lemma 4.2 that $F(x)/\mu$ is an ideal of S/μ and thus $(F/\mu, A)$ is an idealistic soft Γ-ring over S/μ. □

Remark 4.4. *The $(F/\mu, A)$ is called the idealistic soft quotient Γ-ring of (F, A) induced by fuzzy ideal μ.*

Example 4.5. Let $S = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$, and (F, A) be a soft set over S, where $A = \{\overline{0}, \overline{1}, \overline{2}\}$ and $F : A \to P(S)$ is defined by $F(\overline{x}) = \{\overline{y} \in S \mid \overline{x}\rho\overline{y} \Longleftrightarrow \overline{xy} = \overline{0}\}$ for all $\overline{x} \in A$. And let $\Gamma = Z$, define $\overline{x}\alpha\overline{y} = \overline{x\alpha y}$ for all $\overline{x}, \overline{y} \in S, \alpha \in \Gamma$. Then $F(\overline{0}) = Z_6$, $F(\overline{1}) = \{\overline{0}\}$, $F(\overline{2}) = \{\overline{0}, \overline{3}\}$ are ideals of S and (F, A) is an idealistic soft Γ-ring over S.

A fuzzy subset μ in Z_6 is defined by

$$\mu(\overline{x}) = \begin{cases} 0.8, & \overline{x} \in \{\overline{0}, \overline{3}\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

It is easy to verify that μ is a fuzzy ideal of S. We can obtain that $(F/\mu)(\overline{0}) = F(\overline{0})/\mu = \{\{\overline{0}, \overline{3}\}, \{\overline{2}, \overline{5}\}, \{\overline{1}, \overline{4}\}\}$, $(F/\mu)(\overline{1}) = F(\overline{1})/\mu = \{\{\overline{0}\}\}$, $(F/\mu)(\overline{2}) = F(\overline{2})/\mu = \{\{\overline{0}, \overline{3}\}\}$. Then $(F/\mu, A)$ is an idealistic soft quotient Γ-ring of (F, A) induced by fuzzy ideal μ.

Definition 4.6. An idealistic soft Γ-ring over S is said to be commutative if $F(x)$ is a commutative ideal of S for all $x \in A$.

Theorem 4.7. *If μ is a commutative fuzzy ideal of a Γ-ring S, then $(F/\mu, A)$ is commutative.*

Proof. $F(x)/\mu$ is an ideal of S/μ for all $x \in A$. Let $\mu_a, \mu_b \in F(x)/\mu$, since μ is a commutative fuzzy ideal of S, then $\mu_a \gamma \mu_b = \mu_{a\gamma b} = \mu_{b\gamma a} = \mu_b \gamma \mu_a$. Hence, $(F/\mu, A)$ is commutative. □

Lemma 4.8. *Let χ_I be a characteristic function of a subset I of a Γ-ring S. Then χ_I is a fuzzy ideal of S if and only if I is an ideal of S.*

Proof. Assume that I is an ideal of S. For any $x, y \in S$ and $\alpha \in \Gamma$, if $x, y \in I$, then $x - y \in I$ and $x\alpha y \in I$. Thus $\chi_I(x-y) = \chi_I(x\alpha y) = \chi_I(x) = \chi_I(y) = 1$. Therefore $\chi_I(x-y)$=$\min\{\chi_I(x), \chi_I(y)\}$ and $\chi_I(x\alpha y)$=$\max\{\chi_I(x), \chi_I(y)\}$. If one of x and y is not in I, then $x\alpha y \in I$ and one of $\chi_I(x)$ and $\chi_I(y)$ is 0. So, $\chi_I(x-y) \geq \min\{\chi_I(x), \chi_I(y)\}$ and $\chi_I(x\alpha y) \geq \max\{\chi_I(x), \chi_I(y)\}$. If neither of x and y is in I, then $\chi_I(x)$ and $\chi_I(y)$ are both 0, therefore $\chi_I(x-y) \geq \min\{\chi_I(x), \chi_I(y)\}$ and $\chi_I(x\alpha y) \geq \max\{\chi_I(x), \chi_I(y)\}$. Hence, χ_I is a fuzzy ideal of S.

