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Relation diagram between fuzzy n-fold filters in BL-algebras

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Abstract. The aim of this paper is to introduce the notions of fuzzy n-fold normal and fuzzy n-fold Boolean filters in BL-algebras and to investigate their properties. It is shown that fuzzy n-fold Boolean filters are equivalent to fuzzy n-fold positive implicative filters in BL-algebras and the links between all fuzzy n-fold filters in BL-algebras are investigated. Finally, we give the relation diagram between various fuzzy n-fold filters in BL-algebras. These results generalize the corresponding results in the crisp case.

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1. Introduction

In [1], Hajek introduced the basic logic algebra (BL-algebra) as the algebraic structure of his basic logic. Up to now, that algebra have been widely studied and emphasis have been put on filter theory ([2, 4, 10, 12, 15]). It is well known that in various logical systems, the theory of ideals and filters play a fundamental role, ideals or filters correspond to sets of provable formulas and closed with respect to modus ponens. This is to say that ideals and filters are not just abstract concepts, but are mathematically deep and significant concepts with applications in various areas.

Fuzzy filters are useful tool to obtain results on classical filters in BL-algebras. In this paper, we introduce the notions of fuzzy n-fold normal and fuzzy n-fold Boolean filters in BL-algebra and investigate their properties. We also analyze the relation between various fuzzy filters in BL-algebras. In the appendix, we give the relation diagram between fuzzy n-fold filters in BL-algebras. These results generalize the corresponding results in the crisp case [13].

2. Preliminaries

We recollect some definitions and results which will be used in the following and we shall not cite them every time they are used.

Definition 2.1. A BL-algebra is an algebra $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ that satisfies the following conditions for all $x, y, z \in X$:

BL-1. $(X, \wedge, \vee, 0, 1)$ is a bounded lattice;
BL-2. $(X, *, 1)$ is an commutative monoid, i.e., $*$ is commutative and associative with $x * 1 = x$;
BL-3. $(x * y) \leqslant z$ iff $x \leqslant y \rightarrow z$ (Residuation);
BL-4. $x \wedge y = x * (x \rightarrow y)$ (Divisibility);
BL-5. $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (Prelinearity).

Example 2.2. (i) Let X be a nonempty set and let $P(X)$ be the family of all subsets of X. Define operations $*$ and $\rightarrow$ by : $A * B = A \cap B$ and $A \rightarrow B = A^C \cup B$ for all $A, B \in P(X)$, respectively. Then $(P(X), \cap, \cup, *, \rightarrow, \varnothing, X)$ is a BL-algebra called the power BL-algebra of X.

(ii) $L = ([0, 1], \wedge, \vee, *, \rightarrow, 0, 1)$ where $\rightarrow$ is the residuum of a continuous t-norm $*$ is a BL-algebra.

Lemma 2.3 ([14])**.** *The following properties hold in any BL-algebra.*

1. $x \leqslant y$ *iff* $x \rightarrow y = 1$;
2. $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$;
3. $x * y \leqslant x \wedge y$;
4. $(x \rightarrow y) * (y \rightarrow z) \leqslant x \rightarrow z$;
5. $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$;
6. $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
7. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
8. $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$;
9. $y \rightarrow x \leqslant (z \rightarrow y) \rightarrow (z \rightarrow x)$;
10. *If* $x \leqslant y$, *then* $y \rightarrow z \leqslant x \rightarrow z$ *and* $z \rightarrow x \leqslant z \rightarrow y$;
11. *If* $x \vee x^- = 1$, *then* $x \wedge x^- = 0$ *where* $x^- = x \rightarrow 0$;
12. $y \leqslant (y \rightarrow x) \rightarrow x$;
13. $x \leqslant y \rightarrow (x * y)$;
14. $x * (x \rightarrow y) \leqslant y$;
15. $1 \rightarrow x = x$; $x \rightarrow x = 1$; $x \rightarrow 1 = 1$; $x \leqslant y \rightarrow x$.

We denote $(...(x \rightarrow (x \rightarrow (x \rightarrow y)))...)$ by $x^n \rightarrow y$ where x occurs n times for all $x, y \in X$. Using the fact that in any BL-algebra $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$, we can prove by induction that $(...(x \rightarrow (x \rightarrow (x \rightarrow y)))...) = (... * x * x * x) \rightarrow y = x^n \rightarrow y$ where x occurs n times for all $x, y \in X$.

