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(λ, μ)-fuzzy ideals of ordered semigroups

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Abstract. Definitions of (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals of an ordered semigroup were introduced. One obtained that in regular and in intra-regular ordered semigroups, the (λ, μ)-fuzzy ideals and the (λ, μ)-fuzzy interior ideals coincide. The concept of a (λ, μ)-fuzzy simple ordered semigroup was also introduced and one proved that an ordered semigroup is simple if and only if it is (λ, μ)-fuzzy simple.

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1. Introduction and preliminaries

The concept of fuzzy sets was first introduced by Zadeh [10] in 1965 and then the fuzzy sets have been used in the reconsideration of classical mathematics. Recently, Yuan et al. [9] introduced the concept of fuzzy subfield with thresholds. A fuzzy subfield with thresholds λ and μ is also called a (λ, μ)-fuzzy subfield. Yao continued to research (λ, μ)-fuzzy normal subfields, (λ, μ)-fuzzy quotient subfields , (λ, μ)-fuzzy subrings and (λ, μ)-fuzzy ideals in [5, 6, 7, 8]. Feng et al. researched (λ, μ)-fuzzy sublattices and (λ, μ)-fuzzy subhyperlattices in [1].

In this paper, we studied (λ, μ)-fuzzy ideals of ordered semigroups. This can be seen as an application of [8] and as a generalization of [3]. We first introduced definitions of (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals of an ordered semigroup. Then we proved that in regular and in intra-regular ordered semigroups the (λ, μ)-fuzzy ideals and the (λ, μ)-fuzzy interior ideals coincide. Lastly, we introduced the concept of a (λ, μ)-fuzzy simple ordered semigroup, proved that an ordered semigroup is simple if and only if it is (λ, μ)-fuzzy simple and characterized the simple ordered semigroups in terms of (λ, μ)-fuzzy interior ideal.

An ordered semigroup $(S, \circ, \leq)$ is a poset $(S, \leq)$ equipped with a binary operation "$\circ$" such that

(1) $(S, \circ)$ is a semigroup, and

(2) If $x, a, b \in S$, then $a \leq b \Rightarrow \begin{cases} a \circ x \leq b \circ x \\ x \circ a \leq x \circ b. \end{cases}$

If $(S, \circ, \leq)$ is an ordered semigroup, and A is a subset of S, we denote by $(A]$ the subset of S defined as follows:

$$(A] = \{t \in S | t \leq a \text{ for some } a \in A\}.$$

Given an ordered semigroup S, a fuzzy subset of S (or a fuzzy set in S) is an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any $\alpha \in [0, 1]$, f_α is defined by $f_\alpha = \{x \in S | f(x) \geq \alpha\}$.

For each subset A of S, the characteristic function f_A is a fuzzy subset of S defined by

$$f_A(x) = \begin{cases} 1, & \text{if} \quad x \in A \\ 0, & \text{if} \quad x \notin A. \end{cases}$$

In the following, we will use S or $(S, \circ, \leq)$ to denote an ordered semigroup and the multiplication of x, y will be xy instead of $x \circ y$.

In the rest of this paper, we will always assume that $0 \leq \lambda < \mu \leq 1$.

2. (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals

In this section, we first introduce the concepts of (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals of an ordered semigroup. And then show that every (λ, μ)-fuzzy ideal is a (λ, μ)-fuzzy interior ideal.

Definition 2.1. A fuzzy subset f of an ordered semigroup S is called a (λ, μ)-*fuzzy right ideal* of S if

(1) $f(xy) \vee \lambda \geq f(x) \wedge \mu$ for all $x, y \in S$ and

(2) If $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$ for all $x, y \in S$.

A fuzzy subset f of S is called a (λ, μ)-*fuzzy left ideal* of S if

(1) $f(xy) \vee \lambda \geq f(y) \wedge \mu$ for all $x, y \in S$ and

(2) If $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$ for all $x, y \in S$.

