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Some contributions to soft groups

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Abstract. Molodtsove introduced the concept of soft set theory which can be used as a generic mathematical tool for dealing with uncertainty. In this paper, the concept of soft group is extended and in the meantime, some of their properties and structural characteristics are discussed and studied. Furthermore, definitions of normal soft group, cyclic soft group, abelian soft group, product of soft groups, coset of a soft subgroup of a soft group are defined. After that, factor soft group, maximal normal soft group, simple soft group are defined and some important results are proved.

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1. Introduction

Most of the problems in engineering, medical science, economics, environments etc. have various uncertainties. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy sets [10], intuitionistic fuzzy sets [2], rough sets [9], i.e., which can be used as mathematical tools for dealing with uncertainties. As was mentioned [7], these theories have their own difficulties. In 1999, Molodsov [7] initiated a novel concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainty.

A soft set is a parameterized family of subsets of the universal set. We can say that soft sets are neighborhood systems, and that they are a special case of context-dependent fuzzy sets. In soft set theory the problem of setting the membership function, among other related problems, simply does not arise. This makes the theory very convenient and easy to apply in practice. Soft set theory has potential applications in many different fields, including the game theory, operations research, Fiemann integration, Perron integration, probability theory, and measurement theory. After Molodsov's work, some different applications of soft sets were studied in

[3] and [6]. Furthermore Maji, Biswas and Roy worked on soft set theory in [5]. Some corrections were given by M. Irfan at el in [4].

In this paper, we extend the idea of soft groups introduced by Aktas and Cagman in [1] and develop the soft group theory.

This paper is organized as follows. In the preliminaries, we give the concept of soft set and discuss the binary operations on soft sets. In section three, we introduce the definition of restricted and extended intersection, restricted and extended union of soft groups and give some fundamental properties of of soft groups. In the forth and fifth sections, we define normal soft groups, Abelian soft groups and Abelian soft Subgroups of a soft group and check out their properties. In the six and seventh sections, we define restricted product of soft groups, inverse of a soft group, identity soft group and prove some results by using cartesian product of soft groups. In the eight and ninth sections, we define cyclic soft groups, cyclic soft subgroups, soft coset of a soft subgroup of a soft group, order of a soft group, soft index, partition of a soft set over a group and prove their related results. In the second last section, we introduce soft maximal normal subgroups, simple soft groups and factor soft groups. We establish some results as well.

2. Preliminaries

In this section as a beginning, the concept of soft set introduced by Molodsov [7], binary operations on soft sets by Maji et al. [5] and corrections on these binary operations by M. Irfan et al. [4], will be presented.

Definition 2.1 ([7]). Let U be an initial universe and E be a set of parameters. For $A \subseteq E$, the pair (F, A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of U.

It has been interpreted that a soft set is indeed a parameterized family of subsets of U, and thus E is referred to as a set of parameters. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [7] and [8], one of which we present below.

Example 2.2 ([7]). Suppose the following:

U is the set of houses under consideration.

E is the set of parameters. Each parameter is a word or a sentence.

$E = \{$expensive, beautiful, wooden, cheap, in the green surroundings, modern, in good repair, in bad repair$\}$.

In this case, to define a soft set means to point out expensive houses and so on. Let the soft set (F, A) describes the "attractiveness of the houses" which Mr. X (say) is going to buy.

Suppose that there are six houses in the universe U given by

$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $A = \{e_1, e_2, e_3, e_4, e_5\}$ where

e_1 stands for the parameter 'expensive',

e_2 stands for the parameter 'beautiful',

e_3 stands for the parameter 'wooden',

e_4 stands for the parameter 'cheap',

e_5 stands for the parameter 'in the green surroundings', Suppose that

$$\begin{aligned} F(e_1) &= \{h_2,\, h_4\}, \\ F(e_2) &= \{h_1,\, h_3\}, \\ F(e_3) &= \{h_3,\, h_4,\, h_5\}, \\ F(e_4) &= \{h_2,\, h_4,\, h_5\}, \\ F(e_5) &= \{h_1\}. \end{aligned}$$

The soft set (F, A) is a parametrized family $\{F(e_i),\ i = 1,2,3...,5\}$ of subsets of the set U and gives us a collection of approximate description of an object. Thus, we can view the soft set (F, A) as a collection of approximations as below:

$(F, A) = \{$expensive $= \{h_2, h_4\}$, beautiful $= \{h_1, h_3\}$, wooden $= \{h_3, h_4, h_5\}$, cheap $= \{h_2, h_4, h_5\}$, in the green surroundings $= \{h_1\}\}$.

Example 2.3 ([7])**.** For a topological space (X, τ), if $F(x)$ is the family of all open neighbourhoods of a point $x \in X$, i.e., $F(x) = \{v \in \tau : x \in v\}$, then the ordered pair (F, X) indeed a soft set over X.

Definition 2.4 ([7])**.** Let (F, A) and (H, B) be two soft sets over a common universe U, then we say that (H, B) is a soft subset of (F, A) if

1. $B \subseteq A$ and
2. $H(e) \subseteq F(e)$ for all $e \in B$.

We write $(H, B) \widetilde{\subset} (F,B)$, and (H, B) is said to be a soft super set of (F, A), if (F, A) is a soft subset of (H, B). We denote it by $(H, B) \widetilde{\supset} (F,B)$.

Definition 2.5 ([7])**.** Two soft sets (F, A) and (H, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (H, B) and $(H\ ,B)$ is a soft subset of (F, A).

Definition 2.6 ([7])**.** A soft set (F, A) over U is said to be a NULL soft set denoted by Φ if for all $e \in A$, $F(e) = \varnothing$(empty set).

Definition 2.7 ([7])**.** A soft set (F, A) over U is said to be an absolute soft set denoted by $\tilde{A}$ if for all $e \in A$, $F(e) = U$.

Definition 2.8 ([5])**.** Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \Cap (G, B) = (H, C)$ where (H,C) is defined as $H(c) = F(c) \cap G(c)$ for all $c \in C$.

Definition 2.9 ([5])**.** The extended intersection of two soft sets (F, A) and (G,B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$, $H(e)$ is defined as

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cap G(e) & if\ e \in A \cap B. \end{cases}$$

We write $(F, A) \sqcap_{\mathbf{E}} (G, B) = (H, C)$.

Definition 2.10 ([5])**.** The restricted union (H, C) of two soft sets (F, A) and (G, B) over the common universe U is defined as the soft set $(H, C) = (F, A) \cup_R (G, B)$, where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Definition 2.11 ([5])**.** Let (F, A) and (G, B) be any two soft sets over a common universe U. Then extended union (H, C) of two soft sets (F, A) and (G, B) is denoted as the soft set $(H, C) = (F, A) \cup_E (G, B)$ where $C = A \cup B$ and H is defined as

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We write $(F, A) \cup_E (G, B) = (G, B)$.

3. Soft groups

H. Aktas and N. Cagman [1] introduced the notion of soft groups, which extends the notion of group to include the algebraic structures of soft sets. A soft group is a parameterized family of subgroups. Now in this section we introduce the definition of restricted and extended intersection, restricted and extended union of soft groups and give some fundamental properties of of soft groups.

Definition 3.1 ([1])**.** Let G be a group and E be a set of parameters. For $A \subseteq E$, the pair (F, A) is called a soft group over G if and only if $F(a) \leqslant G$ for all $a \in A$, where F is a mapping of A into the set of all subsets of G.

Definition 3.2 ([4])**.** Let (F, A) and (G, B) be two soft groups over a group G such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \Cap (G, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap G(c)$ for all $c \in A \cap B$.

Theorem 3.3. *Let (F, A) and (H, B) be two soft groups over G. Then their restricted intersection $(F, A) \Cap (H, B)$ is a soft group over G.*

Proof. Since (F, A) and (H, B) are soft groups over G, their restricted intersection over G is a soft set (L, C), where $C = A \cap B \neq \phi$ and is defined as

$$L(c) = F(c) \cap H(c) \text{ for all } c \in C.$$

We show that for each $c \in C$, $L(c)$ is a subgroup of G. Since $F(c)$ and $H(c)$ are subgroups of G for all $c \in C$. This implies that $L(c) = F(c) \cap H(c)$ is a subgroup of G for all $c \in C$. Hence $(L, C) = (F, A) \Cap (H, B)$ is a soft group over G. ☐

Definition 3.4. The extended intersection of two soft groups (F, A) and (G, B) over a group G is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$.

