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Rough intuitionistic fuzzy ideals in semigroups

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Abstract. In this paper basic notions of rough intuitionistic fuzzy set in semigroups are given and we discuss some of its basic properties. We introduce the notions of rough intuitionistic fuzzy left (right, two-sided, bi-, (1, 2)-) ideals in a semigroup and give some properties of such ideals.

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1. Introduction

In some sense almost all concepts, we are meeting in everyday life, are vague rather than precise. On the contrary, it is interesting to see that classical mathematics requires that all mathematical notions must be exact, otherwise precise reasoning would be impossible (Pawlak and Skowron 2007)[8]. To reduce this gap between the real world full of vagueness and the traditional mathematics purely concerning precise concepts, some kind of theories were given like theory of fuzzy sets, rough sets etc. The concept of a fuzzy set was initiated by Zadeh[11] in 1965 and the rough set theory was introduced by Pawlak[7] in 1982. Thereafter in 1986, Atanassov[1] presented the intuitionistic fuzzy sets as a generalization of fuzzy sets. In 1994, Biswas and Nanda[2] introduced the notion of rough subgroups and in 1997, Kuroki[4] introduced the notion of a rough ideal in a semigroup. So, many authors contributed different articles on these concepts and applied it on different branches of pure and applied mathematics. Based on an equivalence relation, in 1990, Dubois and Prade[3] introduced the lower and upper approximations of fuzzy sets in a Pawlak approximation space to obtain an extended notion called rough fuzzy sets. In [10], the notions of rough prime ideals and rough fuzzy prime ideals in semigroup are introduced. Kazanci and Davvaz in [6] introduced rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. In

2011, K. V. Thomas and L. S. Nair[9] introduced rough intuitionistic fuzzy sets in a lattice.
In this paper, in section 3, we substitute a semigroup instead of the universe in pawlak approximation space and define lower and upper approximation of an intuitionistic fuzzy set in semigroup and also discuss some important properties. In section 4, we define rough intuitionistic fuzzy left (right, two-sided, bi-, $(1,2)$-) ideals in semigroups and verify its some basic properties and section 5 concludes the paper.

2. Preliminaries

This section contains some basic definitions which will be needed in the sequel. In this paper, unless otherwise stated explicitly, S always denotes a semigroup with identity. The symbol $\square$ marks the end of a proof.

Definition 2.1 ([7]). Let (U, R) be an approximation space, where U is the non-empty universe, R is an equivalence relation and let X be any non-empty subset of U. Then the sets

$$R_*(X) = \{x \in U \mid [x]_R \subseteq X\}$$

$$\textit{and} \qquad R^*(X) = \{x \in U \mid [x]_R \cap X \neq \phi\}$$

are respectively called the lower approximation and upper approximation of the set X with respect to R, where $[x]_R$ denotes the equivalence class containing the element $x \in X$ with respect to R. X is called R-definable if $R_*(X) = R^*(X)$, in the opposite case, i.e. if $R_*(X) \neq R^*(X)$ then X is called rough set with respect to R.

Definition 2.2 ([1]). An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership and degree of nonmembership of the element $x \in X$ to A, respectively and satisfy $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The family of all IFS in X is denoted by $\mathcal{IFS}(X)$.
An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ in X can be identified to an ordered pair (μ_A, ν_A) in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

Definition 2.3 ([1]). If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS in X, then
(1) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$,
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$,
(3) $A^c = (\nu_A, \mu_A)$,
(4) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$, where for all $x \in X$,
$(\mu_A \cap \mu_B)(x) = \mu_A(x) \wedge \mu_B(x)$ and $(\nu_A \cup \nu_B)(x) = \nu_A(x) \vee \nu_B(x)$.
(5) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$, where for all $x \in X$,
$(\mu_A \cup \mu_B)(x) = \mu_A(x) \vee \mu_B(x)$ and $(\nu_A \cap \nu_B)(x) = \nu_A(x) \wedge \nu_B(x)$.

Definition 2.4. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS of S. Then the composition $A \circ B$ is defined by

$$A \circ B = (\mu_A \circ \mu_B, \nu_A \circ \nu_B),$$

where for all $x \in S$,

$$(\mu_A \circ \mu_B)(x) = \bigvee_{x = yz} [\mu_A(y) \wedge \mu_B(z)]$$

and $$(\nu_A \circ \nu_B)(x) = \bigwedge_{x = yz} [\nu_A(y) \vee \nu_B(z)].$$

Definition 2.5 ([5])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy subsemigroup of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \quad and \quad \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

Definition 2.6 ([5])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy left ideal of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(y) \quad and \quad \nu_A(xy) \leq \nu_A(y).$$

Definition 2.7 ([5])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy right ideal of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \quad and \quad \nu_A(xy) \leq \nu_A(x).$$

Definition 2.8 ([5])**.** An IFS $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy two-sided ideal of S if for all $x, y \in S$,

$$\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y) \quad and \quad \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y).$$

Definition 2.9 ([5])**.** An intuitionistic fuzzy subsemigroup $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy bi-ideal of S if for all $x, y, w \in S$,

$$\mu_A(xwy) \geq \mu_A(x) \wedge \mu_A(y) \quad and \quad \nu_A(xwy) \leq \nu_A(x) \vee \nu_A(y).$$