A fuzzy subset f of S is called a (λ, μ)-*fuzzy ideal* of S if it is both a (λ, μ)-fuzzy right and a (λ, μ)-fuzzy left ideal of S.

Example 2.2. Let $(S, *, \leq)$ be an ordered semigroup defined by $e \leq a$ and the following table:

$*$	e	a
e	e	a
a	a	e

If we define a fuzzy set f as following

S	e	a
f	0.5	0.5

Then f is a $(0.5, 0.7)$-fuzzy ideal of S.

Definition 2.3 ([2])**.** If $(S, \circ, \leq)$ is an ordered semigroup, a nonempty subset A of S is called an *interior ideal* of S if

(1) $SAS \subseteq A$ and

(2) If $a \in A, b \in S$ and $b \leq a$, then $b \in A$.

Definition 2.4. If $(S, \circ, \leq)$ is an ordered semigroup, a fuzzy subset f of S is called a (λ, μ)-*fuzzy interior ideal* of S if the following assertions are satisfied:
(1) $f(xay) \vee \lambda \geq f(a) \wedge \mu$ for all $x, a, y \in S$ and
(2) If $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$.

In Example 2.2, it is easy to know that$f$ is also a $(0.5, 0.7)$-fuzzy interior ideal of S.

Theorem 2.5. *Let $(S, \circ, \leq)$ be an ordered semigroup, Then f is a (λ, μ)-fuzzy interior ideal of S if and only if f_α is an interior ideal of S for all $\alpha \in (\lambda, \mu]$.*

Proof. Let f be a (λ, μ)-fuzzy interior ideal of S and $\alpha \in (\lambda, \mu]$. First of all, we need to show that $xay \in f_\alpha$, for all $a \in f_\alpha$, $x, y \in S$. From $f(xay) \vee \lambda \geq f(a) \wedge \mu \geq \alpha \wedge \mu = \alpha$ and $\lambda < \alpha$, we conclude that $f(xay) \geq \alpha$, that is $xay \in f_\alpha$. Then, we need to show that $b \in f_\alpha$ for all $a \in f_\alpha, b \in S$ such that $b \leq a$. From $b \leq a$ we know that $f(b) \vee \lambda \geq f(a) \wedge \mu$ and from $a \in f_\alpha$, we have $f(a) \geq \alpha$. Thus $f(b) \vee \lambda \geq \alpha \wedge \mu = \alpha$. Notice that $\lambda < \alpha$, we conclude that $f(b) \geq \alpha$, that is, $b \in f_\alpha$.

Conversely, let f_α be an interior ideal of S for all $\alpha \in (\lambda, \mu]$. If there are $x_0, a_0, y_0 \in S$, such that $f(x_0 a_0 y_0) \vee \lambda < \alpha = f(a_0) \wedge \mu$, then $\alpha \in (\lambda, \mu], f(a_0) \geq \alpha$ and $f(x_0 a_0 y_0) < \alpha$. That is $a_0 \in f_\alpha$ and $x_0 a_0 y_0 \notin f_\alpha$. This is a contradiction with that f_α is an interior ideal of S. Hence $f(xay) \vee \lambda \geq f(a) \wedge \mu$ holds for all $x, a, y \in S$. If there are $x_0, y_0 \in S$ such that $x_0 \leq y_0$ and $f(x_0) \vee \lambda < \alpha = f(y_0) \wedge \mu$, then $\alpha \in (\lambda, \mu], f(y_0) \geq \alpha$ and $f(x_0) < \alpha$, that is $y_0 \in f_\alpha$ and $x_0 \notin f_\alpha$. This is a contradiction with that f_α is an interior ideal of S. Hence if $x \leq y$, then $f(x) \vee \lambda \geq f(y) \wedge \mu$. ☐

Theorem 2.6. *Let $(S, \circ, \leq)$ be an ordered semigroup and f a (λ, μ)-fuzzy ideal of S. Then f is a (λ, μ)-fuzzy interior ideal of S.*