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cap G(e) & if\ e \in A \cap B. \end{cases}$$

We write $(H, C) = (F, A) \sqcap_{\mathbf{E}} (G, B)$.

Theorem 3.5. *Let (F, A) and (H, B) be two soft groups over G. Then their extended intersection $(F, A) \sqcap_E (H, B)$ is a soft group over G.*

Proof. Since (F, A) and (H, B) are soft groups over G, their extended intersection over G is a soft set (L, C), where $C = A \cup B$ and for all $c \in C$, it is defined as

$$L(c) = \begin{cases} F(c) & if\ c \in A - B \\ H(c) & if\ c \in B - A \\ F(c) \cap H(c) & if\ c \in A \cap B. \end{cases}$$

From here we have, $F(c)$ is a subgroup of G for all $c \in A \backslash B$. This implies that $L(c)$ is a subgroup of G for all $c \in A \backslash B$. Similarly, $L(c) = H(c)$ is a subgroup of G for all $c \in B \backslash A$ and $L(c) = F(c) \cap H(c)$ is a subgroup of G for all $c \in A \cap B$. Thus (L, C) $= (F, A) \sqcap_E (H, B)$ is a soft group over G. □

Definition 3.6. The restricted union (H, C) of two soft groups (F, A) and (G, B) over G is defined as the soft set $(H, C) = (F, A) \cup_R (G, B)$, where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Theorem 3.7. *Let (F, A) and (H, B) be two soft groups over G. Then their restricted union is a soft group over G if and only if either $F(x) \subset H(x)$ or $H(x) \subset F(x)$ for all $x \in A \cap B$.*

Proof. Let $(F, A) \cup_R (H, B) = (L, C)$ be restricted union of two soft sets over G where $A \cap B = C$ and L is defined as $L(x) = F(x) \cup H(x)$ for all $x \in C$. Let (F, A) and (H, B) be two soft groups over G. Suppose either $F(x) \subset H(x)$ or $H(x) \subset F(x)$ for all $x \in C$. Then $F(x) \cup H(x) = H(x)$ or $F(x) \cup H(x) = F(x)$ for all $x \in C$. Since $H(x)$ and $F(x)$ are subgroups of G for all $x \in C$, this implies that $F(x) \cup H(x)$ is a subgroup of G for all $x \in C$. Thus $L(x) = F(x) \cup H(x)$ is a subgroup of G for all x $\in C$. Hence $(L, C) = (F, A) \cup_R (H, A)$ is a soft group over G.

Conversely suppose that $(F, A) \cup_R (H, A)$ is a soft group over G and also that $F(x) \nsubseteq H(x)$ and $H(x) \nsubseteq F(x)$ for some $x \in C$. Then there are elements $a \in H(x) \backslash F(x)$, $b \in F(x) \backslash H(x)$ for some $x \in C$ and both a , $b \in H(x) \cup F(x)$. As $H(x) \cup F(x)$ is a subgroup of G, hence $ab \in H(x) \cup F(x)$. But then $ab \in H(x)$ or $ab \in F(x)$. If $ab \in F(x)$ then $a = (ab)b^{-1} \in F(x)$. If $ab \in H(x)$ then $b = a^{-1}(ab)$ $\in H(x)$. It is a contradiction in both the cases by the choice of a and b. Hence either $H(x) \backslash F(x) = \phi$ or $F(x) \backslash H(x) = \phi$ for all $x \in C$. That is either $H(x) \subset F(x)$ or $F(x) \subset H(x)$ for all $x \in C$. □

Definition 3.8. Let (F, A) and (G, B) be any two soft groups over a group G. Then extended union (H, C) of two soft groups (F, A) and (G, B) is denoted as the soft set $(H, C) = (F, A) \cup_E (G, B)$ where $C = A \cup B$ and H is defined as

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

Theorem 3.9. *Let (F, A) and (H, B) be two soft groups over G. Then their extended union is a soft group over G if and only if either $F(x) \subset H(x)$ or $H(x) \subset F(x)$ for all $x \in A \cap B$.*

Proof. Let (F, A) and (H, B) be any two soft groups over G. Then their extended union (L, C) is denoted as the soft set $(L, C) = (F,A) \cup_E (H, B)$ where $C = A \cup B$ and is defined as, for all $x \in C$

$$L(x) = \begin{cases} F(x) & if\ x \in A - B \\ H(x) & if\ x \in B - A \\ F(x) \cup H(x) & if\ x \in A \cap B. \end{cases}$$

Suppose either $H(x) \subset F(x)$ or $F(x) \subset H(x)$ for all $x \in A \cap B$. Then $F(x) \cup H(x)$ $= L(x)$ for all $x \in A \cap B$. Since each $F(x)$ is a subgroup of G for all $x \in A$ and each $H(x)$ is a subgroup of G for all $x \in B$. Hence, $F(x) \cup H(x)$ is a subgroup of G for all $x \in A \cap B$. Thus, $L(x) = F(x) \cup H(x)$ is a subgroup of G for all $x \in A \cap B$. If $x \in A - B$ then $L(x) = F(x)$ is a subgroup of G. Similarly, $L(x) = G(x)$ is a subgroup of G for all $x \in B - A$. So $(L, C) = (F, A) \cup_E (H, B)$ is a soft group over G for all $x \in C$.

Conversely, suppose $(F, A) \cup_E (H, B)$ be the soft group over G and also that $F(x) \nsubseteq H(x)$ and $H(x) \nsubseteq F(x)$ for some $x \in A \cap B$, then there are elements $a \in H(x) \backslash F(x)$, $b \in F(x) \backslash H(x)$ and both $a, b \in F(x) \cup H(x)$. As $F(x) \cup H(x)$ is a subgroup of G. This implies that $ab \in H(x) \cup F(x)$. But then $ab \in H(x)$ or $ab \in F(x)$. If $ab \in F(x)$ then $a = (ab)b^{-1} \in F(x)$. If $ab \in H(x)$ then $b = a^{-1}(ab) \in H(x)$. It is a contradiction in both the cases by the choice of a and b. Hence, either $H(x) \backslash F(x) = \phi$ or $F(x) \backslash H(x) = \phi$ for all $x \in A \cap B$. That is either $H(x) \subset F(x)$ or $F(x) \subset H(x)$ for all $x \in A \cap B$. □

Definition 3.10 ([1])**.** Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a soft subgroup of (F, A) written as $(H, K) \tilde{<} (F,A)$, if

1. $K \subseteq A$,
2. $H(x) \leqslant F(x)$ for all $x \in K$.

Two soft groups are equal if $(F, A) \tilde{<} (H, K)$ and $(H, K) \tilde{<} (F, A)$.

Theorem 3.11. *Let (F, A) be a soft group over G and $\{(H_i, B_i) : i \in I\}$ be a family of soft subgroups of (F, A), where I is an indexing set. Then restricted intersection $\Cap(H_i,B_i)$ is a soft subgroup of (F, A) over G.*

Proof. Straightforward. □

Theorem 3.12. *Let (F, A) be a soft group over G and $\{(H_i,A_i) : i \in I\}$ be a family of soft subgroups of (F, A) then their extended intersection $\sqcap_E(H_i,K_i)$ is a soft subgroup of (F, A) over G.*

Proof. Straightforward. □

4. Normal soft groups

In this section, we define normal soft group and prove some of their related results.

Definition 4.1. A soft group (F, A) over G is called a normal soft group over G if $F(x)$ is a normal subgroup of G for all $x \in A$.

Example 4.2. Consider the group $D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ which is generated by two elements a and b satisfying the relations $\circ(a) = 4$, $\circ(b) = 2$ and $ba = a^3b$. Consider the set of parameters $A = \{t_1, t_2, t_3\}$. A mapping $F : A \to P(D_4)$ such that $F(t_1) = \{e, a, a^2, a^3\}$, $F(t_2) = \{e, a^2, b, a^2b\}$ and $F(t_3) = \{e, ab, a^2, a^3b\}$. For each parameter, $F(t)$ is a normal subgroup of D_4. This implies that the pair (F, A) is a normal soft group over G.