Lemma 2.4. *The following properties also hold in any BL-algebra.*

1. *If* $x \leqslant y$, *then* $x^n \rightarrow y = 1$;
2. *If* $x \leqslant y$, *then* $y^n \rightarrow z \leqslant x^n \rightarrow z$ *and* $z \rightarrow x^n \leqslant z \rightarrow y^n$;
3. $x^n \rightarrow x = 1$; $x^n \rightarrow 1 = 1$;
4. $x \leqslant y^n \rightarrow x$;
5. $x \rightarrow y \leqslant x^n \rightarrow y$;

6. $x^n \to (y \to z) = y \to (x^n \to z)$.

Definition 2.5 ([14]). Let X and Y are BL-algebras. A function $f : X \longrightarrow Y$ is called homomorphism of BL-algebras if and only if

1. $f(0) = 0$, $f(1) = 1$
2. $f(x * y) = f(x) * f(y)$,
3. $f(x \to y) = f(x) \to f(y)$,
4. $f(x \vee y) = f(x) \vee f(y)$, $f(x \wedge y) = f(x) \wedge f(y)$, for all $x, y \in X$.

We briefly review some fuzzy logic concepts, we refer the reader to [8], [9] and [16] for more details.

Definition 2.6. Let X be a BL-algebra. A fuzzy subset of X is a function

$$A : X \longrightarrow [0, 1].$$

Definition 2.7. A filter of a BL-algebra X is a subset F containing 1 such that

$$\text{if } \ x \to y \in F \ \text{ and } \ x \in F \ \text{ imply } \ y \in F.$$

Definition 2.8. A fuzzy subset A of X is a called

- a fuzzy filter if
$$A(1) \geq A(x) \ \text{ and } \ A(y) \geq \min\{A(x \to y), A(x)\}, \ \forall \ x, y \in X.$$
- a fuzzy n-fold positive implicative filter if $A(1) \geq A(x)$ and
$$A(x) \geq A(x^{n-} \to x), \ where \ x^{n-} = x^n \to 0, \ \forall \ x, y \in X.$$
- a fuzzy n-fold implicative filter if $A(1) \geq A(x)$ and
$$A(x^n \to y) \geq \min\{A(x^n \to (y \to z)), A(x^n \to z)\}, \ \forall x, y, z \in X.$$
- a fuzzy n-fold obstinate filter if
$$\min\{A(x^n \to y), A(y^n \to x)\} \geq \min\{(1 - A(x)), (1 - A(y))\}, \ \forall x, y \in X.$$
- a fuzzy n-fold fantastic filter if $A(1) \geq A(x)$ and
$$A(((x^n \to y) \to y) \to x) \geq \min\{A(z \to (y \to x)), A(z)\}, \ \forall \ x, y, z \in X.$$

Proposition 2.9 ([12]). *Let A be a fuzzy subset of X. A is a fuzzy n-fold fantastic filter if and only if*

$$A(((x^n \to y) \to y) \to x) \geq A(y \to x), \ \forall x, y \in X.$$

Proposition 2.10 ([6]). *Let A be a fuzzy filter of X. A is a fuzzy n-fold implicative filter if and only if $A((x^n \to y) \to (x^n \to z)) \geq A(x^n \to (y \to z))$, for all $x, y, z \in X$.*

Proposition 2.11 ([5]). *A fuzzy filter A of X is a fuzzy n-fold obstinate filter iff*

$$A(x^{n-}) \geq 1 - A(x), \ \forall x, y \in X.$$

The following theorem gives some characterizations of fuzzy filters.

Theorem 2.12 ([3]). *Suppose that A is a fuzzy subset of a BL-algebra X. Then the following conditions are equivalent:*

1. *A is a fuzzy filter;*

2. $\forall\ t \in [0,1]$, *the* t*-level subset* $A_t = \{x \in X : A(x) \geq t\}$ *is a filter of* X *if* $A_t \neq \varnothing$.

Lemma 2.13 ([9]). *The following properties hold if* A *is a fuzzy filter for any BL-algebra* X*, for all* $x, y \in X$*:*

(a) *If* $x \leq y$ *then* $A(y) \geq A(x)$*, that is,* A *is order-preserving;*
(b) *If* $A(x \to y) = A(1)$*, then* $A(y) \geq A(x)$*;*
(c) $A(x * y) = A(x) \wedge A(y)$ *;*
(d) $A(x \wedge y) = A(x) \wedge A(y)$*;*
(e) $A(0) = A(x) \wedge A(x^-)$*;*
(f) $A(x \to z) \geq A(x \to y) \wedge A(y \to z)$*;*
(g) $A(x \to y) \leq A((y \to z) \to (x \to z))$*;*
(h) $A(y \to x) \leq A((z \to y) \to (z \to x))$.