Definition 2.10 ([5])**.** An intuitionistic fuzzy subsemigroup $A = (\mu_A, \nu_A)$ of S is called an intuitionistic fuzzy $(1, 2)$-ideal of S if for all $x, y, z, w \in S$,

$$(i) \ \mu_A(xw(yz)) \geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_A(z),$$

$$(ii) \ \nu_A(xw(yz)) \leq \nu_A(x) \vee \nu_A(y) \vee \nu_A(z).$$

3. Rough Intuitionistic Fuzzy Subsets (RIFS) in a Semigroup

Let θ be a congruence relation on S, that is, θ is an equivalence relation on S such that

$$(a, b) \in \theta \Rightarrow (ax, bx) \in \theta \ and \ (xa, xb) \in \theta \ \forall x \in S.$$

We denote the θ-congruence class containing the element $a \in S$ by $[a]_\theta$. For a congruence relation θ on S, we have $[a]_\theta [b]_\theta \subseteq [ab]_\theta \ \forall a, b \in S$. A congruence relation θ on S is called *complete* if $[a]_\theta [b]_\theta = [ab]_\theta$ for all $a, b \in S$. Let $A = (\mu_A, \nu_A)$ be an *IFS* of S. Then the *IFS* $\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$ and

$\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$ are respectively called *θ-lower* and *θ-upper approximation* of the *IFS* $A = (\mu_A, \nu_A)$, where $\forall x \in S$,

$$\theta_*(\mu_A)(x) = \bigwedge_{a \in [x]_\theta} \mu_A(a), \qquad \theta_*(\nu_A)(x) = \bigvee_{a \in [x]_\theta} \nu_A(a),$$

$$\theta^*(\mu_A)(x) = \bigvee_{a \in [x]_\theta} \mu_A(a), \qquad \theta^*(\nu_A)(x) = \bigwedge_{a \in [x]_\theta} \nu_A(a).$$

For an IFS $A = (\mu_A, \nu_A)$ of S,

$$\theta(A) = (\theta_*(A), \theta^*(A))$$

is called rough intuitionistic fuzzy set with respect to θ if $\theta_*(A) \neq \theta^*(A)$.

Note 3.1.

$$\begin{aligned}\theta_*(\mu_A)(x) + \theta_*(\nu_A)(x) &= \bigwedge_{a \in [x]_\theta} \mu_A(a) + \bigvee_{a \in [x]_\theta} \nu_A(a)\\ &\leq \bigwedge_{a \in [x]_\theta} \mu_A(a) + \bigvee_{a \in [x]_\theta} (1 - \mu_A(a))\\ &= \bigwedge_{a \in [x]_\theta} \mu_A(a) + 1 - \bigwedge_{a \in [x]_\theta} \mu_A(a) = 1.\end{aligned}$$

Similarly, $\theta^*(\mu_A)(x) + \theta^*(\nu_A)(x) \leq 1$ for all $x \in S$. This justify that

$$\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A)) \text{ and } \theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$$

are IFS of S.

Theorem 3.2. *Let θ, ϕ be two congruence relations on S. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS of S, then the following hold:*

(1) $\theta_*(A) \subseteq A \subseteq \theta^*(A)$,
(2) $\theta_*(\theta_*(A)) = \theta_*(A)$,
(3) $\theta^*(\theta^*(A)) = \theta^*(A)$,
(4) $\theta^*(\theta_*(A)) = \theta_*(A)$,
(5) $\theta_*(\theta^*(A)) = \theta^*(A)$,
(6) $(\theta^*(A^c))^c = \theta_*(A)$,
(7) $(\theta_*(A^c))^c = \theta^*(A)$,
(8) $\theta_*(A \cap B) = \theta_*(A) \cap \theta_*(B)$,
(9) $\theta^*(A \cap B) \subseteq \theta^*(A) \cap \theta^*(B)$,
(10) $\theta^*(A \cup B) = \theta^*(A) \cup \theta^*(B)$,
(11) $\theta_*(A \cup B) \supseteq \theta_*(A) \cup \theta_*(B)$,
(12) $A \subseteq B \Rightarrow \theta_*(A) \subseteq \theta_*(B)$,
(13) $A \subseteq B \Rightarrow \theta^*(A) \subseteq \theta^*(B)$,
(14) $\theta \subseteq \phi \Rightarrow \theta_*(A) \supseteq \phi_*(A)$,
(15) $\theta \subseteq \phi \Rightarrow \theta^*(A) \subseteq \phi^*(A)$.