Corollary 4.3. *Let (F, A) be a normal soft group over G. Then (F, A) is commutative with every soft set over G.*

Proof. It is straightforward. □

Theorem 4.4. *Let (F, A) and (H, B) be two normal soft groups over G. Then their restricted intersection $(F, A) \Cap (H, B)$ is a normal soft group over G .*

Proof. It is straightforward. □

Theorem 4.5. *Let (F, A) and (H, B) be two normal soft groups over G. Then their extended intersection $(F, A) \sqcap_E (H, B)$ is a normal soft group over G.*

Corollary 4.6. *Let $\{(F_i, A_i) : i \in I\}$ be a family of the normal soft groups over G. Then their extended intersection $\sqcap_E(F_i, A_i)$ is a normal soft group over G.*

Proof. It is straightforward. □

Theorem 4.7. *Let (F, A) be a soft group over G. If (H, B) is a soft subgroup of (F, A) and (K, B) is a normal soft subgroup of (F, A). Then $(H,B) \Cap (K, B)$ is a normal soft subgroup of (H, B).*

Proof. Let $(H, B) \Cap (K, B) = (L, B)$ be the restricted intersection of (H,B) and (K, B). Since (L, B) is a soft subgroup of (F, A) and obviously $(L, B) \tilde{\subset} (H, B)$. From this, we imply that (L, B) is a soft subgroup of (H, B). Now we show that $(L, B)\tilde{\lhd}(H, B)$, for this we show that $H(b) \cap K(b) \lhd H(b)$ for all $b \in B$. It is well known result that $H(b) \cap K(b)$ is a normal subgroup of $H(b)$. This is hold for all $b \in B$. Hence $(H, B) \Cap (K, B) = (L, B)$ is a normal soft subgroup of (H, B). □

Definition 4.8 ([1])**.** Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a normal soft subgroup of (F, A), written $(H, B)\tilde{\lhd}(F, A)$, if $H(x)$ is a normal subgroup of $F(x)$, for all $x \in B$.

Theorem 4.9. *Let (F, A) be a soft group over G and $\{(H_i, B_i) : i \in I\}$ be a family of the normal soft subgroups of (F,A). Then their restricted intersection $\Cap(H_i,K_i)$ is a normal soft subgroup of (F, A).*

Proof. It is straightforward. □

Theorem 4.10. *Let (F, A) be soft group over G and let $\{(H_i, A_i) : i \in I\}$ be a family of normal soft subgroups of (F, A). Then extended intersection $\sqcap_E(H_i,A_i)_{i\, \in I}$ is a normal soft subgroup of (F, A).*

Proof. Since $\{(H_i,A_i) : i \in I\}$ be a family of the normal soft subgroup of (F, A). Then their extended intersection $\sqcap_E(H_i,A_i)_{i\, \in I}$ is denoted by $\sqcap_E(H_i,A_i)_{i\, \in I} = (H, C)$ where $C = \cup A_i$ and for all $i \in I$, it is defined as

$$H(e) = \begin{cases} H_i(e) & if\ e \in A_i - A_j \text{ for all } i,\, j \in I \\ \cap H_i(e) & if\ e \in \cap A_i \text{ for all } i,\, j \in I \end{cases}$$

Each $H_i(e)$ is a normal subgroup of $F(e)$ for all $e \in A_i - A_j$, for all $i,\, j \in I$ and $i \neq j$. Also $\cap H_i(e)$ is a normal subgroup of $F(e)$ for all $e \in \cap A_i$. This implies that $H(e)$ is a normal subgroup of $F(e)$ for all $e \in C$. Thus $(H, C)\tilde{\lhd}(F, A)$. □

5. Abelian soft groups and Abelian soft subgroups

In this section, we define Abelian soft groups and Abelian soft Subgroups of a soft group and check out their properties.

Definition 5.1. A soft group $(F,\ A)$ over G is said to be abelian soft group over G if each $F(\alpha)$ is an abelian subgroup of G for all $\alpha \in A$.

Example 5.2. Consider the group $S_3 = \{e,\ x,\ y,\ y^2,\ xy,\ xy^2\}$ which is generated by two elements x and y satisfying the relation $\circ(x) = 2$, $\circ(y) = 3$, $\circ(xy) = 2$, $xy = y^2x$ and $yx = xy^2$. Consider the set of parameters $A = \{\alpha,\ \beta,\ \gamma\}$. A mapping $F : A \to P(S_3)$ such that $F(\alpha) = \{e,\ x\}$, $F(\beta) = \{e,\ y,\ y^{\ 2}\}$, $F(\gamma) = \{e,\ xy\}$. Then (F, A) is a soft group over S_3 . Since for each parameter $\alpha \in A$, $F(\alpha)$ is an abelian subgroup of S_3. Hence (F, A) is an abelian soft group over S_3. It is interesting to note that S_3 is not abelian.

Definition 5.3. Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is an abelian soft subgroup of (F,A) if $H(x)$ is an abelian subgroup of $F(x)$ for all $x \in B$.

Example 5.4. Consider the group $S_3 = \{e,\ x,\ y,\ y^2,\ xy,\ xy^2\}$ which is generated by two elements x and y satisfying the relation $\circ(x) = 2$, $\circ(y) = 3$, $\circ(xy) = 2$, $xy = y^2x$ and $yx = xy^2$. Consider the set of parameters $A = \{\alpha,\ \beta\ ,\gamma,\ \lambda\}$. A mapping $F : A \to P(S_3)$ such that $F(\lambda) = \{e,\ x\}$, $F(\beta) = \{e,\ y,\ y^{\ 2}\}$, $F(\gamma) = \{e,\ xy\}$ and $F(\alpha) = \{e,\ x,\ y,\ y^2,\ xy,\ xy^2\}$. Then (F, A) is a soft group over S_3. Let (H, B) be a soft subgroup of (F, A) such that $H(\alpha) = \{e\}$, $H(\beta) = \{e,\ y,\ y^{\ 2}\}$ and $H(\gamma) = \{e,\ xy\}$. Since for each parameter $\alpha \in B$, $H(\alpha)$ is an abelian subgroup of $F(\alpha)$. Hence, (H, B) is an abelian soft subgroup of (F, A).

Theorem 5.5. *Every soft subgroup (H, B) of an abelian soft group (F, A) over G is a normal soft subgroup of (F, A).*

Proof. Since each $F(\alpha)$ for all $\alpha \in A$ is an abelian subgroup of G and each $H(\alpha)$ is a subgroup of $F(\alpha)$ for all $\alpha \in B$. It is well known that a subgroup of an abelian group is a normal subgroup. This implies that $H(\alpha)$ is a normal subgroup $F(\alpha)$ for all $\alpha \in B$. Hence $(H, B)\tilde{\lhd}(F, A)$. □

6. Restricted soft product of soft groups

In this section we define restricted product of two soft groups, inverse of a soft group and prove some results.

Definition 6.1. The restricted product (H, C) of two soft groups (F, A) and (K, B) over G is denoted by the soft set $(H, C) = (F, A)\hat{\circ}(K, B)$ where $C = A \cap B$ and H is a set valued function from C to $P(G)$ and is defined as $H(c) = F(c)K(c)$ for all $c \in C$. The soft set (H, C) is called the restricted soft product of (F, A) and (K, B) over G.

Definition 6.2. Let (F, A) be a soft set over a group G. Then inverse of (F, A) is denoted by $(F, A)^{-1}$ and is defined as follows $(F, A)^{-1} = \{(F(a))^{-1} : a \in A\}$, where $(F(a))^{-1}$ is called the inverse of $F(a)$ and is defined as

$$(F(a))^{-1} = \{x^{-1} : x \in F(a)\}$$

Theorem 6.3. *Let (F, A) and (K, B) be any two soft sets over G. Then*

$$((F, A)\hat{\circ}(K, B))^{-1} = (K, B)^{-1}\hat{\circ}(F, A)^{-1}.$$

Proof. Suppose that the inverse of restricted soft product of (F, A) and (K,B) denoted by $((F, A)\hat{\circ}(K, B))^{-1} = (H, C)$ is defined as $H(c) = (F(c)K(c))^{-1}$ for all $c \in C$ and $(K, B)^{-1}\hat{\circ}(F, A)^{-1} = (L, C)$ and is defined as $L(c) = (K(c))^{-1}(F(c))^{-1}$ for all $c \in C$. But then $(F(c)K(c))^{-1} = (K(c))^{-1}(F(c))^{-1}$ for all $c \in C$. This implies that $L(c) = H(c)$ for all $c \in C$. Thus $((F, A)\hat{\circ}(K, B))^{-1} = (K, B)^{-1}\hat{\circ}(F, A)^{-1}$. $\square$

Theorem 6.4. *If (F, A) is a soft group over G, then $(F, A)^{-1} = (F, A)$ but the converse is not true.*

Proof. By definition $(F, A)^{-1} = \{(F(a))^{-1} : a \in A\}$. For each parameter, $F(\alpha)$ is a subgroup of G, this implies that $F(\alpha) = (F(\alpha))^{-1}$ for all $\alpha \in A$. Hence $(F, A)^{-1} = (F, A)$. But converse is not true. This is proved by the following example. $\square$

Example 6.5. Let $G = \{1, -1, i, -i\}$ be a group under consideration and $A = \{x, y, z\}$ be the set of parameters. Let $F : A \to P(G)$ be a mapping such that

$$\begin{aligned} F(x) &= \{1\} \\ F(y) &= \{-1\} \\ F(z) &= \{-1\}. \end{aligned}$$

Then we have the pair $(F, A) = (F, A)^{-1}$. But (F, A) is not a soft group over G.