In the sequel, $X = (X, \wedge, \vee, *, \to, 0, 1)$ will be a BL-algebra, and n an integer. In the following section, we introduce fuzzy n-fold normal filters in BL-algebras and investigate their related properties.

3. Fuzzy n-fold normal filters

Definition 3.1. Let $A : X \longrightarrow [0,1]$ be a fuzzy subset of X, A is said to be a fuzzy n-fold normal filter if for all $x, y, z \in X$, $A(1) \geq A(x)$ and

$$A((x \to y) \to y) \geq \min\{A(z \to ((y^n \to x) \to x)), A(z)\}.$$

A fuzzy 1-fold normal filter is called a fuzzy normal filter.

Proposition 3.2. *Every fuzzy* n*-fold normal filter is a fuzzy filter.*

Proof. Let A be a fuzzy n-fold normal filter of X and $x, y \in X$. Then

$$A((y \to y) \to y) \geq \min\{A(x \to ((y^n \to y) \to y)), A(x)\}.$$

Hence, $A(y) \geq \min\{A(x \to y), A(x)\}$. □

The following example shows that the converse of the above proposition is not true.

Example 3.3. Let $X = [0,1]$, define $*$ and $\to$ as follows: $x * y = \min(x, y)$ and

$$x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise.} \end{cases}$$

Then $(X, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. Define the fuzzy subset A by $A(x) = 1$ for $x \in [\frac{1}{2}, 1]$ and $A(x) = 0$ if $x \notin [\frac{1}{2}, 1]$. Then A is a fuzzy filter but not a fuzzy n-fold normal filter $\forall n \geq 1$ since

$$A((\tfrac{1}{5} \to \tfrac{1}{4}) \to \tfrac{1}{4}) < \min\{A(1 \to ((\tfrac{1}{4}^n \to \tfrac{1}{5}) \to \tfrac{1}{5})), A(1)\}.$$

We establish some results on fuzzy n-fold normal filters:

Proposition 3.4. *Let* A *be a fuzzy filter of* X*.* A *is fuzzy* n*-fold normal if and only if*

$$A((x \to y) \to y) \geq A((y^n \to x) \to x)\ \forall x, y \in X.$$

Proof. Assume that A is a fuzzy n-fold normal filter of X. By setting $z = 1$ in Definition 3.1, we obtain the result.

Conversely, let $x, y, z \in X$. From the hypothesis, we have

$$A((x \to y) \to y) \geq A((y^n \to x) \to x).$$

Since A is a fuzzy filter, we have

$$A((y^n \to x) \to x) \geq \min\{A(z \to ((y^n \to x) \to x)), A(z)\}$$

and we obtain

$$A((x \to y) \to y) \geq \min\{A(z \to ((y^n \to x) \to x), A(z)\}.$$

Hence, A is a fuzzy n-fold normal filter. $\square$

Now, we describe the transfer principle [3] for fuzzy n-fold normal filter in terms of level subsets as:

Theorem 3.5. *A fuzzy subset A of a BL-algebra X is a fuzzy n-fold normal filter if and only if $A^t = \{x \in X \mid A(x) \geq t\}$ is either empty or an n-fold normal filter for every $t \in [0, 1]$.*

Proof. Assume that A is a fuzzy n-fold normal filter of X. Let $t \in [0, 1]$ and $x \in A^t$ Then $A(x) \geq t$. Since A is a fuzzy filter, $A(1) \geq A(x)$, therefore $1 \in A^t$. Let $x, y, z \in X$ with $z \to ((y^n \to x) \to x) \in A^t$ and $z \in A^t$. We have

$$A(z \to ((y^n \to x) \to x)) \geq t \quad \text{and} \quad A(z) \geq t.$$

Since A is a fuzzy n-fold normal filter, we have

$$A((x \to y) \to y) \geq \min\{A(z \to ((y^n \to x) \to x)), A(z)\} \geq t.$$

Therefore $(x \to y) \to y \in A^t$. This proves that the t-level set A^t is an n-fold normal filter of X.

