Annals of Fuzzy Mathematics and Informatics
Volume 4, No. 1, (July 2012), pp. 143- 154
ISSN 2093–9310
http://www.afmi.or.kr

On generalized open fuzzy sets

Ratna Dev Sarma, A. Sharfuddin, Anubha Bhargava

Received 18 August 2011; Revised 27 October 2011; Accepted 16 November 2011

Abstract. Convergence theory for the generalized fuzzy topological spaces is developed. Some important subclasses of $\Gamma(X)$, the class of monotonic mappings on X, are also discussed.

2010 AMS Classification: 54A20

Keywords: Generalized open fuzzy sets, Interior, Closure, Generalized fuzzy topology, g-net, Convergence classes.

Corresponding Author: Ratna Dev Sarma (ratna_sarma@yahoo.com)

1. Introduction

Three years after Zadeh introduced the notion of a fuzzy set in his seminal paper in 1965 [40], the first paper of fuzzy topology appeared in 1968 [8]. Since then, more than thousand research papers have been published so far in the field of fuzzy topology. The first few years, were formative years for fuzzy topology. During this period, Azad, Chang, Gouguen, Hutton, Lowen, Pu and Liu, Pascali, Wang, Wong, Warren, etc. [4, 5, 8, 15, 18, 19, 20, 21, 36, 37, 38], among others, investigated different aspects of fuzzy topology and gave it a sound footing. The following years were marked by justification for and experiments in fuzzy topology. Why do we need fuzzy topology? What is the best definition of a fuzzy topology? Which extension of a topological notion is to be treated as superior to its competitors? L-fuzzy topology–fixed and variable basis, intuitionistic fuzzy topology etc. may be ascribed to this phase. Extensive use of category theory substantiated the superiority of FTOP over TOP: that fuzzy topology has been able to provide solutions to the hitherto unsolved problems of general topology [22, 23, 25, 26, 34]. The other line of justification came through exploring and establishing the intrinsic fuzziness of several topological notions [1, 2, 3, 13, 21, 24, 26, 35, 39].

The present phase of fuzzy topology may be termed as its expanding phase. At this stage, many existing notions of fuzzy topology are being revisited for further investigations and improvement [6, 7, 9, 12, 14, 16, 26, 28, 29]. Presently, topological ideas are being tried and tested in several areas beyond the domain of pure mathematics. In some of these applications, generalized form of open sets are being used [10, 11, 17, 31, 33]. This enhances the scope as well as acts as motivation for further study of fuzzy topology. As a result, one is naturally prompted to study the generalized form of open sets in fuzzy settings with new perspective. With this background behind, we have introduced and investigated a class of generalized open fuzzy sets in [30, 32]. These weaker forms of open fuzzy sets, under different sets of conditions are found to represent different generalized forms of open fuzzy sets already existing in the literature. Also, there it was found that these generalized open fuzzy sets form a structure which is a generalized form of a fuzzy topology. In fact, these generalized fuzzy topological spaces along with generalized fuzzy continuous mappings form a category which accommodates FTOP as a subcategory. The present paper is a sequel to our ongoing investigations in this direction. Here, we investigate the theory of convergence for the generalized fuzzy topologies. For this purpose, in section 3, we have introduced g-net, a generalized notion for a fuzzy net. We have constructed the convergence classes for the generalized fuzzy topological spaces. It is found that there is a one-to-one correspondence between the convergence classes generated by the g-nets and the generalized fuzzy topologies on a given set X. In the last section, we discuss some important properties of monotonic mappings which generate γ-open fuzzy sets, a special case of generalized open fuzzy sets. We have also discussed few interesting results of some subclasses of these monotonic mappings.

2. Preliminaries

Throughout this paper, fuzzy sets are denoted by A, B, C etc and X, Y, Z etc. denote the ordinary sets. A fuzzy set A on a set X is a mapping $A : X \to [0, 1]$. The constant fuzzy sets which take each member of X to zero and to one respectively are denoted by $\underline{0}$ and $\underline{1}$ respectively. The union and intersection of a family of fuzzy sets $\{A_i\}$, denoted by $\vee A_i$ and $\wedge A_i$ respectively, are defined by

$$\left(\underset{i\in\Delta}{\vee} A_i\right)(x) = \sup\{A_i(\mathbf{x}) : i \in \Delta\} \text{ and}$$
$$\left(\underset{i\in\Delta}{\wedge} A_i\right)(x) = \inf\{A_i(\mathbf{x}) : i \in \Delta\} \text{ respectively.}$$

The complement of a fuzzy set A, denoted by A^c, is defined by

$$A^c(x) = 1 - A(x) \quad \text{for all } x \in X.$$

A fuzzy point x_α, with support x and value α, $0 < \alpha \leq 1$, is a fuzzy set which takes value α at x and 0 at every other point of X. While $x_\alpha \leq A$ implies $\alpha \leq A(x)$; $x_\alpha \in A$ implies $\alpha < A(x)$. For fuzzy sets A and B, $A \leq B$ implies $A(x) \leq B(x)$ for each x. The dual fuzzy point of a fuzzy point x_α, where $0 < \alpha < 1$, is defined to be the fuzzy point $x_{1-\alpha}$ and is denoted by $x_{\alpha'}$.

Definition 2.1. Two fuzzy sets A and B are said to overlap, denoted by AqB, if there exists x in X such that $A(x) + B(x) > 1$.

For other notations and definitions used in this paper, please refer to [3, 21].

3. Convergence Classes for Generalized Fuzzy Topological Spaces

A generalized fuzzy topological space on a set X has been defined in the following way [30, 32]:

Definition 3.1. Let X be a nonempty set. Let ζ be a collection of fuzzy sets on X such that

(1) $\underline{0} \in \zeta$;
(2) For $G_i \in \zeta$, $i \in \Delta$, $\underset{i\in\Delta}{\vee} G_i \in \zeta$.