Theorem 6.6. *Let (H, A) and (K, B) be two soft groups over G. Then their restricted product $(F,C) = (H, A)\hat{\circ}(K, B)$ is a soft group over G if and only if $H(c)K(c) = K(c)H(c)$ for all $c \in C$.*

Proof. Suppose $H(c)K(c) = K(c)H(c)$ for all $c \in C$. We show that $(F,C) = (H, A)\hat{\circ}(K, B)$, which is defined as $F(c) = H(c)K(c)$ for all $c \in C$, is a soft group over G. To prove this, it is sufficient to prove that $(H(c)K(c))(H(c)K(c))^{-1} = H(c)K(c)$ for all $c \in C$. We have

$$\begin{aligned} (H(c)K(c)).(H(c)(K(c))^{-1} &= (H(c)K(c))((K(c))^{-1}(H(c))^{-1}) \\ &= H(c)(K(c)(K(c))^{-1})(H(c))^{-1}) \\ &= (H(c)K(c))H(c) \\ &= (K(c)H(c))H(c)) \\ &= (K(c)H(c))H(c) \\ &= K(c)(H(c)H(c)) \\ &= K(c)(H(c) \\ &= H(c)K(c) \end{aligned}$$

This is hold for all $c \in C$. Hence, $F(c) = H(c)K(c)$ for all $c \in C$ is a subgroup of G. This implies that $(H, A)\hat{\circ}(K, B)$ is a soft group over G.

Conversely suppose that $(H, A)\hat{\circ}(K, B)$ is a soft group over G . Then $F(c) = H(c)K(c)$ is a subgroup of G for all $c \in C$. Consider

$$\begin{aligned} (H(c)K(c))^{-1} &= H(c)K(c) \\ (K(c))^{-1}(H(c))^{-1} &= H(c)K(c) \\ K(c)H(c) &= H(c)K(c) \end{aligned}$$

This holds for all $c \in C$. Hence, $H(c)K(c) = K(c)H(c)$ for all $c \in C$. □

Theorem 6.7. *Let (F, A) and (K, B) be two soft groups over an abelian group G. Then their restricted product $(L, C) = (F, A)\hat{\circ}(K, B)$ is a soft group over G.*

Proof. Let the restricted soft product of (F, A) and (K, B) be denoted by$(H,C) = (F, A)\hat{\circ}(K, B)$ and is defined as $H(c) = F(c)K(c)$ for all $c \in C$. Since G is abelian, we have $F(c)K(c) = K(c)F(c)$ for all $c \in C$. This implies that $(F, A) \hat{\circ} (K, B)$ is a soft group over G. □

Corollary 6.8. *Let (H, A) be a soft group over G and (K, B) be a normal soft group over G. Then $(H, A) \hat{\circ} (K, B)$ is a soft group over G.*

Proof. Let the restricted soft product of (H, A) and (K, B) be denoted by $(H ,A)\hat{\circ}(K, B) = (F,C)$ and it is defined as $F(c) = H(c)K(c)$ for all $c \in C$. But we have $K(c)$ is a normal subgroup of G for all $c \in C$, this implies that $H(c)K(c) = K(c)H(c)$ for all $c \in C$. Hence, $(H, A)\hat{\circ}(K, B)$ is a soft group over G. □

Corollary 6.9. *If (F, A) is any soft group over G, then $(F, A) \hat{\circ} (F,A) = (F, A)$.*

Proof. It is straightforward. □

Corollary 6.10. *Let (N,A) and (M,B) be two normal soft groups over G. Then $(N,A)\hat{\circ}(M,B)$ is also a normal soft group over G.*

Proof. It is straightforward. □

7. Cartesian product of soft groups

In this section, we define the identity soft group and prove some results by using cartesian product of soft groups.

Definition 7.1 ([1])**.** Let (F, A) and (H, B) be two soft sets over G and K, respectively, and let $f : G \to K$ and $g : A \to B$ be two functions. Then we say that (f,g) is a soft homomorphism, if the following conditions are satisfied:
1. f is a homomorphism from G onto K;
2. g is mapping from A onto B;
3. $f(F(x)) = H(g(x))$.

In this definition, if f is an isomorphism G to K and g is a one-to-one mapping from A on to B, then we say that (f,g) is a soft isomorphism and that (F, A) is soft isomorphic to (H, B). The latter is denoted by $(F, A) \simeq (H, B)$

Definition 7.2 ([1])**.** Let (F, A) and (H, B) be two soft groups over G and K, respectively. Then the cartesian product of soft groups (F, A) and (H, B) is denoted by $(F, A) \tilde{\times} (H, B) = (U,A \times B)$ and U is defined as $U(a, b) = F(a)\tilde{\times}H(b)$ for all $(a,b) \in A \times B$.

Theorem 7.3. *Let (F, A) and (H, B) be two soft groups over G and K, respectively. Then the cartesian product $(F, A)\tilde{\times}(H, B)$ is a soft group over $G \times K$ and $(F, A)\tilde{\times}(H, B)$ is soft isomorphic to $(H, B) \tilde{\times} (F, A)$.*

Proof. First had proved in [1]. We will prove next part. Now we show that (f, g) : $(F, A)\tilde{\times}(H, B) \to (H, B)\tilde{\times}(F, A)$ is a soft isomorphism That is $(f,\ g) : (U, A \times B)$ $\to (W, B \times A)$ is a soft isomorphism where $W(b, a)$ is defined as $W(b, a) = H(b) \times F(a)$. Actually, we prove three conditions.

1. We show that $f : G \times K \to K \times G$ is an isomorphism. Let f be a function defined by $f(g, k) = (k, g)$. Then obviously f is an isomorphism.

2. Now we show that $g : A \times B \to B \times A$ is a bijective mapping. The mapping g is defined by $g(a,\ b) = (b, a)$ then obviously g is a bijective mapping.

3.

$$\begin{aligned}
f(U(a,\ b)) &= f(F(a) \times H(b)) \\
&= f(\{(g,\ k) : g \in F(a),\ k \in H(b)\}) \\
&= \{(k,\ g) : k \in H(b),\ g \in F(a)\} \\
&= H(b) \times F(a) \\
&= W(b,\ a) \\
&= W(g(a,\ b)) \text{ for all } (a,\ b) \in A \times B.
\end{aligned}$$

This implies that $(f,\ g) : (F, A)\tilde{\times}(H, B) \to (H, B)\tilde{\times}(F, A)$ is a soft isomorphism. Hence, $(F, A)\tilde{\times}(H, B) \simeq (H, B)\times\ (F,A)$. ∎

Corollary 7.4. *Let (F, A) and (H, B) be two normal soft groups over G and K respectively. Then the cartesian product $(F, A)\tilde{\times}(H, B)$ is a normal soft group over $G \times K$ and $(F, A) \times (H, B)$ is soft isomorphic to $(H, B) \tilde{\times} (F, A)$.*

Definition 7.5 ([1])**.** Let (F, A) be a soft group over G. Then (F, A) is said to be an identity soft group over G if $F(x) = \{e\}$ for all $x \in A$, where e is the identity element of G. Identity soft group is represented by (I,A).