Proof. Let $x, a, y \in S$. Since f is a (λ, μ)-fuzzy left ideal of S and $x, ay \in S$, we have that

$$f(x(ay)) \vee \lambda \geq f(ay) \wedge \mu. \tag{1}$$

Since f is a (λ, μ)-fuzzy right ideal of S, we have that

$$f(ay) \vee \lambda \geq f(a) \wedge \mu. \tag{2}$$

From (1) and (2) we know that $f(xay) \vee \lambda = (f(x(ay)) \vee \lambda) \vee \lambda \geq (f(ay) \wedge \mu) \vee \lambda = (f(ay) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(a) \wedge \mu$. ☐

3. (λ, μ)-fuzzy interior ideals of regular/intra-regular ordered semigroups

We prove here that in regular and in intra-regular ordered semigroups the (λ, μ)-fuzzy ideals and the (λ, μ)-fuzzy interior ideals coincide.

Definition 3.1 ([3])**.** An ordered semigroup $(S, \circ, \leq)$ is called *regular* if for all $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Definition 3.2 ([3])**.** An ordered semigroup $(S, \circ, \leq)$ is called *intra-regular* if for all $a \in S$ there exists $x, y \in S$ such that $a \leq xa^2y$.

Theorem 3.3. *Let $(S, \circ, \leq)$ be a regular ordered semigroup and f a (λ, μ)-fuzzy interior ideal of S. Then f is a (λ, μ)-fuzzy ideal of S.*

Proof. Let $x, y \in S$. Then $f(xy) \vee \lambda \geq f(x) \wedge \mu$. Indeed, since S is regular and $x \in S$, there exist $z \in S$ such that $x \leq xzx$. Thus we have that $xy \leq (xzx)y = (xz)xy$. So

$$f(xy) \vee \lambda \geq f((xz)xy) \wedge \mu \tag{3}$$

for f is a (λ, μ)-fuzzy interior ideal. Again since f is a (λ, μ)-fuzzy interior ideal of S, we have

$$f((xz)xy) \vee \lambda \geq f(x) \wedge \mu. \tag{4}$$

From (3) and (4) we have that $f(xy) \vee \lambda = (f(xy) \vee \lambda) \vee \lambda \geq (f((xz)xy) \wedge \mu) \vee \lambda = (f((xz)xy) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(x) \wedge \mu$, and f is a (λ, μ)-fuzzy right ideal of S. In a similar way, we can prove that f is a (λ, μ)-fuzzy left ideal of S. Thus f is a (λ, μ)-fuzzy ideal of S. □

Theorem 3.4. *Let $(S, \circ, \leq)$ be a intra-regular ordered semigroup and f a (λ, μ)-fuzzy interior ideal of S. Then f is a (λ, μ)-fuzzy ideal of S.*

Proof. Let $a, b \in S$. Then $f(ab) \vee \lambda \geq f(a) \wedge \mu$. Indeed, since S is intra-regular and $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$. Then $ab \leq (xa^2y)b$. Since f is a (λ, μ)-fuzzy interior ideal of S, we have that $f(ab) \vee \lambda = (f(ab) \vee \lambda) \vee \lambda \geq (f(xa^2yb) \wedge \mu) \vee \lambda = (f(xa^2yb) \vee \lambda) \wedge (\mu \vee \lambda)$. Again since f is a (λ, μ)-fuzzy interior ideal of S, we have $f(xa^2yb) \vee \lambda = f((xa)a(yb)) \vee \lambda \geq f(a) \wedge \mu$. Thus we have that $f(ab) \vee \lambda \geq f(a) \wedge \mu$, and f is a (λ, μ)-fuzzy right ideal of S. In a similar way we can prove that f is a (λ, μ)-fuzzy left ideal of S. Therefore, f is a (λ, μ)-fuzzy ideal of S. □

Remark 3.5. From previous theorems we know that in regular or intra-regular ordered semigroups the concepts of (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals coincide.