Then ζ is called a *generalized fuzzy topology* (GFT, in short) on X and (X, ζ) is called a *generalized fuzzy topological space* (GFTS, in brief). The collection of all GFT's on X is denoted by $\zeta(X)$. Members of ζ are called *generalized open fuzzy sets* and their complements are called *generalized closed fuzzy sets* on X.

Example 3.2. Consider $X = \{a, b, c\}$ and let

$$\zeta = \{\underline{0}, a_{1/2} \vee b_{3/4}, b_{3/4} \vee c_{1/4}, a_{1/2} \vee b_{3/4} \vee c_{1/4}\}.$$

Then ζ is a GFT on X.

Further, let (X, ζ) and (Y, ζ') be two GFTS's. A mapping $f : X \to Y$ is called *generalized fuzzy continuous* if f brings back generalized open fuzzy sets of Y to generalized open fuzzy sets of X. It can be verified that the families of all generalized fuzzy topological spaces as objects along with generalized fuzzy continuous mappings as morphisms form a category. The category FTOP of fuzzy topological spaces and fuzzy continuous mappings turns out to be a full subcategory of this new category.

Lemma 3.3. *For a fuzzy set A, $A = \underset{x_\alpha \le A}{\vee} x_\alpha = \underset{x_\alpha \in A}{\vee} x_\alpha$.*

The closure and interior of a fuzzy set in a GFT may be defined as in a fuzzy topological space. A fuzzy set A is called *generalized Quasi-neighbourhood* (*g-Qnbhd*, in short) of a fuzzy point x_α if there exists a generalized open fuzzy set G such that $x_\alpha q G \le A$.

Proposition 3.4. *For a GFT ζ on X, $A \in I^X$ and a fuzzy point x_α, $x_\alpha \le \mathrm{cl}(A)$ iff every g-Qnbhd of x_α overlaps with A.*

Since net theory is a very important tool for investigating the topological structures, in the following, we develop the theory of convergence for generalized fuzzy topological spaces. We also construct the convergence class for a generalized fuzzy topology.

Definition 3.5. (i) A *pre-ordered* set is a pair $(D, \ge)$, where D is a non-empty set and $\ge$ is a binary relation in D which is reflexive and transitive.

(ii) Let X be a nonempty set and $(D, \ge)$ be a pre-ordered set. A mapping $S : D \to F_P(x)$, where $F_P(x)$ is the collection of all fuzzy points of X, is called a *generalized fuzzy net* (g-net, in brief) in X. A g-net is usually denoted by $\{x^n_{\alpha_n}\}_{n\in D}$, where x^n is the support and α_n is the value of its nth member. If A is a fuzzy set of $X$and $x^n_{\alpha_n} \le A$ for each $n \in D$, then we say that the g-net $\{x^n_{\alpha_n}\}_{n\in D}$ is in A.

Remark 3.6. Let D denote the collection of all fuzzy sets on X. Then D is a pre-ordered set under the ordering of inverse set inclusion, that is, under $\geq$ where $U \geq V$ iff U is contained in V.

Definition 3.7. Let $\{x^n_{\alpha_n}\}_{n\in D}$ be a g-net in X and A be a fuzzy set of X. We say, $\{x^n_{\alpha_n}\}_{n\in D}$ is *eventually overlapping with* A (*eventually contained in* A, respectively) if there exists $m \in D$ such that $x^n_{\alpha_n} qA$ ($x^n_{\alpha_n} \leq A$, respectively) for every $n \geq m$ in D. On the other hand, $\{x^n_{\alpha_n}\}_{n\in D}$ is said to be *frequently overlapping with* A (*frequently in* A, respectively) if for each $m \in D$, there exists $n \geq m$ in D such that $x^n_{\alpha_n} qA$ ($x^n_{\alpha_n} \leq A$, respectively).

Definition 3.8. A g-net $S = \{x^n_{\alpha_n}\}_{n\in D}$ is said to *converge to a fuzzy point* x_α if S eventually overlaps with every g-Qnbhd of x_α.

Example 3.9. Let X be a GFTS and x_α be any fuzzy point in X. Let $\mathcal{U}$ be the collection of all g-Qnbhds of x_α. Then $(\mathcal{U} \geq)$ is a pre-ordered set under inverse set inclusion. For $U \in \mathcal{U}$, we have $U(x) + \alpha > 1$, so that there exists a positive real number, say $\varepsilon(U)$, such that $U(x) + \alpha - \varepsilon(U) > 1$. We write $\alpha - \varepsilon(U) = \alpha_U$. Then $\{x^U_{\alpha_U}\}_{U\in\mathcal{U}}$ is a g-net, where $x^U = x$ for each $U \in \mathcal{U}$. Since $U(x) + \alpha_V > 1$ for each $V \geq U$, this g-net converges to x_α.

A subnet of a g-net may be defined in the same way as it is done in case of a fuzzy net in a fuzzy topology [21, 24].