Theorem 7.6. *Let (F, A) and (H, B) be two normal soft groups over G and K. Let (I,A) and (I, B) be the identity soft groups over G and K, respectively. Then*

1. *$(F, A)\ \tilde{\times}\ (I, B)$ is a normal soft group over $G\times\ K$ and $(I,A)\ \tilde{\times}\ (H, B)$ is a normal soft group over $G \times K$.*

2. *Every element of $(F, A)\ \tilde{\times}\ (I, B)$ commutes with every element of $(I,A) \times (H, B)$.*

Proof. 1. The cartesian product of soft groups (F, A) and (I, B) denoted by

$$(F, A)\tilde{\times}(I, B) = (U, A \times B),$$

and U is defined as $U(a, b) = F(a) \times I(b)$ for all $(a, b) \in A \times B$. For each parameter, $I(b)$ is identity subgroup of K. Hence, $U(a, b) = F(a) \times \{e_2\}$ for all $(a, b) \in A \times B$. We show that $U(a, b)$ is a normal subgroup of $G \times K$ for all $(a,b) \in A \times B$. By Theorem 7.3, $U(a, b)$ is a normal subgroup of $G \times K$ for all $(a, b) \in A \times B$. This implies that $(F, A)\ \tilde{\times}\ (I, B)$ is a normal soft group over $G \times K$. Similarly, we can prove that $(I,A)\ \tilde{\times}\ (H, B)$ is a normal soft group over $G \times K$.

2. Let the product of soft groups (F, A) and (I, B) be denoted by $(U,\ A \times B) = (F, A) \times (I, B)$ and it is defined as $U(a, b) = F(a) \times I(b) = F(a) \times \{e_2\}$ for all $(a, b) \in A \times B$, where $e_2 \in K$. Also, $(W,\ A \times B)$ be the cartesian product of soft groups (I,A) and (H,B) denoted by $(W, A \times B) = (I,A) \times (H, B)$ and it is defined as $W(a, b) = I(a) \times H(b) = \{e_1\} \times H(b)$ for all $(a, b) \in A \times B$, where $e_1 \in G$. Let $(g,\ e_2) \in\ F(a) \times \{e_2\}$ for all $(a,b) \in A \times B$ and $(e_1,\ k) \in \{e_1\} \times H(b)$ for all $(a, b) \in A \times B$. Then $(g,\ e_2)(e_1,\ k) = (ge_1,\ e_2k) = (e_1g,\ ke_2) = (e_1,\ k)(g,\ e_2)$ for all $(a, b) \in A \times B$. Hence, every element of $(F,\ A)\tilde{\times}(I,\ B)$ commutes with every element of $(I,\ A)\ \tilde{\times}\ (H,\ B)$. □

8. Cyclic soft groups

In this section, we define cyclic soft groups, cyclic soft subgroups and check out their properties.

Definition 8.1. A soft group (F, A) over G is called a cyclic soft group over G if each $F(\alpha)$ is a cyclic subgroup of G for all $\alpha \in A$.

Example 8.2. Consider the group $D_4 = \{e,\ a,\ a^2,\ a^3,\ b,\ ab,\ a^2b,\ a^3b\}$ which is generated by two elements a and b satisfying the relations $\circ(a) = 4$, $\circ(b) = 2$, $ba = a^3b$. Consider the set of parameters $A = \{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6\}$. A mapping $F : A \to P(D_4)$ such that $F(t_1) = \{e,\ a^2\}$, $F(t_2) = \{e,\ b\}$, $F(t_3) = \{e,\ ab\}$, $F(t_4) = \{e,\ a^2b\}$, $F(t_5) = \{e,\ a^3b\}$, $F(t_6) = \{e,\ a,\ a^2,\ a^3\}$ then (F, A) is a soft group over D_4. Since for each parameter $\alpha \in A$, $F(\alpha)$ is a cyclic subgroup of D_4 hence (F, A) is a cyclic soft group over D_4.

Definition 8.3. Let (F, A) be a soft group over G. Then a soft subgroup (H, B) of (F, A) is called a cyclic soft subgroup of (F, A) if $H(\alpha)$ is a cyclic subgroup of $F(\alpha)$ for all $\alpha \in A$.

Example 8.4. Consider the group $S_3 = \{e,\ x,\ y,\ y^2,\ xy,\ xy^2\}$ which is generated by two elements x and y satisfying the relation $\circ(x) = 2$, $\circ(y) = 3$, $\circ(xy) = 2$, $xy = y^2x$ and $yx = xy^2$. Consider the set of parameters $A = \{\alpha,\ \beta\ ,\gamma,\ \lambda\}$. A mapping $F : A \to P(S_3)$ such that $F(\alpha) = \{e,x\}$, $F(\beta) = \{e,\ y,\ y^{\ 2}\}$, $F(\gamma) = \{e,\ xy\}$ and $F(\lambda) = \{e,\ x,\ y,\ y^2,\ xy,\ xy^2\}$. Then (F, A) is a soft group over S_3. Let (H, B) be a soft subgroup of (F, A) such that $H(\alpha) = \{e\}$, $H(\beta) = \{e,\ y,\ y^{\ 2}\}$ and $H(\gamma) = \{e,\ xy\}$. Since for each parameter $\alpha \in B$, $H(\alpha)$ is a cyclic subgroup of $F(\alpha)$, hence (H, B) is a cyclic soft subgroup of (F, A). It is important to note that (F, A) is not a cyclic soft group over G.

Theorem 8.5. *Every cyclic soft group (F, A) over G is an abelian soft group over G.*

Proof. Let (F, A) be a cyclic soft group over G. Then each $F(\alpha)$ is a cyclic subgroup of G for all $\alpha \in A$, but then each cyclic subgroup is abelian subgroup of G. Hence each $F(\alpha)$ is an abelian subgroup of G for all $\alpha \in A$. This implies that $(F\ ,A)$ is an abelian soft group over G. □

Theorem 8.6. *Let (F, A) be a soft group over G. If a cyclic soft subgroup (H, B) of (F, A) is a normal soft subgroup of (F, A), then every soft subgroup of (H, B) is a normal soft subgroup of (F, A).*

Proof. Suppose cyclic soft subgroup (H, B) of (F, A) is a normal soft subgroup of (F, A). Let (K,C) be a soft subgroup of (H, B), this implies that (K,C) is a cyclic subgroup of (H, B) and hence a cyclic subgroup of (F, A). We show that (K, B) is a normal soft subgroup of (F, A). We have $H(\alpha)$ is a cyclic subgroup of $F(\alpha)$ for all $\alpha \in B$. But, $H(\alpha)$ is a normal subgroup of $F(\alpha)$ for all $\alpha \in B$. Hence, (K,C), a soft subgroup of (H, B) is a normal soft subgroup of (F, A). $\square$

9. Soft coset of a soft group

In this section, we define the soft coset of a soft subgroup of a soft group, order of a soft group, soft index, partition of a soft group and prove their related results.

Definition 9.1. Let (F, A) be a soft group over G and (H, B) is any soft subgroup of (F, A). Let $a \in \cap F(x)$. Then the soft set (H_a,B) defined as $(H_a(b) = (H(b))a$ for all $b \in B$ is called a soft right coset of (H, B) in (F, A) generated by a. Similarly, the soft set $({}_aH, B)$ is called a soft left coset of (H, B) in (F, A) generated by a.

Example 9.2. Consider the group $D_8 = \{e,\, a,\, a^2,\, a^3,\, b,\, ab,\, a^2b,\, a^3b\}$ which is generated by two elements a, b satisfying the relation $\circ(a) = 4$, $a^2 = b^2$ and $ba = a^3b$. Consider the set of parameters $A = \{t_1,\, t_2,\, t_3,\, t_4\}$. A mapping $F : A \to P(D_8)$ such that $F(t_1) = \{e,\, a^2\}$, $F(t_2) = \{e,\, a,\, a^2,\, a^3\}$, $F(t_3) = \{e,\, ab,\, a^2,\, a^3b\}$, $F(t_4) = \{e,\, b,\, a^2,\, a^2b\}$. For each parameter, $F(t)$ is subgroup of D_8. Hence, (F, A) is a soft group over D_8. Since $\cap F(t_i) = \{e,\, a^2\}$ for all $t_i \in A$. Let $B = \{t_2,\, t_3,\, t_4\}$ and consider (H, B) be any soft subgroup of (F, A) such that $H(t_2) = \{e,\, a^2\}$, $H(t_3) = \{e,\, ab,\, a^2,\, a^3b\}$, $H(t_4) = \{e,\, b,\, a^2,\, a^2b\}$. Now we find soft left coset as follows: For identity element we have,

$$({}_eH, B) = (H, B)$$

and

$$({}_{a^2}H, B) = \{a^2(H(t)) : \forall t \in B\}$$

Now we find $a^2H(t)$ for all $t \in B$ as follows $a^2(H(t_2)) = H(t_2)$, $a^2(H(t_3)) = H(t_3)$, $a^2(H(t_4)) = H(t_4)$. This implies that $a^2(H, B) = (H, B)$. Hence, the only soft left coset of (H, B) in (F, A) is itself (H, B). Similarly, we can find soft right coset which is (H, B) itself.