4. (λ, μ)-fuzzy simple ordered semigroups

In this section, we introduce the concept of (λ, μ)-fuzzy simple ordered semigroups and characterize this type of ordered semigroups in terms of (λ, μ)-fuzzy interior ideals.

Definition 4.1 ([3])**.** An ordered semigroup S is called *simple* if it does not contain proper ideals, that is, for any ideal $A \neq \varnothing$ of S, we have $A = S$.

Definition 4.2. An ordered semigroup S is called (λ, μ)-*fuzzy simple* if for any (λ, μ)-fuzzy ideal f of S, we have $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$.

Remark 4.3. In [3], Kehayopulu and Tsingelis studied $(0, 1)$-fuzzy simple ordered semigroup, which was called fuzzy simple ordered semigroup. (see Definition 3.1 of [3])

Theorem 4.4. *Let S be an ordered semigroup. Then S is (λ, μ)-fuzzy simple if and only if for any (λ, μ)-fuzzy ideal f of S, if $f_\alpha \neq \varnothing$, then $f_\alpha = S$, for all $\alpha \in (\lambda, \mu]$.*

Proof. For any (λ, μ)-fuzzy ideal f of S, suppose that $f_\alpha \neq \varnothing$. We need to prove that $x \in f_\alpha$ for all $x \in S$, where $\alpha \in (\lambda, \mu]$. Since $f_\alpha \neq \varnothing$, we can suppose that there exists $y \in f_\alpha$, that is $f(y) \geq \alpha$. So $f(x) \vee \lambda \geq f(y) \wedge \mu \geq \alpha \wedge \mu = \alpha$. Notice that $\lambda < \alpha$, we have that $f(x) \geq \alpha$, that is $x \in f_\alpha$.

Conversely, for any (λ, μ)-fuzzy ideal f of S, suppose that $f_\alpha = S$, for all $\alpha \in (\lambda, \mu]$. We need to prove that $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$. If there exist $a_0, b_0 \in S$, such that $f(a_0) \vee \lambda < \alpha = f(b_0) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $f(a_0) < \alpha$ and $f(b_0) \geq \alpha$. Thus $a_0 \notin f_\alpha = S$. This is a contradiction. So $f(a) \vee \lambda \geq f(b) \wedge \mu$ holds, for all $a, b \in S$. □

Proposition 4.5. *Let S be an ordered semigroup and f a (λ, μ)-fuzzy right ideal of S. Then $I_a = \{b \in S | f(b) \vee \lambda \geq f(a) \wedge \mu\}$ is a right ideal of S for every $a \in S$.*

Proof. Let $a \in S$. Then $I_a \neq \varnothing$ since $a \in I_a$.

(1) Let $b \in I_a$ and $s \in S$, then $bs \in I_a$. Indeed, since f is a (λ, μ)-fuzzy right ideal of S and $b, s \in S$, we have

$$f(bs) \vee \lambda \geq f(b) \wedge \mu. \tag{5}$$

Since $b \in I_a$, we have that

$$f(b) \vee \lambda \geq f(a) \wedge \mu. \tag{6}$$

From (5) and (6) we conclude that $f(bs) \vee \lambda = (f(bs) \vee \lambda) \vee \lambda \geq (f(b) \wedge \mu) \vee \lambda = (f(b) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(a) \wedge \mu$. So $bs \in I_a$.