Proof. This is easily obtained from Definition 2.3 and Definition of lower and upper approximation of an intuitionistic fuzzy set. □

Theorem 3.3. *Let θ be a congruence relation on S. If A, B are any two IFS of S, then $\theta^*(A) \circ \theta^*(B) \subseteq \theta^*(A \circ B)$.*

Proof. Since θ is a congruence relation on S,

$$[a]_\theta [b]_\theta \subseteq [ab]_\theta \quad \forall\, a, b \in S.$$

Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be any two IFS of S. Then

$$\theta^*(A) \circ \theta^*(B) = (\theta^*(\mu_A) \circ \theta^*(\mu_B), \theta^*(\nu_A) \circ \theta^*(\nu_B))$$

and $\theta^*(A \circ B) = (\theta^*(\mu_A \circ \mu_B), \theta^*(\nu_A \circ \nu_B))$. To show

$$\theta^*(A) \circ \theta^*(B) \subseteq \theta^*(A \circ B),$$

we have to prove that for all $x \in S$,

$$(\theta^*(\mu_A) \circ \theta^*(\mu_B))(x) \leq \theta^*(\mu_A \circ \mu_B)(x)$$

and $$(\theta^*(\nu_A) \circ \theta^*(\nu_B))(x) \geq \theta^*(\nu_A \circ \nu_B)(x).$$

Now for all $x \in S$,

$$\begin{aligned}
&(\theta^*(\mu_A) \circ \theta^*(\mu_B))(x) = \bigvee_{x=yz} [\theta^*(\mu_A)(y) \wedge \theta^*(\mu_B)(z)] \\
&= \bigvee_{x=yz} \Big[\Big(\bigvee_{a \in [y]_\theta} \mu_A(a) \Big) \wedge \Big(\bigvee_{b \in [z]_\theta} \mu_B(b) \Big) \Big] \\
&= \bigvee_{x=yz} \Big[\bigvee_{a \in [y]_\theta,\, b \in [z]_\theta} (\mu_A(a) \wedge \mu_B(b)) \Big] \\
&\leq \bigvee_{x=yz} \Big[\bigvee_{ab \in [yz]_\theta} (\mu_A(a) \wedge \mu_B(b)) \Big], \quad \text{since } ab \in [y]_\theta [z]_\theta \subseteq [yz]_\theta \\
&= \bigvee_{ab \in [x]_\theta} (\mu_A(a) \wedge \mu_B(b)), \quad \text{since } yz = x \\
&= \bigvee_{\alpha \in [x]_\theta,\, \alpha = ab} (\mu_A(a) \wedge \mu_B(b)) \\
&= \bigvee_{\alpha \in [x]_\theta} \bigvee_{\alpha = ab} (\mu_A(a) \wedge \mu_B(b)) \\
&= \bigvee_{\alpha \in [x]_\theta} (\mu_A \circ \mu_B)(\alpha) = \theta^*(\mu_A \circ \mu_B)(x).
\end{aligned}$$

Again,

$$\begin{aligned}
&(\theta^*(\nu_A) \circ \theta^*(\nu_B))(x) = \bigwedge_{x=yz} [\theta^*(\nu_A)(y) \vee \theta^*(\nu_B)(z)] \\
&= \bigwedge_{x=yz} \Big[\Big(\bigwedge_{a \in [y]_\theta} \nu_A(a) \Big) \vee \Big(\bigwedge_{b \in [z]_\theta} \nu_B(b) \Big) \Big] \\
&= \bigwedge_{x=yz} \Big[\bigwedge_{a \in [y]_\theta,\, b \in [z]_\theta} (\nu_A(a) \vee \nu_B(b)) \Big] \\
&\geq \bigwedge_{x=yz} \Big[\bigwedge_{ab \in [yz]_\theta} (\nu_A(a) \vee \nu_B(b)) \Big], \quad \text{since } ab \in [y]_\theta [z]_\theta \subseteq [yz]_\theta \\
&= \bigwedge_{ab \in [x]_\theta} (\nu_A(a) \vee \nu_B(b)), \quad \text{since } yz = x \\
&= \bigwedge_{\alpha \in [x]_\theta,\, \alpha = ab} (\nu_A(a) \vee \nu_B(b)) \\
&= \bigwedge_{\alpha \in [x]_\theta} \bigwedge_{\alpha = ab} (\nu_A(a) \vee \nu_B(b)) \\
&= \bigwedge_{\alpha \in [x]_\theta} (\nu_A \circ \nu_B)(\alpha) = \theta^*(\nu_A \circ \nu_B)(x).
\end{aligned}$$

Thus we have $\theta^*(A) \circ \theta^*(B) \subseteq \theta^*(A \circ B)$. □

Theorem 3.4. *Let θ be a complete congruence relation on S. If A and B are any two IFS of S, then $\theta_*(A) \circ \theta_*(B) \subseteq \theta_*(A \circ B)$.*

Proof. Since θ is complete congruence relation on S,

$$[a]_\theta [b]_\theta = [ab]_\theta \quad \forall\, a, b \in S.$$

Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be any two IFS of S. Then

$$\theta_*(A) \circ \theta_*(B) = (\theta_*(\mu_A) \circ \theta_*(\mu_B), \theta_*(\nu_A) \circ \theta_*(\nu_B))$$

and $\theta_*(A \circ B) = (\theta_*(\mu_A \circ \mu_B), \theta_*(\nu_A \circ \nu_B))$. To show

$$\theta_*(A) \circ \theta_*(B) \subseteq \theta_*(A \circ B),$$

we have to prove that for all $x \in S$,

$$(\theta_*(\mu_A) \circ \theta_*(\mu_B))(x) \leq \theta_*(\mu_A \circ \mu_B)(x)$$

and $$(\theta_*(\nu_A) \circ \theta_*(\nu_B))(x) \geq \theta_*(\nu_A \circ \nu_B)(x).$$