The closure of a fuzzy set can be characterized in a GFT using g-net in the following way:

Proposition 3.10. *Let X be a nonempty set and ζ be a GFT on X. Then for a fuzzy set A and a fuzzy point x_α in X, $x_\alpha \leq \mathrm{cl}(A)$ iff there is a g-net in A converging to x_α.*

Proof. Let $x_\alpha \leq \mathrm{cl}(A)$. Then for each g-Qnbhd U of x_α, we have UqA. Suppose U and A overlap at x^U. Let D denote the collection of all g-Qnbhd of x_α. We define '$\geq$' in D by $U \geq V$ iff U is contained in V. Then D is a pre-ordered set and hence $\{x^U_{\alpha_U}\}_{U\in D}$, where $\alpha_U = A(x^U)$, is a g-net in A. We show that this g-net converges to x_α. Let U_0 be any g-Qnbhd of x_α. Then for any g-Qnbhd U of x_α such that $U \geq U_0$, $U(x^U) + \alpha_U > 1$ so that $U_0(x^U) + \alpha_U > 1$. Hence $\{x^U_{\alpha_U}\}_{U\in D}$ eventually overlaps with U_0. Hence $\{x^U_{\alpha_U}\}_{U\in D}$ converges to x_α. The converse part follows trivially. ☐

Proposition 3.11. *Let S be a g-net and $\mathcal{F}$ be a family of fuzzy sets such that S is frequently in each member of $\mathcal{F}$. Then there is a subnet of S which is eventually in each member of $\mathcal{F}$.*

Proof. In $\mathcal{F}$, we introduce a relation $\geq$ by $A \geq B$ iff A is contained in B. Then $\mathcal{F}$ is a pre-ordered set under this relation. Let $S = \{x^n_{\alpha_n}\}_{n\in D}$ be a g-net which is frequently in each member of $\mathcal{F}$. Let E be the set of all ordered pairs of the type (m, A) such that $m \in D$, $A \in \mathcal{F}$ and $x^m_{\alpha_m} \leq A$. We introduce a relation '$\geq$' in E by $(m_1, A_1) \geq (m_2, A_2)$ iff $m_1 \geq m_2$ and $A_1 \leq A_2$. Then $E = \{(m, A)\}$ is a pre-ordered set. Let $N : E \to D$ be defined by $N((m, A)) = m$. Now for $n \in D$, there exists $m \geq n$ such that $x^m_{\alpha_m} \leq A$, as $\{x^n_{\alpha_n}\}_{n\in D}$ is frequently in $\mathcal{F}$. For this m

and $(p, A_1) \geq (m, A)$, we have $N((p, A_1)) = p$ and $p \geq n$. Hence $S \circ N$ is a subnet of S. Finally for $A \in \mathcal{F}$, let $m \in D$ such that $x^m_{\alpha_m} \leq A$. Now, for $(n, B) \geq (m, A)$, we have $n \geq m$ and $B \leq A$. Hence $(S \circ N)\ ((n, B)) = S(n) = x^n_{\alpha_n} \leq B \leq A$. Thus $S \circ N$ is eventually in A. □

For iterated limit of g-nets, we have the following result:

Proposition 3.12. *Let X be a GFT and D be a pre-ordered set and E_n be a pre-ordered set for each $n \in D$. Let $x^{(n,m)}_{\alpha(n,m)}$ be defined for each $n \in D$, $m \in E_n$. Let $\{x^{(n,m)}_{\alpha(n,m)}\}_{m \in E_n}$ converge to $x^n_{\alpha_n}$ for each n and $\{x^n_{\alpha_n}\}_{n \in D}$ converge to x_α. Let $D \times \prod\{E_m : m \in D\}$ be ordered by $(n, g) \geq (p, f)$ iff $n \geq p$ and $g(m) \geq f(m)$ for each $m \in D$. Then the g-net $\{x^{(n,g(n))}_{\alpha(n,g(n))}\}$ converges to x_α.*

Proof. Let U be any g-Qnbhd of x_α. Then, there exists $p \in D$ such that $x^n_{\alpha_n} qU$ for all $n \geq p$. As $\{x^{(n,m)}_{\alpha(n,m)}\}$ converges to $x^n_{\alpha_n}$, there exists $f(n) \in E_n$ such that $x^{(n,m)}_{\alpha(n,m)} qU$ for all $(n, m) \geq (p, f(n))$. Now, for $(n, g) \geq (p, f)$, we have $n \geq p$, $g(n) \geq f(n)$ and hence $\{x^{(n,g(n))}_{\alpha(n,g(n))}\} qU$. Consequently, $\{x^{(n,g(n))}_{\alpha(n,g(n))}\}$ converges to x_α. □

The development so far suggests that the theory of convergence for generalized fuzzy topological spaces may be developed by using g-nets. We further establish this point by showing below that g-nets indeed give rise to the convergence classes for the generalized fuzzy topologies.

Definition 3.13. Let $\mathcal{C}$ be the class consisting of ordered pairs (S, x_α) where S is a g-net in X and x_α is a fuzzy point in X. We say that $\mathcal{C}$ is a *generalized convergence class* for X iff it satisfies the conditions listed below: For convenience, we write S $\mathcal{C}$-converges to x_α whenever $(S, x_\alpha) \in \mathcal{C}$.

(1) If $S = \{x^n_{\alpha_n}\}_{n \in D}$ is a g-net such that $x^n_{\alpha_n} = x_\alpha$ for each n, then $(S, x_\alpha) \in \mathcal{C}$;
(2) If $(S, x_\alpha) \in \mathcal{C}$, then for every subnet T of S, $(T, x_\alpha) \in \mathcal{C}$;
(3) If $(S, x_\alpha) \notin \mathcal{C}$, then there exists a subnet T of S such that for no subnet R of T, $(R, x_\alpha) \in \mathcal{C}$;
(4) Let D be a pre-ordered set. Let E_n be a pre-ordered set for each $n \in D$ and $S = \{x^{(n,m)}_{\alpha(n,m)}\}_{\substack{n \in D \\ m \in E_n}}$ be an iterated g-net in X. For each $n \in D$, let $\{x^{(n,m)}_{\alpha(n,m)}\}_{m \in E_n}$ be a g-net which $\mathcal{C}$-converges to $x^n_{\alpha_n}$. Let the g-net $\{x^n_{\alpha_n}\}_{n \in D}$ $\mathcal{C}$-converge to x_α. Further, let $D \times \prod\{E_m : m \in D\}$ be ordered by $(n, g) \geq (p, f)$ iff $n \geq p$ and $g(m) \geq f(m)$ for each $m \in D$. Then the g-net $\{x^{(n,g(n))}_{\alpha(n,g(n))}\}_{\substack{n \in D \\ g(n) \in \Pi E_n}}$ $\mathcal{C}$-converges to x_α.