Conversely, assume that for every $t \in [0, 1]$, $A^t = \{x \in X \mid A(x) \geq t\}$ is an n-fold normal filter of X. We will prove that A is a fuzzy n-fold normal filter. It is easy to prove that $\forall\, x \in X$, $A(1) \geq A(x)$. Let $x, y, z \in X$. We need to show that

$$A((x \to y) \to y) \geq \min\{A(z \to ((y^n \to x) \to x)), A(z)\}.$$

If not, then there exist $a, b, c \in X$ such that

$$A((a \to b) \to b) < \min(A(c \to ((b^n \to a) \to a)), A(c)).$$

Setting

$$t_0 = 1/2[A((a \to b) \to b) + \min(A(c \to ((b^n \to a) \to a)), A(c))],$$

we have

$$A((a \to b) \to b) < t_0 < \min(A(c \to ((b^n \to a) \to a)), A(c)).$$

We obtain $(a \to b) \to b \notin A^{t_0}$, but $c \to ((b^n \to a) \to a) \in A^{t_0}$ and $c \in A^{t_0}$ which is a contradiction since A^{t_0} is an n-fold normal filter of X. Therefore A is a fuzzy n-fold normal filter and the proof is complete. $\square$

Corollary 3.6. *A non empty subset F of X is an n-fold normal filter if and only if the characteristic function χ_F is a fuzzy n-fold normal filter.*

Corollary 3.7. *Let A be a fuzzy filter of a BL-algebra X. The level filter*

$$I = \{x \in X \mid A(x) = A(1)\}$$

is an n-fold normal filter if A is a fuzzy n-fold normal filter.

In the following we analyze the relation between fuzzy n-fold normal filters and fuzzy $(n+k)$-fold normal filters in BL-algebras, for $k \geq 1$.

Proposition 3.8. *Every fuzzy n-fold normal filter is a fuzzy (n+1)-fold normal filter.*

Proof. Let A be a fuzzy n-fold normal filter. We have to show that

$$A((x \to y) \to y) \geq A((y^{n+1} \to x) \to x).$$

Since $(y^{n+1} \to x) \to x \leq (y^n \to x) \to x$, we apply the hypothesis and Lemma 2.3. and obtain $A((x \to y) \to y) \geq A((y^n \to x) \to x) \geq A((y^{n+1} \to x) \to x)$. □

It is easy to prove by induction that every fuzzy n-fold normal filter is a fuzzy $n+k$-fold normal filter for all integer $k \geq 1$. Now, we investigate the relations between fuzzy n-fold normal filters and some other type of fuzzy filters in BL-algebras.

Proposition 3.9. *Every fuzzy n-fold positive implicative filter is a fuzzy n-fold normal filter.*

Proof. Assume that A is a fuzzy n-fold positive implicative filter of a BL-algebra X. Let $a = (y \to x) \to x$ for any $x, y \in X$. By few computations, we obtain that

$$(a^n \to y) \to a \geq (x^n \to y) \to y.$$

Since A is order preserving, we have

$$A((a^n \to y) \to a) \geq A((x^n \to y) \to y).$$

By hypothesis, we have $A(a) \geq A((a^n \to y) \to a)$. Hence

$$A((y \to x) \to x) \geq A((x^n \to y) \to y)$$

and we conclude that A is a fuzzy n-fold normal filter. □

The following example shows that the converse of the above proposition is not true.

Example 3.10. Let $X = \{0, a, b, 1\}$ be a chain. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(X, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. Define the fuzzy filter A of X by $A(1) = t_1$ and $A(0) = A(a) = A(b) = t_2$, with $t_1 > t_2$. Then A is a fuzzy 2-fold normal filter of X but not a fuzzy 2-fold positive implicative filter since

$$A(b) = t_2 < t_1 = A(1) = A((b^2 \to 0) \to b).$$

Proposition 3.11. *Every fuzzy n-fold obstinate filter A such that $A(x) < \frac{1}{2}$, for all $x \in X$ is a fuzzy n-fold positive implicative filter.*