Definition 9.3. Let (F, A) be a soft group over G. Then the order of a soft group (F, A) is the number of distinct subgroups of G.

Definition 9.4. The number of distinct soft left (or right) cosets of a soft subgroup (H,B) of a soft group (F, A) over G is called the soft index of (H,B) in (F, A) and is denoted by $[(F, A) : (H, B)]$.

Theorem 9.5. *A soft subgroup (H, B) of a soft group (F, A) over G is a normal soft subgroup of (F, A) if and only if each soft left coset of (H, B) in (F, A) is a soft right coset of (H, B) in (F, A).*

Proof. Let (H, B) be a normal soft subgroup of (F, A). Then obviously $({}_xH,B) = (H_x,B)$ for all $x \in \cap F(a)$. Thus, each soft left coset of (H, B) in (F, A) is a soft right cost of (H, B) in (F,A).

Conversely, suppose that each soft left coset of (H, B) is a soft right coset. Then we have $({}_xH, B) = (H_x, B)$ for some $x \in \cap F(a)$. This implies that $x(H(b)) = (H(b))y$ for all $b \in B$. Since $e \in H(b)$ for all $b \in B$, so $xe = x \in x(H(b))$. Therefore, we have $x \in (H(b))x$ for all $b \in B$. This implies that $x \in (H(b))x$ and hence we have $x(H(b)) = (H(b))x$ for all $b \in B$. This shows that $({}_xH, B) = (H_x, B)$. Hence, (H, B) is a normal and obviously, normal soft subgroup of $(F\ , A)$. This completes the proof. □

Theorem 9.6. *A soft subgroup (H, B) of a soft group (F, A) over G is a normal soft subgroup of (F, A) if and only if the product of two soft right cosets of (H, B) in (F, A) is again a soft right coset of (H, B) in (F, A).*

Proof. Let (H, B) be a normal soft subgroup of (F, A). Let (H_x, B) and (H_y, B) be two soft right cosets of (H, B) in (F, A), where x, $y \in \cap F(a)$. We denote the product of two soft right coset of (H, B) by $(L, B) = (H_x, B)\hat{\circ}(H_y, B)$ and L is defined as $L(b) = ((H(b))x)((H(b))y) = H(b)(x(H(b))y = (H(b)H(b))xy = (H(b))xy$. Since $x, y \in \cap F(a) \Rightarrow xy \in \cap F(a)$. Therefore, $(H(b))xy$ is also a right coset. This is hold for all $b \in B$. Hence, (L, B), the product of two soft right coset is again a soft right coset.

Conversely, suppose that (H, B) a soft subgroup of a soft group (F, A) such that the product of two soft right coset of (H, B) in (F, A) is again a soft right coset of (H, B) in (F, A). Let $x \in \cap F(a)$. Then $x^{-1} \in \cap F(a)$. Therefore, (H_x, B) and $(H_{x^{-1}}, B)$ are two soft right cosets of (H, B) in (F, A). Consequently, by hypothesis $(H_x, B)\hat{\circ}(H_{x^{-1}}, B) = (H\ , B)$ is a soft right coset, which is defined by $H(b) = (H(b))x(H(b))x^{-1}$. We show that $H(b)$ is a normal subgroup of G. Since $e \in H(b)$ for all $b \in B$. Therefore, $exex^{-1} \in (H(b))x(H(b))x^{-1}$ for all $b \in B$. But $H(b)$ itself is a right coset of G. This implies that

$$\begin{aligned} (H(b))x(H(b))x^{-1} &= H(b) \text{ for all } b \in B \\ &\Longrightarrow h_1xhx^{-1} \in H(b) \\ &\Longrightarrow h_1^{-1}(h_1xhx^{-1}) \in h_1^{-1}(H(b)) \text{ for all } b \in B \\ &\Longrightarrow xhx^{-1} \in H(b) \text{ for all } b \in B. \end{aligned}$$

This implies that $H(b)$ is a normal subgroup of G for all $b \in B$. Hence, (H, B) is a normal soft subgroup of (F, A). □

Theorem 9.7. *Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then there is one one correspondence between any two soft right(left) cosets of (H, B).*

Proof. Let (H_x, B) and (H_y, B) be any two soft right cosets of a soft subgroup (H, B), where x, $y \in \cap F(b)$. Let $f : (H_x, B) \to (H_y, B)$ be a function defined by $f((H(b))x) = (H(b))y$ for all $x \in B$. Let

$$\begin{aligned} H(b_1)x &= H(b_2)x \text{ for all } b_1, b_2 \in B \\ &\Longleftrightarrow H(b_1) = H(b_2) \\ &\Longleftrightarrow H(b_1)y = H(b_2)y \\ &\Longleftrightarrow f(H(b_1)x) = f(H(b_2)y). \end{aligned}$$

Thus, f is well defined and injective. Now, we show that f is onto. Let $(H(b))x$ be an arbitrary element of (H_x,B). Then there exists $(H(b))y \in (H_y,B)$ such that $f(H(b))x = (H(b))y$ for all $b \in B$. Hence, f is onto. □

Corollary 9.8. *Let (F,A) be a soft group over G and (H,B) be a soft subgroup of (F,A). Then for all $x \in \cap F(a)$, $|({}_xH,B)| = |(H_x,B)| = |(H,B)|$.*

Proof. It is straightforward. □

Theorem 9.9. *Let (F,A) be a soft group over G and (H,B) be a soft subgroup of (F,A). Then for all $x,\ y \in \cap\ F(a)$ and for all $a \in A$*

(1). $({}_xH,B) = ({}_yH,B)$ if and only if $y^{-1}x \in H(b)$ for all $b \in B$.

(2). $(H_x,B) = (H_y,B)$ if and only if $xy^{-1} \in H(x)$ for all $b \in B$.

Proof. (1). Suppose $({}_xH,B) = ({}_yH,B)$. This implies that $x(H(b)) = y(H(b))$ for all $b \in B$. Hence, $y^{-1}x(H(b)) = H(b)$, this implies that $y^{-1}x \in H(b)$ for all $b \in B$.

Conversely, suppose that $y^{-1}x \in H(b)$ for all $b \in B$. Tthen, $y^{-1}x(H(b)) = H(b)$ for all $b \in B$, this shows that $x(H(b)) = y(H(b))$ for all $x \in B$. Hence, $({}_xH,B) = ({}_yH,B)$.