It can be verified that convergence of g-nets in a GFT satisfies all the above conditions. Thus every generalized fuzzy topology generates a generalized convergence class $\mathcal{C}$ in the sense that $(S, x_\alpha) \in \mathcal{C}$ iff S converges to x_α in the generalized fuzzy topology. For the converse part, we proceed in the following way:

Proposition 3.14. *Let $\mathcal{C}$ be a generalized convergence class in a nonempty set X. For each $A \in I^X$, let $\overline{A}$ denote the union of all fuzzy points x_α such that, for some g-net S in A, S $\mathcal{C}$-converges to x_α. Let $\zeta = \{G : G = (\overline{A})^C,\ A \in I^X\}$. Then ζ is a generalized fuzzy topology and $(S, x_\alpha) \in \mathcal{C}$ iff S converges to x_α relative to ζ.*

Proof. As $\overline{(\underline{1})} = \underline{1}$, hence $\underline{0} \in \zeta$. Let $G_i \in \zeta$, $i \in \Lambda$. We show that $\vee G_i \in \zeta$. Now,

$$\vee G_i = \vee\{(\overline{A_i})^c : A_i \in I^X\} = (\wedge \overline{A}_i)^c.$$

Claim: $\wedge \overline{A}_i = \overline{(\wedge \overline{A}_i)}$:

The inequality $\wedge \overline{A}_i \leq \overline{(\wedge \overline{A}_i)}$ is obvious. For the reverse inequality, let $x_\alpha \leq \overline{(\wedge \overline{A}_i)}$. Then there exists a g-net $S = \{x^n_{\alpha_n}\}_{n \in D}$ in $\wedge \overline{A}_i$ such that S $\mathcal{C}$-converges to x_α. For any fixed $i \in \Lambda$, $x^m_{\alpha_m} \leq \overline{A}_i$ for each $m \in D$. Hence for each $m \in D$, there is a pre-ordered set E_m such that $\{x^{(m,n)}_{\alpha_{(m,n)}}\}_{n \in E_m}$ is a g-net in A_i which $\mathcal{C}$-converge to $x^m_{\alpha_m}$. Hence by part (iv) of Definition 3.13, the g-net $\{x^{(ng(n))}_{\alpha_{(n,g(n))}}\}_{n \in D}$ $\mathcal{C}$-converges to x_α. Since the g-net is in A_i, we get, $x_\alpha \leq \overline{A}_i$. Hence, $\overline{(\wedge \overline{A}_i)} \leq \wedge \overline{A}_i$. Consequently, $\wedge \overline{A}_i = \overline{(\wedge \overline{A}_i)}$. Thus we get, $\vee G_i = (\wedge \overline{A}_i)^c \in \zeta$. Consequently ζ is a generalized fuzzy topology. The remaining part can be proved the way it is done for a fuzzy topology. This completes the proof. □

The above study suggests that the net theoretic results of generalized fuzzy topological spaces should be parallel to their counterparts in Chang's and Lowen's category [21, 27].

4. Some special subclasses of generalized open fuzzy sets

In our earlier study, it was found that a class of mappings, called the monotonic mappings, can be used for generating generalized open fuzzy sets on a set X [30, 32]. These generalized open fuzzy sets, called γ-open fuzzy sets, are found to represent various already existing weaker forms of open fuzzy sets of fuzzy topology. In this section, we further investigate some interesting subclasses of the family of γ-open fuzzy sets on X.

Let X be a non empty set and $I = [0, 1]$. A mapping $\gamma : I^X \to I^X$ is called a *monotonic mapping* if $\gamma(A) \leq \gamma(B)$, whenever $A \leq B$. The collection of all such monotonic mappings on I^X is denoted by $\Gamma(X)$.

Clearly, a fuzzy interior operator on X is an example of a monotonic mapping. This motivates us to define a generalized form of open fuzzy sets, using a monotonic mappings, as follows:

Definition 4.1. Let $\gamma \in \Gamma(X)$. A fuzzy set A on X is γ-*open* if $A \leq \gamma(A)$. The complement of a γ-open fuzzy set is called a γ-*closed fuzzy set*. It is evident that $\underline{0}$ is always γ-open, on the other hand, $\underline{1}$ is γ-open iff $\underline{1} = \gamma(\underline{1})$.

Below we provide an example which shows that the monotonic mappings are quite abundant.

Example 4.2. Let X be any nonempty set and A be any fuzzy set on X. For $\alpha, \beta \in (0, 1]$, define γ_α and γ_β on I^X by

$$\gamma_\alpha(A)(x) = \begin{cases} A(x), & \text{if } A(x) < \alpha \\ 1, & \text{if } A(x) \geq \alpha; \end{cases}$$

and

$$\gamma_\beta(A)(x) = \begin{cases} \beta, & \text{if } A(x) < \beta \\ A(x), & \text{if } A(x) \geq \beta. \end{cases}$$

Then γ_α and γ_β are monotonic mappings for every value of α and β. Also every fuzzy set on X is γ_α-open as well as γ_β-open.