Proof. Assume that A is a fuzzy n-fold obstinate filter and let $x \in X$ such that $A(x) < \frac{1}{2}$. Since A is a fuzzy filter, we have

$$A(x) \geq \min\{A(x^{n-} \to x), A(x^{n-})\}.$$

By hypothesis and Proposition 2.11, it implies $A(x) \geq \min\{A(x^{n-} \to x), (1-A(x))\}$. If $\min\{A(x^{n-} \to x), (1 - A(x))\} = 1 - A(x)$, then $A(x) \geq \frac{1}{2}$ which contradicts the hypothesis. Hence $\min\{A(x^{n-} \to x), (1 - A(x))\} = A(x^{n-} \to x)$ and we obtain $A(x) \geq A(x^{n-} \to x)$. Thus A is a fuzzy n-fold positive implicative filter. □

By the Propositions 3.9 and 3.11, we obtain the following result:

Proposition 3.12. *Every fuzzy n-fold obstinate filter A such that $A(x) < \frac{1}{2}$ for all $x \in X$ is a fuzzy n-fold normal filter.*

By the following example, we show that the converse of the above proposition is not true.

Example 3.13. Let $X = \{0, a, b, c, d, 1\}$ such that $0 < b < a < 1$, $0 < d < c < 1$ and $0 < d < a < 1$. Define $*$ and $\to$ by :

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	0
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then $(X, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra. We define the fuzzy set A by $A(0) = 0.3$, $A(a) = A(c) = A(d) = 0.4$ and $A(b) = 0.35$. Then it is easy to check that A is a fuzzy n-fold normal filter but not a fuzzy n-fold obstinate filter since $A(a^{2-}) = A(0) = 0.3 < 0.6 = 1 - A(a)$.

4. Fuzzy n-fold Boolean filters

Definition 4.1. A fuzzy filter A of X is a fuzzy n-fold Boolean filter if for all $x \in X$,

$$A(x \vee x^{n-}) = A(1).$$

A fuzzy 1-fold Boolean filter is called a fuzzy Boolean filter.

Example 4.2. Let $X = \{0, a, b, 1\}$ be a chain. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Then $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. The fuzzy set A define by $A(1) = A(b) = A(a) > A(0)$ is a fuzzy n-fold Boolean filter of X for all $n \geq 2$. On the other hand, the fuzzy set B defined by $A(a) = A(b) = A(0) = t_0$, $A(1) = t_1$, where $t_0 < t_1$ is a fuzzy filter of X but not a fuzzy 2-fold Boolean filter of X since $A(b \vee b^{2-}) = A(b \vee 0) = A(b) = t_0 \neq A(1)$. Hence, any fuzzy filter may not be a fuzzy n-fold Boolean filter.

Proposition 4.3. *Every fuzzy n-fold Boolean filter is a fuzzy (n+1)-fold Boolean filter.*

Proof. Let A be a fuzzy n-fold Boolean filter and let $x \in X$. By Lemma 2.3, $x^{n+1} \leq x^n$; then $x^n \rightarrow 0 \leq x^{n+1} \rightarrow 0$. Hence, $x^{n-} \leq x^{(n+1)-}$ and so, $x \vee x^{n-} \leq x \vee x^{(n+1)-}$. Hence, $A(x \vee x^{(n+1)-}) \geq A(x \vee x^{n-}) = A(1)$ by hypothesis. Thus $A(x \vee x^{(n+1)-}) = A(1)$. □

Remark 4.4. By finite induction, it is easy to prove that every fuzzy n-fold Boolean filter is a fuzzy $(n + k)$-fold Boolean filter for any integer $k \geq 0$.

The converse of the proposition is not true as seen in the following example:

Example 4.5. In Example 3.10, define the fuzzy subset A by

$$A(1) > A(b) = A(a) = A(0).$$

A is a fuzzy n-fold Boolean filter ($n \geq 3$), but not a fuzzy 2-fold Boolean filter since $A(b \vee b^{2-}) = A(b \vee (b^2 \rightarrow 0)) = A(b \vee b) = A(b) \neq A(1)$.