(2). Similarly, we can prove part (2). □

Theorem 9.10. *Let (F,A) be a soft group over G and (H,B) be a soft subgroup of (F,A). Then the elements of (H,B) are in one one correspondence with the elements of any soft left(right) coset of (H,B).*

Proof. It is straightforward. □

Theorem 9.11. *Let (F,A) be a soft group over G and (H,B) be a soft subgroup of (F,A). Then there is one to one correspondence between the set of all soft left cosets of (H,B) and the set of all soft right cosets of (H,B).*

Proof. Let $L = \{({}_xH,B) : x \in \cap F(a)\}$ and $T = \{(H_x,B) : x \in \cap\ F(a)\}$ be the set of all soft left and soft right cosets of (H,B). We show that there exists a bijective mapping between L and T. Define a function $f : L \to T$ by $f(({}_xH,B)) = (H_{x^{-1}},B)$ for all $x \in \cap F(a)$. For $x,\ y \in \cap\ F(a)$, we suppose that

$$\begin{aligned}
({}_xH,B) &= ({}_yH,B) \\
&\Longrightarrow x(H(b)) = y(H(b)) \text{ for all } b \in B \\
&\Longrightarrow y^{-1}x \in H(b) \text{ for all } b \in B \\
&\Longrightarrow y^{-1}(x^{-1})^{-1} \in H(b) \text{ for all } b \in B \\
&\Longrightarrow (H(b))x^{-1} = (H(b))y^{-1} \text{ for all } b \in B \\
&\Longrightarrow (H_{x^{-1}},B) = (H_{y^{-1}},B).
\end{aligned}$$

This implies that $f\ ((_xH,\ B)) = f((_yH,\ B)$ for all $x,\ y \in \cap F(a)$. Hence, f is well defined. Let $f((_xH, B)) = f((_yH, B))$. Then

$$\begin{aligned}
& (H_{x^{-1}}, B) = (H_{y^{-1}}, B) \\
\Longrightarrow\ & (H(b))x^{-1} = (H(b))y^{-1} \text{ for all } b \in B \\
\Longrightarrow\ & H(b)x^{-1}y = H(b) \text{ for all } b \in B \\
\Longrightarrow\ & x^{-1}y \in H(b) \text{ for all } b \in B \\
\Longrightarrow\ & x^{-1}(y^{-1})^{-1} \in H(b) \text{ for all } b \in B \\
\Longrightarrow\ & x^{-1}y(H(b)) = H(b) \text{ for all } b \in B \\
\Longrightarrow\ & y(H(b)) = x(H(b)) \text{ for all } b \in B \\
\Longrightarrow\ & x(H(b)) = y(H(b)) \text{ for all } b \in B \\
\Longrightarrow\ & (_xH, B) = (_yH, B).
\end{aligned}$$

Hence, f is injective. To show that f is onto, let $(H_x,B) \in T$ be any soft right coset. Then $(_{x^{-1}}H, B)$ is a soft left coset of (H, B). Also, $f((_{x^{-1}}H, B)) = (H_{(x^{-1})^{-1}},B)$ $= (H_x,B)$. Thus, each right coset of (H, B) is the image of some left coset (H, B). Hence, f is surjective. □

Definition 9.12. Let (F, A) be a soft set over U. Then by partition of (F, A) we mean a collection of all such soft subsets of (F, A), that is $\{(F_i, A_i) : i \in I\}$ such that $(F, A) = \sqcup_E(F_i,A_i)$ and $(F_i,A_i) \sqcap_E (F_j,A_j) = \Phi$ for $i \neq j$ and Φ is a null soft set.

Theorem 9.13. *Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then for all $x,\ y \in \cap F(a)$ either $(_xH, B) = (_yH, B)$ or $(_xH, B) \sqcap_E (_yH, B)$ $= \Phi$.*

Proof. Let $x, y \in \cap F(a)$. Suppose that $(_xH, B) \sqcap_E (_yH, B) \neq \Phi$ (null soft set). Then $(_xH, B) \sqcap_E (_yH,B) = (L, B)$ and L is defined as $L(b) = x(H(b)) \cap y(H(b))$ for all b $\in B$. Let $t \in L(b) = x(H(b)) \cap y(H(b))$ for all $b \in B$. Then $t \in x(H(b)) \cap y(H(b))$ for all $b \in B$. This implies that $t \in x(H(b))$ and $t \in y(H(b))$ for all $b \in B$. Thus, t $= xy_i$ such that $y_i \in H(x)$ and $t = yy_j$ such that $y_j \in H(b)$, this implies that xy_i $= yy_j$ for $y_i,\ y_j \in H(b)$. This implies that $y^{-1}x = y_jy_i^{-1} \in H(b)H(b)$, this implies that $y^{-1}x \in H(b)$ for all $b \in B$. Then we have $y(H(b)) = x(H(b))$ for all $b \in B$, that is $x(H(b)) = y(H(b))$ for all $b \in B$. Hence, $(_xH, B) = (_yH, B)$. Thus, for all x, $y \in \cap F(b)$ either $(_xH, B) = (_yH, B)$ or $(_xH, B) \sqcap_E (_yH, B) = \Phi$. □

Theorem 9.14. *Let (F, A) be a soft group over G and (H, A) be a soft subgroup of (F, A). Then $\{(_xH, B) : x \in \cap F(a)\}$ forms a partition of (F, A) over G.*

Proof. Let $S = \{(_xH, B) : x \in \cap F(a)\}$ be the set of all soft left cosets of (H, B). Then for all $x,\ y \in \cap F(a)$ and $(_xH, B)$, $(_yH, B) \in S$ either $(_xH, B) = (_yH\ ,B)$ or $(_xH, B) \sqcap_E (_yH, B) = \Phi$ by Theorem 9.13. Now, we show that $(F, A) = \sqcup_E(_xH, B)$. Since $(_xH, B) \mathrel{\widetilde{\subset}} (F, A)$ for all $x \in \cap F(a)$. So, $\sqcup_E(_xH, B) \mathrel{\widetilde{\subset}} (F, A)$. Also, $x \in F(a)$ for some $a \in A$ and $x \in \cup x(H(b))$ for all $b \in B$. This implies that $F(a) \subset \cup x(H(b))$ for all $a \in A$ and hence $(F, A) \mathrel{\widetilde{\subset}} \sqcup_E(_xH,B)$. Thus, $(F, A) = \sqcup_E(_xH, B)$. □

Theorem 9.15. *Let (H, B) be a soft subgroup of a soft group (F, A) over a finite group G. Then the order of (H, B) divides the order of (F, A) over G. In particular $|(F, A)| = [(F, A) : (H, B)]\,|(H, B)|$.*

Proof. Since G is finite so collection of subgroups of G is finite. Hence, the order of (F, A) is finite and the number of soft left cosets of (H, B) in (F, A) is finite. Let $\{({}_{x_1}H, B), ({}_{x_2}H, B), ({}_{x_3}H, B)..., ({}_{x_r}H, B)\}$be the set of all distinct left cosets of (H, B), where $x_i \in \cap F(x)\ _{i=1,2...r}$. Then by Theorem 9.14, we have $(F, A) = \sqcup_E ({}_{x_i}H, B)$ and $({}_{x_i}H, B) \sqcap_E ({}_{y_j}H, B) = \Phi$ for all $i \neq j$, $1 \leq i\ ,\ j \leq r$. Hence, $[(F, A) : (H, B)] = r$ and

$$|(F, A)| = |({}_{x_1}H, B)| + |({}_{x_2}H, B)| + |({}_{x_3}H, B)| + ... + |({}_{x_r}H, B)|$$

By the Corollary 9.8, we have $|(H, A)| = |({}_{x_i}H, B)|$ for all $1 \leq i \leq r$. Thus $|(F, A)|$ $= |(H, B)| + |(H, B)| + ... + |(H, B)|$, r times, that is, $|(F, A)| = r\ |(H, B)|$. This implies that $|(F, A)| = [(F, A) : (H, B)]\, |(H, B)|$. Thus, the order of (H, B) divides the order of (F, A). □

10. Factor of a soft group

In this section, we introduce soft maximal normal subgroups, simple soft groups and factor soft groups. We establish some results as well.

Definition 10.1. Let (F, A) be a soft group over G. Then a normal soft subgroup (H,B) of (F, A) is said to be a soft maximal normal subgroup of (F, A) if there exists no proper normal soft subgroup (K, B) of (F, A), which properly contains (H, B). Thus, a normal soft subgroup (H, B) of (F,A) is soft maximal if and only if there exists no normal soft subgroup (K, B) of (F, A) such that $(H, B) \mathrel{\widetilde{\subset}} (K, B) \mathrel{\widetilde{\subset}} (F, A)$.