We have the following result:

Proposition 4.3. *An arbitrary union of γ-open fuzzy sets on X is γ-open. On the other hand, arbitrary intersection of γ-closed fuzzy sets is γ-closed.*

Definition 4.4. The union of all γ-open fuzzy sets contained in a fuzzy set A in X is called the γ-*interior* of A and is denoted by $\mathrm{int}_\gamma(A)$. Similarly, the intersection of all γ-closed fuzzy sets containing a fuzzy set A is called the γ-*closure* of A and is denoted by $\mathrm{cl}_\gamma(A)$.

Like the interior of a fuzzy set in a fuzzy topological space, γ-interior of A is also the largest γ-open fuzzy set contained in A. We have,

Proposition 4.5. *For a fuzzy set A on X, $\mathrm{int}_\gamma(A)$ is the largest γ-open fuzzy set contained in A and $\mathrm{cl}_\gamma(A)$ is the smallest γ-closed fuzzy set containing A.*

Corollary 4.6. *A fuzzy set A is γ-open iff $A = \mathrm{int}_\gamma A$; similarly A is γ-closed iff $A = \mathrm{cl}_\gamma(A)$.*

Remark 4.7. (1) If X is a fuzzy topological space, the operators int(cl), cl(int), int(cl(int)), cl(int(cl)) all belong to $\Gamma(X)$. For $\gamma =$ int(cl), γ-open fuzzy sets coincide with pre-open fuzzy sets. Similarly, we obtain the semi open fuzzy sets for $\gamma =$ cl(int), the α-open fuzzy sets for $\gamma =$ int(cl(int)), the β-open fuzzy sets for $\gamma =$ cl(int(cl)). The corresponding closed sets are pre-closed, semi-closed, α-closed and β-closed respectively.

(2) It may be mentioned that in [9], an operator $L_\omega : L^X \to L^X$ has been defined to generalize closure operator. The operator γ defined above is more general in nature. The operator L_ω becomes a particular case of γ under certain restrictions. Also motivation and further development of this paper is entirely different from that of [9] and is, in fact, sequel to our papers [30, 32].

(3) We have observed that the γ-open fuzzy sets are closed under arbitrary union and $\underline{0}$ is always γ-open. Hence they form a generalized fuzzy topology on X. Thus the monotonic mappings on a set X are a readily available source for generalized fuzzy topologies on X. In fact, one can establish a one to one correspondence between the members of $\Gamma(X)$ and the generalized fuzzy topological spaces on X.

It is found that monotonic mappings are quite abundant. Also under specified conditions, they generate γ-open fuzzy sets which have interesting properties. In the following, we investigate few mappings of such types. The subclass of $\Gamma(X)$ consisting of the monotonic mappings with the property $\gamma(\underline{0}) = \underline{0}$ is denoted by Γ_0.

Few other subclasses, denoted in the following way, are

$$\begin{aligned}
&\Gamma_1 : \gamma(\underline{1}) = \underline{1}\\
&\Gamma_2 : \gamma^2(A) = \gamma(A)\\
&\Gamma_+ : A \leq \gamma(A)\\
&\Gamma_- : A \geq \gamma(A),
\end{aligned}$$

where $A \in I^X$. Thus if $\gamma \in \Gamma_+$, then every fuzzy set on X is γ-open. If $\gamma \in \Gamma_-$, then a fuzzy set A is γ-open iff A= $\gamma(A)$. If X is a fuzzy topological space, then the interior operator 'int' belongs to $\Gamma(X)$ and the int-open fuzzy sets are precisely the open fuzzy sets of X. In general, we have the following result:

Proposition 4.8. *For any $\gamma \in \Gamma$, $\mathrm{int}_\gamma \in \Gamma_2 \cap \Gamma_- \cap \Gamma_0$. Further, $\mathrm{int}_\gamma \in \Gamma_1$ iff $\gamma \in \Gamma_1$. On the other hand, if $\gamma \in \Gamma_2 \cap \Gamma_- \cap \Gamma_0$, then $\gamma = \mathrm{int}_\gamma$.*

Proof. It is obvious that $\mathrm{int}_\gamma \in \Gamma$, $\mathrm{int}_\gamma(\underline{0}) = \underline{0}$, and $\mathrm{int}_\gamma(A) \leq A$. Also, $\mathrm{int}_\gamma(A)$ being the largest γ-open fuzzy set contained in A is γ-open and hence, $\mathrm{int}_\gamma(\mathrm{int}_\gamma A) = \mathrm{int}_\gamma(A)$. This implies that $\mathrm{int}_\gamma \in \Gamma_2$ and hence, $\mathrm{int}_\gamma \in \Gamma_2 \cap \Gamma_- \cap \Gamma_0$. Further, if $\gamma \in \Gamma_1$, then $\underline{1} \leq \gamma(\underline{1})$ so that $\underline{1}$ is γ-open; whence $\mathrm{int}_\gamma(\underline{1}) = \underline{1}$. Hence, $\mathrm{int}_\gamma \in \Gamma_1$. On the other hand, if $\mathrm{int}_\gamma \in \Gamma_1$, then $\mathrm{int}_\gamma(\underline{1}) = \underline{1}$, that is, the largest γ-open fuzzy set contained in $\underline{1}$ is $\underline{1}$ itself. Therefore, $\underline{1}$ is open. But then $\underline{1} \leq \gamma(\underline{1})$, so that $\underline{1} = \gamma(\underline{1})$, and hence $\gamma \in \Gamma_1$. Also, if $\gamma \in \Gamma_2 \cap \Gamma_- \cap \Gamma_0$, then $\gamma(A) \leq \gamma(\gamma(A))$, $\gamma(A) \leq A$. Hence, $\gamma(A)$ is open and is contained in A. If B is another γ-open fuzzy set contained in A then, $B \leq \gamma(B) \leq \gamma(A)$. Thus $\gamma(A)$ is the largest open fuzzy set in A. Consequently, $\gamma(A) = \mathrm{int}_\gamma(A)$. □

Corollary 4.9. *A fuzzy set A is γ-open iff $A = \mathrm{int}_\gamma A$ iff A is int_γ-open.*

We now define a special type of monotonic mappings denoted by γ^*.