Now for all $x \in S$,

$$\begin{aligned}
&(\theta_*(\mu_A) \circ \theta_*(\mu_B))(x) = \bigvee_{x=yz} [\theta_*(\mu_A)(y) \wedge \theta_*(\mu_B)(z)] \\
&= \bigvee_{x=yz} [(\bigwedge_{a \in [y]_\theta} \mu_A(a)) \wedge (\bigwedge_{b \in [z]_\theta} \mu_B(b))] \\
&= \bigvee_{x=yz} [\bigwedge_{a \in [y]_\theta,\, b \in [z]_\theta} (\mu_A(a) \wedge \mu_B(b))] \\
&\leq \bigvee_{x=yz} [\bigwedge_{a \in [y]_\theta,\, b \in [z]_\theta} \bigvee_{ab=\alpha\beta} (\mu_A(\alpha) \wedge \mu_B(\beta))] \quad \text{where } \alpha, \beta \in S \\
&= \bigvee_{x=yz} [\bigwedge_{a \in [y]_\theta,\, b \in [z]_\theta} (\mu_A \circ \mu_B)(ab)] \\
&= \bigvee_{x=yz} [\bigwedge_{ab \in [yz]_\theta} (\mu_A \circ \mu_B)(ab)], \quad \text{since } ab \in [y]_\theta [z]_\theta = [yz]_\theta \\
&= \bigvee_{x=yz} \theta_*(\mu_A \circ \mu_B)(yz) = \theta_*(\mu_A \circ \mu_B)(x).
\end{aligned}$$

Again,

$$\begin{aligned}
&(\theta_*(\nu_A) \circ \theta_*(\nu_B))(x) = \bigwedge_{x=yz} [\theta_*(\nu_A)(y) \vee \theta_*(\nu_B)(z)] \\
&= \bigwedge_{x=yz} [(\bigvee_{a \in [y]_\theta} \nu_A(a)) \vee (\bigvee_{b \in [z]_\theta} \nu_B(b))] \\
&= \bigwedge_{x=yz} [\bigvee_{a \in [y]_\theta,\, b \in [z]_\theta} (\nu_A(a) \vee \nu_B(b))] \\
&\geq \bigwedge_{x=yz} [\bigvee_{a \in [y]_\theta,\, b \in [z]_\theta} \bigwedge_{ab=\alpha\beta} (\nu_A(\alpha) \vee \nu_B(\beta))] \quad \text{where } \alpha, \beta \in S \\
&= \bigwedge_{x=yz} [\bigvee_{a \in [y]_\theta,\, b \in [z]_\theta} (\nu_A \circ \nu_B)(ab)] \\
&= \bigwedge_{x=yz} [\bigvee_{ab \in [yz]_\theta} (\nu_A \circ \nu_B)(ab)], \quad \text{since } ab \in [y]_\theta [z]_\theta = [yz]_\theta \\
&= \bigwedge_{x=yz} \theta_*(\nu_A \circ \nu_B)(yz) = \theta_*(\nu_A \circ \nu_B)(x).
\end{aligned}$$

Thus we have $\theta_*(A) \circ \theta_*(B) \subseteq \theta_*(A \circ B)$. □

Theorem 3.5. *Let θ, ϕ be two congruence relations on S. If A is an IFS of S, then $(\theta \cap \phi)^*(A) \subseteq \theta^*(A) \cap \phi^*(A)$.*

Proof. It is easy to observe that $\theta \cap \phi$ is also a congruence relation of S and it is also clear that $\theta \cap \phi \subseteq \theta$ and $\theta \cap \phi \subseteq \phi$. Let $A = (\mu_A, \nu_A)$ be an IFS of S. Then by using Theorem 3.2 (15), we obtain

$$(\theta \cap \phi)^*(A) \subseteq \theta^*(A) \text{ and } (\theta \cap \phi)^*(A) \subseteq \phi^*(A).$$

Therefore, using definition 2.3 (1), (4), we have $(\theta \cap \phi)^*(A) \subseteq \theta^*(A) \cap \phi^*(A)$. □

Theorem 3.6. *Let θ, ϕ be two congruence relations on S. If A is an IFS of S, then*

$$(\theta \cap \phi)_*(A) \supseteq \theta_*(A) \cup \phi_*(A).$$

Proof. It is easy to observe that $\theta \cap \phi$ is also a congruence relation of S and it is also clear that $\theta \cap \phi \subseteq \theta$ and $\theta \cap \phi \subseteq \phi$. Let $A = (\mu_A, \nu_A)$ be an IFS of S. Then by using Theorem 3.2 (14), we obtain

$$(\theta \cap \phi)_*(A) \supseteq \theta_*(A) \text{ and } (\theta \cap \phi)_*(A) \supseteq \phi_*(A).$$

Therefore, using definition 2.3 (1), (5), we get $(\theta \cap \phi)_*(A) \supseteq \theta_*(A) \cup \phi_*(A)$. □

4. Rough intuitionistic fuzzy ideals (RIFI) in a semigroup

Definition 4.1. Let θ be a congruence relation on S. An IFS A of S is called an upper (lower) rough intuitionistic fuzzy subsemigroup of S if $\theta^*(A)$ ($\theta_*(A)$) is an intuitionistic fuzzy subsemigroup of S.
A is called a rough intuitionistic fuzzy subsemigroup of S if $\theta^*(A)$ and $\theta_*(A)$ are both intuitionistic fuzzy subsemigroup of S.