Conversely, suppose that χ_I is a fuzzy ideal of S. If $x, y \in I$, $z \in S$, we have $\chi_I(x) = \chi_I(y) = 1$. Since $\chi_I(x-y) \geq \min\{\chi_I(x), \chi_I(y)\}$, $\chi_I(x\alpha z) \geq \max\{\chi_I(x), \chi_I(z)\}$ and $\chi_I(z\alpha x) \geq \max\{\chi_I(x), \chi_I(z)\}$. Then $\chi_I(x-y) = 1$, $\chi_I(x\alpha z) = \chi_I(z\alpha x) = 1$. Then $x - y \in I$, $x\alpha z \in I$ and $z\alpha x \in I$. Thence, I is an ideal of S. □

Let I be an ideal of a Γ-ring S. The quotient Γ-ring S/I induced by an ideal I is determined by an equivalent relation $\sim$, where $x \sim y$ is defined by $x - y \in I$. For our convenience, we write $x \sim y(I)$ to show that x is equivalent to y with respect to the ideal I, and $x \sim y(\chi_I)$ to mean that x is equivalent to y with respect to the fuzzy ideal χ_I.

Lemma 4.9. *Suppose that I is an ideal of a Γ-ring S. Then $x \sim y(I)$ if and only if $x \sim y(\chi_I)$.*

Proof. $x \sim y(I) \Longleftrightarrow x - y \in I \Longleftrightarrow \chi_I(x-y) = 1 \Longleftrightarrow \chi_I(x-y) = \chi_I(0) \Longleftrightarrow x \sim y(\chi_I)$. □

Corollary 4.10. Let I be a commutative ideal of a Γ-ring S, then $(F/\chi_I, A)$ is commutative.

Corollary 4.11. Let I be a commutative ideal of a Γ-ring S and $I \subset F(x)$ for all $x \in A$, then $(F/I, A)$ is commutative.

Definition 4.12 ([25])**.** Let (F, A) and (G, B) be two soft Γ-rings over Γ-rings M and N, respectively. Let $f : M \to N$ and $g : A \to B$ be two functions. Then the pair (f, g) is called a soft Γ-homomorphism if it satisfies the following conditions:

(i) f is a Γ-epimorphism,
(ii) g is onto and
(iii) $f(F(x)) = G(g(x))$ for all $x \in A$.

We say that (F, A) is soft Γ-homomorphic to (G, B), and is denoted by $(F, A) \sim_\Gamma (H, B)$. Moreover, if f is an Γ-isomorphism and g is a bijective mapping, then (f, g) is called a soft Γ-isomorphism, which is denoted by $(F, A) \simeq_\Gamma (H, B)$.

Theorem 4.13 ([25])**.** *Let $f : S \to K$ be a Γ-epimorphism of Γ-rings. If (F, A) is an idealistic soft Γ-ring over S, then $(f(F), A)$ is an idealistic soft Γ-ring over K.*

Lemma 4.14 ([11]). *If $f : S \to K$ is a Γ-epimorphism of Γ-rings and μ a fuzzy ideal of S, then $f(\mu)$ is a fuzzy ideal of K.*

Theorem 4.15. *Let $f : S \to K$ be a Γ-epimorphism of Γ-rings and μ a fuzzy ideal of S. If (F, A) is an idealistic soft Γ-ring over S. then $(f(F)/f(\mu), A)$ is an idealistic soft Γ-ring over $K/f(\mu)$.*

Proof. We have $(f(F), A)$ is an idealistic soft Γ-ring over K by Theorem 4.13 and $f(\mu)$ is a fuzzy ideal of K by Lemma 4.14. Then we can deduce that $(f(F)/f(\mu), A)$ is an idealistic soft Γ-ring over $K/f(\mu)$ from Theorem 4.3. □

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