Definition 4.10. For a given mapping $\gamma \in \Gamma(X)$ and $A \in I^X$, γ^* is defined by

$$\gamma^*(A) = \underline{1} - \gamma(\underline{1} - A) = (\gamma(A^c))^c.$$

Remark 4.11. It is clear that for $\gamma \in \Gamma(X)$, $\gamma(A) = \underline{1} - \gamma^*(\underline{1} - A)$.

In what follows, we prove the interrelationship between the mappings γ and γ^*:

Proposition 4.12. (i) *If $\gamma \in \Gamma$, then $\gamma^* \in \Gamma$ and $(\gamma^*)^* = \gamma$. Further,*

(ii) $\gamma \in \Gamma_0$ *iff* $\gamma^* \in \Gamma_1$*;*
(iii) $\gamma \in \Gamma_1$ *iff* $\gamma^* \in \Gamma_0$*;*
(iv) $\gamma \in \Gamma_2$ *iff* $\gamma^* \in \Gamma_2$*;*
(v) $\gamma \in \Gamma_+$ *iff* $\gamma^* \in \Gamma_-$*;*
(vi) $(\mathrm{int}_\gamma)^* = \mathrm{cl}_\gamma$.

Proof. (i) As $\gamma \in \Gamma$, for $A \leq B \leq \underline{1}$, we have

$$\begin{aligned}
&& \underline{1} - A \geq \underline{1} - B\\
\Rightarrow && \gamma(\underline{1} - B) \leq \gamma(\underline{1} - A)\\
\Rightarrow && \underline{1} - \gamma(\underline{1} - A) \leq \underline{1} - \gamma(\underline{1} - B).
\end{aligned}$$

Thus, $\gamma^*(A) \leq \gamma^*(B)$ and consequently, $\gamma^* \in \Gamma$. Also,

$$\begin{aligned} (\gamma^*)^*(A) &= \underline{1} - \gamma^*(\underline{1} - A) \\ &= \gamma(A) \\ \Rightarrow \qquad (\gamma^*)^* &= \gamma\,. \end{aligned}$$

The proofs of (ii) and (iii) are obvious. Since $(\gamma^*)^* = \gamma$, we have

$$\gamma(\gamma(A)) = (\gamma^*)^*(\gamma(A)) = \underline{1} - \gamma^*(\underline{1} - \gamma(A)).$$

To prove (iv), let $\gamma \in \Gamma_2$. Then

$$\begin{aligned} (\gamma^*)^2(A) &= \gamma^*(\gamma^*(A)) = \gamma(A) = \gamma^2(A) = \gamma(\gamma(A)) \\ &= \underline{1} - \gamma^*(\underline{1} - \gamma(A)) = \underline{1} - \gamma^*(\gamma^*(\underline{1} - A)) \\ &= \underline{1} - \gamma^{*^2}(\underline{1} - A) = \underline{1} - \gamma(\underline{1} - A) = \gamma^*(A)\,. \end{aligned}$$

Thus, $\gamma \in \Gamma_2 \Rightarrow \gamma^* \in \Gamma_2$.

Conversely, let $\gamma^* \in \Gamma_2$. Then,

$$\begin{aligned} \gamma^2(A) &= \gamma(\gamma(A)) = \gamma(\underline{1} - \gamma^*(\underline{1} - A)) \\ &= \underline{1} - \gamma^*(\gamma^*(\underline{1} - A)) = \underline{1} - (\gamma^*)^2(\underline{1} - A) \\ &= \underline{1} - \gamma^*(\underline{1} - A) = \gamma(A) \end{aligned}$$

Thus $\gamma \in \Gamma_2 \Rightarrow \gamma^* \in \Gamma_2$. Hence, $\gamma \in \Gamma_2$ iff $\gamma^* \in \Gamma_2$. To prove (v), let $\gamma \in \Gamma_+$. Then

$$\begin{aligned} \gamma^*(A) &= \underline{1} - \gamma(\underline{1} - A) \\ &\leq \underline{1} - (\underline{1} - A) = A \end{aligned}$$

Therefore, $\gamma^* \in \Gamma_-$.

Conversely, if $\gamma^* \in \Gamma_-$, then

$$\gamma(A) = \underline{1} - \gamma^*(\underline{1} - A) \geq \underline{1} - (\underline{1} - A) = A \Rightarrow \gamma \in \Gamma_+$$

Finally, since $\mathrm{int}_\gamma(\underline{1} - A)$ is the largest γ-open fuzzy set contained in $\underline{1} - A$, its complement coincides with the smallest γ-closed fuzzy set containing A, that is, $\underline{1} - \mathrm{int}_\gamma(\underline{1} - A) = \mathrm{cl}_\gamma A$. Thus $(\mathrm{int}_\gamma)^* = \mathrm{cl}_\gamma$. □

It may be easily verified that in a fuzzy topological space, $(\mathrm{int})^* = \mathrm{cl}$. Moreover, for such γ^* defined above, a fuzzy set A in X is γ^*-closed iff $\gamma(A) \leq A$ that is, iff A is γ-closed.