Definition 4.2. Let θ be a congruence relation on S. An IFS A of S is called an upper rough intuitionistic fuzzy left (right, two-sided) ideal of S if $\theta^*(A)$ is an intuitionistic fuzzy left (right, two-sided) ideal of S.

Definition 4.3. Let θ be a congruence relation on S. An IFS A of S is called a lower rough intuitionistic fuzzy left (right, two-sided) ideal of S if $\theta_*(A)$ is an intuitionistic fuzzy left (right, two-sided) ideal of S.

Definition 4.4. Let θ be a congruence relation on S. An IFS A of S is called an upper (lower) rough intuitionistic fuzzy bi-ideal of S if $\theta^*(A)$ ($\theta_*(A)$) is an intuitionistic fuzzy bi-ideal of S.

Definition 4.5. Let θ be a congruence relation on S. An IFS A of S is called an upper (lower) rough intuitionistic fuzzy $(1, 2)$-ideal of S if $\theta^*(A)$ ($\theta_*(A)$) is an intuitionistic fuzzy $(1, 2)$-ideal of S.

Definition 4.6. Let θ be a congruence relation on S. An IFS A of S is called a rough intutionistic fuzzy left (right, two-sided, bi-, $(1, 2)$-) ideal of S if $\theta^*(A)$ and $\theta_*(A)$ are both intuitionistic fuzzy left (right, two-sided, bi-, $(1, 2)$-) ideal of S.

Theorem 4.7. *Let θ be a congruence relation on S. Then*

(1) *If A is an intuitionistic fuzzy subsemigroup of S, then A is an upper rough intuitionistic fuzzy subsemigroup of S.*

(2) *If A is an intuitionistic fuzzy left (right, two-sided) ideal of S, then A is an upper rough intuitionistic fuzzy left (right, two-sided) ideal of S.*

Proof. Since θ is a congruence relation on S,

$$[a]_\theta [b]_\theta \subseteq [ab]_\theta \quad \forall a, b \in S.$$

(1) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subsemigroup of S and $x, y \in S$. Now $\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$. Then

$$\theta^*(\mu_A)(xy) = \bigvee_{z \in [xy]_\theta} \mu_A(z) \geq \bigvee_{z \in [x]_\theta [y]_\theta} \mu_A(z) = \bigvee_{ab \in [x]_\theta [y]_\theta} \mu_A(ab)$$

$$\geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)] = \bigvee_{a \in [x]_\theta} \mu_A(a) \wedge \bigvee_{b \in [y]_\theta} \mu_A(b)$$

$$= \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y).$$

and

$$\theta^*(\nu_A)(xy) = \bigwedge_{z \in [xy]_\theta} \nu_A(z) \leq \bigwedge_{z \in [x]_\theta [y]_\theta} \nu_A(z) = \bigwedge_{ab \in [x]_\theta [y]_\theta} \nu_A(ab)$$

$$\leq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} [\nu_A(a) \vee \nu_A(b)] = \bigwedge_{a \in [x]_\theta} \nu_A(a) \vee \bigwedge_{b \in [y]_\theta} \nu_A(b)$$

$$= \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y).$$

This shows that $\theta^*(A)$ is an intuitionistic fuzzy subsemigroup of S. Therefore, A is an upper rough intuitionistic fuzzy subsemigroup of S.

(2) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy left ideal of S and $x, y \in S$. Now $\theta^*(A) = (\theta^*(\mu_A), \theta^*(\nu_A))$. Then

$$\theta^*(\mu_A)(xy) = \bigvee_{z \in [xy]_\theta} \mu_A(z) \geq \bigvee_{z \in [x]_\theta [y]_\theta} \mu_A(z) = \bigvee_{ab \in [x]_\theta [y]_\theta} \mu_A(ab)$$

$$\geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_A(b) = \bigvee_{b \in [y]_\theta} \mu_A(b) = \theta^*(\mu_A)(y).$$

and

$$\theta^*(\nu_A)(xy) = \bigwedge_{z \in [xy]_\theta} \nu_A(z) \leq \bigwedge_{z \in [x]_\theta [y]_\theta} \nu_A(z) = \bigwedge_{ab \in [x]_\theta [y]_\theta} \nu_A(ab)$$

$$\leq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \nu_A(b) = \bigwedge_{b \in [y]_\theta} \nu_A(b) = \theta^*(\nu_A)(y).$$

This shows that $\theta^*(A)$ is an intuitionistic fuzzy left ideal of S. Therefore, A is an upper rough intuitionistic fuzzy left ideal of S. Similarly we can prove the other cases. □

The following example shows that the converse of Theorem 4.7 does not hold in general.