Theorem 4.6. (Extension theorem for fuzzy n-fold Boolean filters) *Let A and B be two fuzzy filters of X such that $A \subseteq B$. If A is a fuzzy n-fold Boolean filter, then so is B.*

Proof. Suppose that A is a fuzzy n-fold Boolean filter and that $A \subseteq B$. For all $x \in X$, $B(x \vee x^{n-}) \geq A(x \vee x^{n-}) = A(1)$. So B is a fuzzy n-fold Boolean filter. □

Theorem 4.7. *Let A be a fuzzy subset of X. A is a fuzzy n-fold Boolean filter of X if it is both a fuzzy n-fold normal and a fuzzy n-fold implicative filter of X.*

Proof. Suppose that A is a fuzzy n-fold normal and a fuzzy n-fold implicative filter. Since A is fuzzy normal, we have

$$A((x^{n-} \rightarrow x) \rightarrow x) \geq A((x^n \rightarrow x^{n-}) \rightarrow x^{n-}) = A((x^n \rightarrow x^{n-}) \rightarrow (x^n \rightarrow 0)).$$

Since A is fuzzy n-fold implicative, we have by Proposition 2.10:

$$\begin{aligned} A(((x^n \rightarrow x^{n-}) \rightarrow (x^n \rightarrow 0))) &\geq A(x^n \rightarrow (x^{n-} \rightarrow 0)) \\ &= A(x^{n-} \rightarrow (x^n \rightarrow 0)) = A(x^{n-} \rightarrow x^{n-}) = A(1). \end{aligned}$$

Hence $A((x^{n-} \rightarrow x) \rightarrow x) = A(1)$. By the fact that

$$\begin{aligned} A(x \vee x^{n-}) &= A(\min\{((x \rightarrow x^{n-}) \rightarrow x^{n-}), (x^{n-} \rightarrow x) \rightarrow x\}) \\ &= \min\{A(((x \rightarrow x^{n-}) \rightarrow x^{n-}), A((x^{n-} \rightarrow x) \rightarrow x)\} \\ &= \min\{A((x \rightarrow x^{n-}) \rightarrow x^{n-}), A(1)\} \\ &= A(((x \rightarrow x^{n-}) \rightarrow x^{n-})). \end{aligned}$$

Moreover, $A((x \to x^{n-}) \to x^{n-}) \geq A((x^n \to x^{n-}) \to x^{n-}) = A(1)$. Hence

$$A(x \vee x^{n-}) = A(1),$$

and furthermore A is a fuzzy n-fold Boolean filter. □

Theorem 4.8. *Let A be a fuzzy filter of X. A is fuzzy n-fold Boolean if and only if it is fuzzy n-fold positive implicative.*

Proof. Suppose that A is a fuzzy n-fold Boolean filter. Then

$$A((x^{n-} \to x) \to x) \geq \min\{A((x \to x^{n-}) \to x^{n-}), A((x^{n-} \to x) \to x)\}.$$

Hence we have

$$\begin{aligned} A((x^{n-} \to x) \to x) &\geq A(\min\{((x \to x^{n-}) \to x^{n-}), ((x^{n-} \to x) \to x)\}) \\ &= A(x \vee x^{n-}) = A(1) \end{aligned}$$

by hypothesis. Since A is a fuzzy filter, we have

$$A(x) \geq \min\{A((x^{n-} \to x) \to x), A(x^{n-} \to x)\}.$$

Hence $A(x) \geq \min\{A(1), A(x^{n-} \to x)\}$ and so $A(x) \geq A(x^{n-} \to x)$. Thus A is a fuzzy n-fold positive implicative filter.

For the converse, suppose that A is fuzzy n-fold positive implicative. From [6], A is fuzzy n-fold implicative. Moreover, it follows from Proposition 3.9 that A is fuzzy n-fold normal. By Theorem 4.7 it follows that A is fuzzy n-fold Boolean. □

Theorem 4.9. *Every fuzzy n-fold positive implicative filter is a fuzzy n-fold fantastic filter.*

Proof. Assume that A is a fuzzy n-fold positive implicative filter of X. Let $x, y \in X$ and $a = ((x^n \to y) \to y) \to x$. Then by few computation [11], we have $a^n \to y \leq x^n \to y$. By hypothesis, we have

$$\begin{aligned} A(a) &\geq A((a^n \to y) \to a) \geq A((x^n \to y) \to a) \\ &\geq A((x^n \to y) \to (((x^n \to y) \to y) \to x)) \\ &\geq A(((x^n \to y) \to y) \to ((x^n \to y) \to x) \\ &\geq A(y \to x) \end{aligned}$$

by Lemma 2.3. Hence $A(((x^n \to y) \to y) \to x) \geq A(y \to x)$ and so A is a fuzzy n-fold fantastic filter. □

Corollary 4.10. *Every fuzzy n-fold Boolean filter is a fuzzy n-fold fantastic filter.*

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Appendix

Relations between fuzzy n-fold filters in BL-algebras