Proposition 4.13. $\mathrm{cl}_\gamma \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_+$, *for any* $\gamma \in \Gamma$. *On the other hand, if* $\gamma \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_+$ *then,* $\gamma = \mathrm{cl}_\gamma$. *Further* $\mathrm{cl}_\gamma \in \Gamma_0$ *iff* $\gamma \in \Gamma_1$.

Proof. For $A \leq B$, we have $\mathrm{cl}_\gamma(A) \leq \mathrm{cl}_\gamma(B)$. Thus $\mathrm{cl}_\gamma \in \Gamma$. Since $\underline{0}$ is γ-open, its complement $\underline{1}$ is γ-closed, that is, $\mathrm{cl}_\gamma(\underline{1}) = \underline{1}$. Again, $\mathrm{cl}_\gamma(\mathrm{cl}_\gamma(A))$ is the smallest γ-closed fuzzy set containing $\mathrm{cl}_\gamma(A)$ and since $\mathrm{cl}_\gamma(A)$ is itself closed, we get $\mathrm{cl}_\gamma \in \Gamma_2$. Then using the fact that $A \leq \mathrm{cl}_\gamma(A)$, we get that, $\mathrm{cl}_\gamma \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_+$. Also, let $\gamma \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_+$, then $\gamma(\gamma(A)) \leq \gamma(A)$ as $\gamma \in \Gamma_2$. Hence, $\gamma(A)$ is γ^*-closed. Also, $A \leq \gamma(A)$ as $\gamma \in \Gamma_+$. Now if B is γ^*-closed and $A \leq B$ then $\gamma(A) \leq \gamma(B)$ as well as $\gamma(B) \leq B$. Thus $\gamma(A)$ is the smallest γ^*-closed fuzzy set containing A. Hence, $\gamma(A) = \mathrm{cl}_{\gamma^*}(A)$. We have that $\mathrm{cl}_\gamma(\underline{0}) = \underline{1} - \mathrm{int}_\gamma(\underline{1})$. Now $\mathrm{int}_\gamma \in \Gamma_1$ iff $\gamma \in \Gamma_1$ (in view of Proposition 4.8). Therefore $\mathrm{cl}_\gamma(\underline{0}) = \underline{0}$ iff $\gamma \in \Gamma_1$. Thus, $\mathrm{cl}_\gamma \in \Gamma_0$ iff $\gamma \in \Gamma_1$. □

The following result is straight forward:

Proposition 4.14. *A fuzzy set A is γ-closed iff $A = \mathrm{cl}_\gamma(A)$ iff A is int_γ-closed.*

The classes Γ and Γ_n, $n = 0, 1, 2, +$ and $-$, are found to be closed under composition of mappings. Here is the result:

Proposition 4.15. *The classes Γ and Γ_n, $n = 0, 1, 2, +$ and $-$, are closed under composition of mappings. Further, for $\gamma_1, \gamma_2 \in \Gamma_n$, we have, $(\gamma_2\gamma_1)^* = \gamma_2^*\gamma_1^*$. Here $\gamma_2\gamma_1$ denotes the composition of γ_1 and γ_2.*

Proof. For any fuzzy sets A, B on X

$$\begin{aligned} A \le B &\Rightarrow \gamma_1(A) \le \gamma_1(B) \qquad\qquad (\text{as } \gamma_1 \in \Gamma) \\ &\Rightarrow \gamma_2\left(\gamma_1\left(A\right)\right) \le \gamma_2\left(\gamma_1\left(B\right)\right) \\ &\Rightarrow \gamma_2\gamma_1\left(A\right) \le \gamma_2\gamma_1\left(B\right) \end{aligned}$$

Therefore, $\gamma_2\gamma_1 \in \Gamma$. Now, let $\gamma_1, \gamma_2 \in \Gamma_0$. Then

$$\gamma_2\gamma_1(\underline{0}) = \gamma_2(\gamma_1(\underline{0})) = \gamma_2(\underline{0}) = \underline{0}$$

and hence, $\gamma_2\gamma_1 \in \Gamma_0$. Similarly, we can prove that $\gamma_2\gamma_1 \in \Gamma_n$ for $\gamma_1, \gamma_2 \in \Gamma_n$ when $n = 1, 2, +$ and $-$. Further,

$$\begin{aligned} \left(\gamma_2\gamma_1\right)^* A &= \underline{1} - \gamma_2\gamma_1\left(\underline{1} - A\right) \\ &= \underline{1} - \gamma_2\left(\gamma_1\left(\underline{1} - A\right)\right) \\ &= \underline{1} - \gamma_2\left(\underline{1} - \gamma_1^* A\right) \\ &= \underline{1} - \left(\underline{1} - \gamma_2^*\left(\gamma_1^*\left(A\right)\right)\right) = \gamma_2^*\gamma_1^*\left(A\right) \end{aligned}$$

which proves the result. □

Remark 4.16. In a fuzzy topological space the operators int cl, cl int, int cl int, cl int cl all belong to $\Gamma_0 \cap \Gamma_1$.

The development so far demands for further studies of generalized fuzzy topological spaces, particularly, with respect to their topological properties. Further investigations in this direction are being taken up in our subsequent papers.

Acknowledgements. 1. The authors thank the referees of this paper for their valuable suggestions.

2. This paper is part of a minor research project financed by the University Grants Commission, India.

References

[1] N. Ajmal and J. K. Kohli, Connectedness in fuzzy topological spaces, Fuzzy Sets and Systems 31 (1989) 369–388.

[2] N. Ajmal and S. K. Azad, Fuzzy almost continuity and its pointwise characterization by dual point and fuzzy nets, Fuzzy Sets and Systems 34 (1990) 81–101.