Example 4.8. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Let θ be a congruence relation on S such that the θ-congruence classes are the subsets $\{a\}$, $\{d\}$, $\{b, c\}$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in S\}$, be an intuitionistic fuzzy subset of S, defined by

$$A = \{\langle a, 0.3, 0.4\rangle, \langle b, 0.2, 0.3\rangle, \langle c, 0.4, 0.5\rangle, \langle d, 0.3, 0.4\rangle\}$$

Since for every $x \in S$, $\theta^*(\mu_A)(x) = \bigvee_{\alpha \in [x]_\theta} \mu_A(\alpha)$ and $\theta^*(\nu_A)(x) = \bigwedge_{\alpha \in [x]_\theta} \nu_A(\alpha)$, so the upper approximation $\theta^*(A) = \{\langle x, \theta^*(\mu_A)(x), \theta^*(\nu_A)(x)\rangle \mid x \in S\}$ is given by

$$\theta^*(A) = \{\langle a, 0.3, 0.4\rangle, \langle b, 0.4, 0.3\rangle, \langle c, 0.4, 0.3\rangle, \langle d, 0.3, 0.4\rangle\}$$

Then it can be easily verified that

$$\theta^*(\mu_A)(xy) \geq \theta^*(\mu_A)(x) \vee \theta^*(\mu_A)(y),$$

$$\theta^*(\nu_A)(xy) \leq \theta^*(\nu_A)(x) \wedge \theta^*(\nu_A)(y)$$

for all $x, y \in S$. Therefore $\theta^*(A)$ is an intutionistic fuzzy two-sided ideal of S.

Now $a.b = b$ (by composition table given above). So, $\mu_A(ab) = \mu_A(b) = 0.2$ and $\mu_A(a) = 0.3$. Hence $\mu_A(ab) < \mu_A(a) \vee \mu_A(b)$.

Again, $a.c = c$ (by composition table given above). So, $\nu_A(ac) = \nu_A(c) = 0.5$ and $\nu_A(a) = 0.4$. Hence $\nu_A(ac) > \nu_A(a) \wedge \nu_A(c)$. So, this shows that

$$(\mu_A)(xy) \geq (\mu_A)(x) \vee (\mu_A)(y),$$

$$(\nu_A)(xy) \leq (\nu_A)(x) \wedge (\nu_A)(y)$$

are not hold for all $x, y \in S$. So, A is not an intuitionistic fuzzy two-sided ideal of S but A is an upper rough intutionistic fuzzy two-sided of S.

Theorem 4.9. *Let θ be a complete congruence relation on S. Then*

(1) *If A is an intuitionistic fuzzy subsemigroup of S, then A is a lower rough intuitionistic fuzzy subsemigroup of S.*

(2) *If A is an intuitionistic fuzzy left (right, two-sided) ideal of S, then A is a lower rough intuitionistic fuzzy left (right, two-sided) ideal of S.*

Proof. Since θ is a complete congruence relation on S,

$$[a]_\theta [b]_\theta = [ab]_\theta \quad \forall\, a, b \in S.$$

(1) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subsemigroup of S and $x, y \in S$. Now $\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$. Then

$$\theta_*(\mu_A)(xy) = \bigwedge_{z \in [xy]_\theta} \mu_A(z) = \bigwedge_{z \in [x]_\theta [y]_\theta} \mu_A(z) = \bigwedge_{ab \in [x]_\theta [y]_\theta} \mu_A(ab)$$

$$\geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)] = \bigwedge_{a \in [x]_\theta} \mu_A(a) \wedge \bigwedge_{b \in [y]_\theta} \mu_A(b)$$

$$= \theta_*(\mu_A)(x) \wedge \theta_*(\mu_A)(y)$$

and

$$\theta_*(\nu_A)(xy) = \bigvee_{z \in [xy]_\theta} \nu_A(z) = \bigvee_{z \in [x]_\theta [y]_\theta} \nu_A(z) = \bigvee_{ab \in [x]_\theta [y]_\theta} \nu_A(ab)$$

$$\leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} [\nu_A(a) \vee \nu_A(b)] = \bigvee_{a \in [x]_\theta} \nu_A(a) \vee \bigvee_{b \in [y]_\theta} \nu_A(b)$$
$$= \theta_*(\nu_A)(x) \vee \theta_*(\nu_A)(y).$$

This shows that $\theta_*(A)$ is an intuitionistic fuzzy subsemigroup of S. Therefore, A is an lower rough intuitionistic fuzzy subsemigroup of S.

(2) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy right ideal of S and $x, y \in S$. Now $\theta_*(A) = (\theta_*(\mu_A), \theta_*(\nu_A))$. Then

$$\begin{aligned} \theta_*(\mu_A)(xy) &= \bigwedge_{z \in [xy]_\theta} \mu_A(z) = \bigwedge_{z \in [x]_\theta[y]_\theta} \mu_A(z) = \bigwedge_{ab \in [x]_\theta[y]_\theta} \mu_A(ab) \\ &\geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu_A(a) = \bigwedge_{a \in [x]_\theta} \mu_A(a) = \theta_*(\mu_A)(x) \end{aligned}$$

and

$$\begin{aligned} \theta_*(\nu_A)(xy) &= \bigvee_{z \in [xy]_\theta} \nu_A(z) = \bigvee_{z \in [x]_\theta[y]_\theta} \nu_A(z) = \bigvee_{ab \in [x]_\theta[y]_\theta} \nu_A(ab) \\ &\leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \nu_A(a) = \bigvee_{a \in [x]_\theta} \nu_A(a) \\ &= \theta_*(\nu_A)(x) \end{aligned}$$

This shows that $\theta_*(A)$ is an intuitionistic fuzzy right ideal of S. Therefore, A is an lower rough intuitionistic fuzzy right ideal of S. Similarly we can prove the other cases. □