Example 10.2. Let $G = \left\{e,\ a,\ a^2,\ a^3,\ b,\ ab,\ a^2b,\ a^3b\right\}$ be a dihedral group of degree 4 generated by the elements a and b such that $\circ(a) = 4$, $\circ(b) = 2$ and $ba = a^3b$. Consider the set of parameters $A = \{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8\}$. A mapping $F : A \to P(G)$ defined by

$$\begin{aligned} F(t_1) &= \{e\}, F(t_2) = \left\{e,\ a^2\right\}, F(t_3) = \{e,\ b\}, F(t_4) = \left\{e,\ a^2b\right\} \\ F(t_5) &= \left\{e,\ a,\ a^2,\ a^3,\ b,\ ab,\ a^2b,\ a^3b\right\}, F(t_6) = \left\{e,\ a,\ a^2,\ a^3\right\} \\ F(t_7) &= \left\{e,\ a^2,\ b,\ a^2b\right\}, F(t_8) = \left\{e,\ ab,\ a^2,\ a^3b\right\}. \end{aligned}$$

For each parameter $t \in A$, we have subgroups of G. This implies that (F, A) is a soft group over G. Let (H, B) be an other soft group over G such that $B = \{t_5,\ t_6,\ t_7,\ t_8\}$ and

$$\begin{aligned} H(t_5) &= \left\{e,\ a,\ a^2,\ a^3,\ b,\ ab,\ a^2b,\ a^3b\right\}, H(t_6) = \left\{e,\ a,\ a^2,\ a^3\right\} \\ H(t_7) &= \left\{e,\ a^2,\ b,\ a^2b\right\}, H(t_8) = \left\{e,\ ab,\ a^2,\ a^3b\right\}. \end{aligned}$$

This implies that (H, B) is a normal soft subgroup of (F, A). This is a soft maximal normal subgroup of (F, A).

Definition 10.3. Let (F, A) be a soft group over G. Then (F, A) is called a simple soft group over G if for any normal soft subgroup (H, B) of (F, A) we have either $(H, B) = (I, B)$ or $(H, B) = (F, A)$.

Example 10.4. Consider the group $S_3 = \left\{e,\ x,\ y,\ y^2,\ xy,\ xy^2\right\}$ which is generated by two elements x and y satisfying the relations $\circ(x) = 2$, $\circ(y) = 3$, $\circ(xy) = 2$, $xy = y^2x$ and $yx = xy^2$. Consider the set of parameters $A = \{\alpha,\ \beta\ ,\gamma\}$. A mapping F

$: A \to P(S_3)$ such that $F(\alpha) = \{e, x\}$, $F(\beta) = \{e, y, y^{\,2}\}$, $F(\gamma) = \{e, xy\}$. Then (F, A) is a soft group over S_3. This is a simple soft group over S_3.

Definition 10.5. Let (F, A) be a soft group over G and (H, B) be a normal soft subgroup of (F, A). Then **quotient or factor** of (F, A) by $(H\ ,B)$ is denoted by $(F, A)/(H, B) = (L, B)$ and L is defined as $L(b) = F(b)/H(b)$ for all $b \in B$. Since each $F(b)/H(b)$ is a group under coset multiplication and is written as $L(b) = \{x(H(b))\ :\ x\ \in\ F(b)\ \forall\, b \in B\}$. Then (L, B) is called a factor soft group of (F, A) by (H, B).

Theorem 10.6. *Let (F, A) be a soft group over G and (H, B) a normal soft subgroup of (F, A). If (K, B) is a normal soft subgroup of (F,A) containing (H, B), then the factor soft group $(K, A)/(H, B)$ is a normal soft subgroup of the factor soft group $(F, A)/(H, B)$. Conversely, if $(K, A)/(H, B)$ is a normal soft subgroup of the factor soft group $(F,A)/(H, B)$, then (K, A) is a normal soft subgroup of (F, A) containing (H, B).*

Proof. It is given that (H, B) is a normal soft subgroup of (F, A) and $(H,B) \mathrel{\widetilde{\subset}} (K, B)$, where (K, B) is itself a normal soft subgroup of (F, A). Therefore, (H, B) is also a normal soft subgroup of (K, B) and consequently $(K, A)/(H, B)$ is a factor soft group. Let $(K, B)/(H, B) = (L, B)$ and $L(x) = \{tH(x) : t \in K(x)\ \}$ for all $x \in B$. Also, suppose that $(F, A)/(H, B) = (M,B)$ and $M(x) = \{rH(x) : r \in F(x)\}$ for all $x \in B$. We show that $(K, A)/(H, B)$ is a normal soft subgroup of the factor soft group $(F, A)/(H, B)$. First we show that $L(x)$ is a subgroup of $M(x)$ for all $x \in B$. Let $a \in L(x)$. Then $a = tH(x)$, since $t \in K(x)$ and $K(x) \subseteq F(x)$ for all $x \in B$. This implies that $t \in F(x)$ for all $x \in B$. Hence, $a = tH(x) \in M(x)$ for all $x \in B$. That is $L(x)$ is a subgroup of $M(x)$ for all $x \in B$. Thus, $(L, B) \mathrel{\tilde{<}} (M,B)$. Now, we show that $(L, B)\tilde{\lhd}(M,B)$. We show that $L(x)$ is a normal subgroup of $M(x)$ for all $x \in B$. Let $q \in L(x)$ and $p \in M(x)$, where $q = t(H(x))$ such that $t \in K(x)$ and $p = r(H(x))$ such that $r \in F(x)$. Consider

$$pqp^{-1} = r(H(x))t(H(x))(r(H(x)))^{-1} = r(H(x))t(H(x))r^{-1}(H(x)) = (rtr^{-1})(H(x)).$$

Since $K(x)$ is a normal subgroup of $F(x)$. This implies that $t_1 = rtr^{-1} \in K(x)$. Hence, $pqp^{-1} = t_1H(x)$. This implies that $pqp^{-1} \in L(x)$. That is $L(x)$ is a normal subgroup of $M(x)$ for all $x \in B$. Hence, $(K, A)/(H, B)$ is a normal soft subgroup of the factor soft group $(F, A)/(H, B)$.

Conversely, if $(K, A)/(H, B)$ is a normal soft subgroup of the factor soft group $(F, A)/(H, B)$. We show that (K, A) is a normal soft subgroup of (F, A) containing (H, B). We have $(K, A)/(H, B)$ is a normal soft subgroup of $(F, A)/(H, B)$. Consider $t(H(x)) \in K(x)/H(x)$ and $r(H(x)) \in F(x)/H(x)$. Then by supposition

$$\begin{aligned} r(H(x))t(H(x))r(H(x))^{-1} \quad &\in \quad K(x)/H(x) \\ &\Longrightarrow \quad r(H(x))t(H(x))r^{-1}(H(x)) \in K(x)/H(x) \\ &\Longrightarrow \quad rtr^{-1}(H(x)) \in K(x)/H(x) \\ &\Longrightarrow \quad rtr^{-1} \in K(x) \text{ for all } x \in B. \end{aligned}$$

Hence, $K(x)$ is a normal subgroup of $F(x)$ for all $x \in B$. Thus, $(K, A)\tilde{\lhd}(F, A)$. Now, we show that $(H, B) \mathrel{\widetilde{\subset}} (K, B)$. Since $(K, B)/(H, B)$ is a factor soft group.

This implies that $(H,B)\tilde{\lhd}(K,A)$ and hence $(H,B) \tilde{\subset} (K,A)$. This completes the proof. ☐

Theorem 10.7. *Let (F,A) be a soft group over G. A normal soft subgroup (H,B) of (F,A) is soft maximal if and only if the factor soft group $(F,A)/(H,B)$ is soft simple group.*

Proof. Suppose (H,B) is soft maximal and $(F,A)/(H,B)$ is not soft simple group that is $(F,A)/(H,B)$ possesses proper normal soft subgroups. Let $(K,B)/(H,B)$ be a proper normal soft subgroup of $(F,A)/(H,B)$. Then (K,B) will be a normal soft subgroup of (F,A) containing (H,B). Since $(K,B)/(H,B)$ is a proper soft subgroup of $(F,A)/(H,B)$, therefore $(H,B) \tilde{\subset} (K,B) \tilde{\subset} (F,A)$. Thus $(K,B)$is a normal soft subgroup of (F,A) and $(H,B) \tilde{\subset} (K,B) \tilde{\subset} (F,A)$. Therefore (H,B) is not soft maximal. This contradicts the hypothesis that (H,B) is maximal in (F,A). Hence $(F,A)/(H,B)$ must be soft simple.

Conversely, let $(F,A)/(H,B)$ be a soft simple and let (H,B) be not soft maximal. Since (H,B) is not soft maximal, therefore there exists a normal soft subgroup (K,B) of (F,A) such that $(H,B) \tilde{\subset} (K,B) \tilde{\subset} (F,A)$. Then by Theorem 10.6, $(K,B)/(H,B)$ is a normal soft subgroup of $(F,A)/(H,B)$. This contradicts the hypothesis, hence (H,B) must be soft maximum. ☐

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