[3] N. Ajmal and R. D. Sarma, Various fuzzy topological notions and fuzzy almost continuity, Fuzzy Sets and Systems 50 (1992) 209–223.

[4] K. K. Azad, On fuzzy semicontuinity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.

[5] K. K. Azad, Fuzzy hausdroff spaces and fuzzy perfect mappings, J. Math. Anal. Appl. 82(1981) 297–305.

[6] H. Boutisque, R. N. Mohapatra and G. Richardson, Lattice valued fuzzy interior operator, Fuzzy Sets and Systems 160(20) (2009) 2947–2955.

[7] S. C. Carlson, The quest for a fuzzy Tychonoff theorem, Amer. Math. Monthly 115(10) (2008) 871–887.

[8] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[9] S. Hui-Li Chen and Ji-Shu Cheng, On L_ω-spaces, in Fuzzy Logic, Soft computing and Computational Intelligence, 11th International Fuzzy Systems Association World Congress, Vol. I (2005) 257–261.

[10] A. Csaszar, Generalized open sets, Acta Math. Hungar. 75(1-2) (1997) 65–87.

[11] A. Csaszar, Generalized topology, Generalized continuity, Acta Math. Hungar. 96(4) (2002) 351–357.

[12] N. R. Das and J. Sharma, Some aspects of pairwise fuzzy preopen sets in fuzzy bi- topological spaces, J. Fuzzy Math. 17(4) (2009) 823–834.

[13] S. Ganguly and S. Saha, A note on semi open sets in fuzzy topological spaces, Fuzzy Sets and Systems 18 (1986) 83–96.

[14] Reny George and R.A. Rashwan, On dislocated fuzzy topologies and fixed points, J. Mathematics Research, to appear.

[15] J. Goguen, The fuzzy tychonoff theorem, J. Math. Anal. Appl. 43 (1973) 734–742.

[16] I. M. Hanafy, Fuzzy β-compactness and fuzzy-closed spaces, Turkish J. Math. 28 (2004) 281–293.

[17] P. Hitzler and A. K. Seda, Dislocated topologies, J. Elec. Eng. 51(10) (2000) 3–7.

[18] B. Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl. 50 (1975) 74–79.

[19] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.

[20] C.de Metri and E. Pascali, Characterization of fuzzy topologies from neighbourhood of fuzzy points, J. Math. Anal. Appl. 93 (1983) 324–327.

[21] Pu and Liu, Fuzzy topology I: Neighbourhood structure of a fuzzy point and Moore Smith Convergence, J. Math. Anal. Appl. 76 (1980) 571–599.

[22] S. E. Rodabaugh, Categorical accommodation of various notions of fuzzy topology, Fuzzy Sets and Systems 9 (1983) 241–265.

[23] S. E. Rodabaugh, Point-set lattice theoretic topology, Fuzzy Sets and Systems 40 (1991) 297–345.

[24] R. D. Sarma, Fuzzy nets and their applications, Fuzzy Sets and Systems 51 (1992) 41–51.

[25] R. D. Sarma and N. Ajmal, Category N and functions with closed graph, Fuzzy Sets and Systems 63 (1994) 219–226.

[26] R. D. Sarma and N. Ajmal, Fuzzy subcontinuity, Inverse fuzzy subcontinuity and a new category of fuzzy topological spaces, Fuzzy Sets and Systems 73 (1995) 13–24.

[27] R. D. Sarma and P. K. Jha, Fuzzy net theoretic approach and Lowen's category, J. Fuzzy Math. 5 (1997) 159–163.

[28] R. D. Sarma, D. Hazarika and A. Sahrffudin, Some aspects of c_i-connectedness in L topological spaces, J. Fuzzy Math. 13(4) (2005) 827–843.

[29] R. D. Sarma, The theory of localization for intuitionistic fuzzy topological spaces, J. Fuzzy Math. 15(2) (2007) 1–13.

[30] R. D. Sarma, A. Sharfuddin and Anubha Bhargava, Generalized open fuzzy sets in generalized fuzzy topology, Int. J. Math. Sci. 9(3–4) (2010) 343–355.

[31] R. D. Sarma, Relational toplogy, Int. J. Math. Sci. 9(3–4) (2010) 333–342.

[32] R. D. Sarma, A. Sharfuddin and Anubha Bhargava, Seperation axioms for generalized fuzzy topological Spaces, J. Comb. Inf. Syst. Sci. 35(3–4) (2010) 329–340.

[33] A. K. Seda, Topology and semantics of logic programming, Fund. Inform. 19(1) (1997) 97–117.

[34] A. P. Shostak, Two decades of fuzzy topology: basic ideas, notions and results, Russian Math. Surveys 44(6) (1989) 125–186.

[35] R. Srivastava, S. N. Lal and A. K. Srivastava, Fuzzy Hausdroff topological spaces, J. Math. Anal. Appl. 81 (1981) 497–506.

[36] G. Wang, A new fuzzy compactness defined by fuzzy nets, J. Math. Anal. Appl. 94 (1983) 1–23.
[37] R. Warren, Neighbourhood, base and continuity in fuzzy topological spaces, Rockey Mountain J. Math. 8 (1978) 459–470.
[38] C. K. Wong, Fuzzy points and local properties of fuzzy topology, J. Math. Anal. Appl. 46 (1974) 316–328.
[39] T. H. Yalvac, Fuzzy sets and functions on fuzzy spaces, J. Math. Anal. Appl. 126 (1987) 409–423.
[40] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

Ratna Dev Sarma (ratna_sarma@yahoo.com)
Department of Mathematics, Rajdhani College, University of Delhi, Delhi 110015, India

A. Sharfuddin
Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India

Anubha Bhargava (linktoanubha@yahoo.co.in)
Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India