Theorem 4.10. *Let θ be a congruence relation on S. If A is an intuitionistic fuzzy bi-ideal of S, then it is upper rough intuitionistic fuzzy bi-ideal of S.*

Proof. Since θ is a congruence relation on S,

$$[a]_\theta [b]_\theta \subseteq [ab]_\theta \quad \forall a, b \in S.$$

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy bi-ideal of S and $x, y, w \in S$. Then

$$\theta^*(\mu_A)(xwy) = \bigvee_{z \in [xwy]_\theta} \mu_A(z) \geq \bigvee_{z \in [x]_\theta[w]_\theta[y]_\theta} \mu_A(z)$$
$$= \bigvee_{asb \in [x]_\theta[w]_\theta[y]_\theta} \mu_A(asb) \geq \bigvee_{a \in [x]_\theta, s \in [w]_\theta b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)]$$
$$= \bigvee_{a \in [x]_\theta b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)] = \bigvee_{a \in [x]_\theta} \mu_A(a) \wedge \bigvee_{b \in [y]_\theta} \mu_A(b)$$
$$= \theta^*(\mu_A)(x) \wedge \theta^*(\mu_A)(y).$$

Similarly we have $\theta^*(\nu_A)(xwy) \leq \theta^*(\nu_A)(x) \vee \theta^*(\nu_A)(y)$. This shows that $\theta^*(A)$ is an intuitionistic fuzzy bi-ideal of S. Therefore, A is an upper rough intuitionistic fuzzy bi-ideal of S. □

Theorem 4.11. *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy bi-ideal of S, then it is lower rough intuitionistic fuzzy bi-ideal of S.*

Proof. Since θ is a complete congruence relation on S,

$$[a]_\theta [b]_\theta = [ab]_\theta \quad \forall a, b \in S.$$

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy bi-ideal of S and $x, y, w \in S$. Then

$$\begin{aligned}
\theta_*(\mu_A)(xwy) &= \bigwedge_{z \in [xwy]_\theta} \mu_A(z) = \bigwedge_{z \in [x]_\theta [w]_\theta [y]_\theta} \mu_A(z) \\
&= \bigwedge_{asb \in [x]_\theta [w]_\theta [y]_\theta} \mu_A(asb) \\
&\geq \bigwedge_{a \in [x]_\theta, s \in [w]_\theta, b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)] \\
&= \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} [\mu_A(a) \wedge \mu_A(b)] \\
&= \bigwedge_{a \in [x]_\theta} \mu_A(a) \wedge \bigwedge_{b \in [y]_\theta} \mu_A(b) \\
&= \theta_*(\mu_A)(x) \wedge \theta_*(\mu_A)(y)
\end{aligned}$$

Similarly we have $\theta_*(\nu_A)(xwy) \leq \theta_*(\nu_A)(x) \vee \theta_*(\nu_A)(y)$. This shows that $\theta_*(A)$ is an intuitionistic fuzzy bi-ideal of S. Therefore, A is an lower rough intuitionistic fuzzy bi-ideal of S. □

Theorem 4.12. *Let θ be a congruence relation on S. If A is an intuitionistic fuzzy $(1, 2)$-ideal of S, then it is upper rough intuitionistic fuzzy $(1, 2)$-ideal of S.*

Proof. Proof is similar to Theorem 4.10. □

Theorem 4.13. *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy $(1, 2)$-ideal of S, then it is lower rough intuitionistic fuzzy $(1, 2)$-ideal of S.*

Proof. Proof is similar to Theorem 4.11. □

Corollary 4.14. *Let θ be a complete congruence relation on S. If A is an intuitionistic fuzzy left (right, two-sided, bi-, $(1, 2)$-) ideal of S, then A is a rough intuitionistic fuzzy left (right, two-sided, bi-, $(1, 2)$-) ideal of S.*

Proof. This follows from Definition 4.6 and Theorems 4.7, 4.9, 4.10, 4.11, 4.12, 4.13. □

Theorem 4.15. *Let θ be a congruence relation on S. If A and B are an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta^*(A \circ B) \subseteq \theta^*(A) \cap \theta^*(B).$$

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively and $x \in S$. Then

$$\theta^*(A \circ B) = (\theta^*(\mu_A \circ \mu_B), \theta^*(\nu_A \circ \nu_B))$$

and

$$\theta^*(A) \cap \theta^*(B) = (\theta^*(\mu_A) \cap \theta^*(\mu_B), \theta^*(\nu_A) \cup \theta^*(\nu_B)).$$

To show $\theta^*(A \circ B) \subseteq \theta^*(A) \cap \theta^*(B)$, we have to prove that for all $x \in S$

$$\theta^*(\mu_A \circ \mu_B)(x) \leq (\theta^*(\mu_A) \cap \theta^*(\mu_B))(x)$$

and

$$\theta^*(\nu_A \circ \nu_B)(x) \geq (\theta^*(\nu_A) \cup \theta^*(\nu_B))(x).$$

Now for all $x \in S$,

$$\begin{aligned}
\theta^*(\mu_A \circ \mu_B)(x) &= \bigvee_{a \in [x]_\theta} (\mu_A \circ \mu_B)(a) \\
&= \bigvee_{a \in [x]_\theta} \bigvee_{a = yz} [\mu_A(y) \wedge \mu_B(z)] \\
&\leq \bigvee_{a \in [x]_\theta} \bigvee_{a = yz} [\mu_A(yz) \wedge \mu_B(yz)] \\
&\quad \text{(since } A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \text{ are intuitionistic} \\
&\quad \text{ fuzzy right and intuitionistic fuzzy left ideal, respectively)} \\
&= \bigvee_{a \in [x]_\theta} [\mu_A(a) \wedge \mu_B(a)] \\
&\leq \bigvee_{a \in [x]_\theta, b \in [x]_\theta} [\mu_A(a) \wedge \mu_B(b)] \\
&= \bigvee_{a \in [x]_\theta} \mu_A(a) \wedge \bigvee_{b \in [x]_\theta} \mu_B(b) \\
&= \theta^*(\mu_A)(x) \wedge \theta^*(\mu_B)(x) \\
&= (\theta^*(\mu_A) \cap \theta^*(\mu_B))(x)
\end{aligned}$$

Similarly we have $\theta^*(\nu_A \circ \nu_B)(x) \geq (\theta^*(\nu_A) \cup \theta^*(\nu_B))(x)$. Therefore, $\theta^*(A \circ B) \subseteq \theta^*(A) \cap \theta^*(B)$. □

Theorem 4.16. *Let θ be a congruence relation on S. If A and B are an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively, then*

$$\theta_*(A \circ B) \subseteq \theta_*(A) \cap \theta_*(B).$$

Proof. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy right ideal and an intuitionistic fuzzy left ideal of S, respectively and $x \in S$.
Then

$$\theta_*(A \circ B) = (\theta_*(\mu_A \circ \mu_B), \theta_*(\nu_A \circ \nu_B))$$

and

$$\theta_*(A) \cap \theta_*(B) = (\theta_*(\mu_A) \cap \theta_*(\mu_B), \theta_*(\nu_A) \cup \theta_*(\nu_B))$$

To show $\theta_*(A \circ B) \subseteq \theta_*(A) \cap \theta_*(B)$, we have to prove that for all $x \in S$

$$\theta_*(\mu_A \circ \mu_B)(x) \leq (\theta_*(\mu_A) \cap \theta_*(\mu_B))(x)$$

and $$\theta_*(\nu_A \circ \nu_B)(x) \geq (\theta_*(\nu_A) \cup \theta_*(\nu_B))(x).$$

Now for all $x \in S$,

$$
\begin{aligned}
\theta_*(\mu_A \circ \mu_B)(x) &= \bigwedge_{a \in [x]_\theta} (\mu_A \circ \mu_B)(a) \\
&= \bigwedge_{a \in [x]_\theta} \bigvee_{a = yz} [\mu_A(y) \wedge \mu_B(z)] \\
&\leq \bigwedge_{a \in [x]_\theta} \bigvee_{a = yz} [\mu_A(yz) \wedge \mu_B(yz)] \\
&\quad \text{(since} A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \text{ are intuitionistic} \\
&\quad \text{fuzzy right and intuitionistic fuzzy left ideal, respectively)} \\
&= \bigwedge_{a \in [x]_\theta} [\mu_A(a) \wedge \mu_B(a)] \\
&= \bigwedge_{a \in [x]_\theta} \mu_A(a) \wedge \bigwedge_{b \in [x]_\theta} \mu_B(b) \\
&= \theta_*(\mu_A)(x) \wedge \theta_*(\mu_B)(x) \\
&= (\theta_*(\mu_A) \cap \theta_*(\mu_B))(x)
\end{aligned}
$$

Similarly we have $\theta_*(\nu_A \circ \nu_B)(x) \geq (\theta_*(\nu_A) \cup \theta_*(\nu_B))(x)$. Therefore, $\theta_*(A \circ B) \subseteq \theta_*(A) \cap \theta_*(B)$. □

5. Conclusions

In the present paper, we substitute a universe set by a semigroup with identity and introduce rough intuitionistic fuzzy left(right, two-sided, bi-, $(1,2)$-) ideals and focus on some of its properties from theoretical point of view. It will be natural to continue this work by studying other algebraic structures.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42 (1994) 251–254.
[3] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (1990) 191–209.
[4] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100 (1997) 139–163.
[5] K. H. Kim and Y. B. Jun, Intuitionistic fuzzy ideals of semigroups, Indian J. Pure Appl. Math. 33(4) (2002) 443–449.
[6] Osman Kazanci and B. Davvaz, On the structure of rough prime(primary) ideals and rough fuzzy prime(primary) ideals in commutative rings, Inform. Sci. 178 (2008) 1343–1354.
[7] Z. Pawlak, Rough sets, Internat. J. Comput. Inform. Sci. 11 (1982) 341–356.
[8] Z. Pawlak and A. Skowron, Rudiments of rough sets, Inform. Sci. 177 (2007) 3–27.
[9] K. V. Thomas, L. S. Nair, Rough intutionistic fuzzy sets in a lattice, Int. Math. Forum 6(27) (2011) 1327–1335.
[10] Q. M. Xiao, Z. L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semigroups, Inform. Sci. 176 (2006) 725–733.
[11] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

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