(2) Let $b \in I_a$ and $S \ni s \leq b$, then $s \in I_a$. Indeed, since f is a (λ, μ)-fuzzy right ideal of S , $s, b \in S$ and $s \leq b$, we have

$$f(s) \vee \lambda \geq f(b) \wedge \mu. \tag{7}$$

Since $b \in I_a$, we have

$$f(b) \vee \lambda \geq f(a) \wedge \mu. \tag{8}$$

From (7) and (8) we obtain that $f(s) \vee \lambda = (f(s) \vee \lambda) \vee \lambda \geq (f(b) \wedge \mu) \vee \lambda = (f(b) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(a) \wedge \mu$. So $s \in I_a$. □

Similarly, we have

Proposition 4.6. *Let S be an ordered semigroup and f a (λ, μ)-fuzzy left ideal of S. Then $I_a = \{b \in S \mid f(b) \vee \lambda \geq f(a) \wedge \mu\}$ is a left ideal of S for every $a \in S$.*

By the previous propositions, we have

Proposition 4.7. *Let S be an ordered semigroup and f a (λ, μ)-fuzzy ideal of S. Then $I_a = \{b \in S \mid f(b) \vee \lambda \geq f(a) \wedge \mu\}$ is an ideal of S for every $a \in S$.*

Lemma 4.8. *Let S be an ordered semigroup and $\varnothing \neq I \subseteq S$, then I is an ideal of S if and only if the characteristic function f_I is a (λ, μ)-fuzzy ideal of S.*

Proof. Similar to the proof of Theorem 1 of Section 2. One can also see the proof of Proposition 3.2 of [8]. □

Theorem 4.9. *An ordered semigroup S is simple is simple if and only if it is (λ, μ)-fuzzy simple.*

Proof. Suppose S is simple, let f be a (λ, μ)-fuzzy ideal of S and $a, b \in S$. By previous proposition, the set I_a is an ideal of S. Since S is simple, we have $I_a = S$. Then $b \in I_a$, from which we have that $f(b) \vee \lambda \geq f(a) \wedge \mu$. Thus S is (λ, μ)-fuzzy simple.

Conversely, suppose S contains proper ideals and let I be such ideal of S. By the previous lemma, we know that f_I is a (λ, μ)-fuzzy ideal of S. We have that $S \subseteq I$. Indeed, let $x \in S$. Since S is (λ, μ)-fuzzy simple, $f_I(x) \vee \lambda \geq f_I(b) \wedge \mu$ for all $b \in S$. Now let $a \in I$. Then we have $f_I(x) \vee \lambda \geq f_I(a) \wedge \mu = 1 \wedge \mu = \mu$. Notice that $\lambda < \mu$, we conclude that $f_I(x) \geq \mu$, which implies that $f_I(x) = 1$, that is $x \in I$. Thus we have that $S \subseteq I$, and so $S = I$. We get a contradiction. □

Lemma 4.10 ([3, 4]). *An ordered semigroup S is simple if and only if for every $a \in S$, we have $S = (SaS]$.*

Theorem 4.11. *Let S be an ordered semigroup. Then S is simple if and only if for every (λ, μ)-fuzzy interior ideal f of S, we have $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$.*

Proof. Suppose S is simple. Let f be a (λ, μ)-fuzzy interior ideal of S and $a, b \in S$. Since S is simple and $b \in S$, by the previous lemma, we have that $S = (SbS]$. Since $a \in S$, we have that $a \in (SbS]$. Then there exist $x, y \in S$ such that $a \leq xby$. Since $a, xby \in S, a \leq xby$ and f is a (λ, μ)-fuzzy interior ideal of S, we have that

$$f(a) \vee \lambda \geq f(xby) \wedge \mu. \tag{9}$$

Since $x, b, y \in S$ and f is a (λ, μ)-fuzzy interior ideal of S, we have that

$$f(xby) \vee \lambda \geq f(b) \wedge \mu. \tag{10}$$

From (9) and (10) we conclude that $f(a) \vee \lambda = (f(a) \vee \lambda) \vee \lambda \geq (f(xby) \wedge \mu) \vee \lambda = (f(xby) \vee \lambda) \wedge (\mu \vee \lambda) \geq f(b) \wedge \mu$.

Conversely, Suppose that for every (λ, μ)-fuzzy interior ideal f of S, we have $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$. Now let f be any (λ, μ)-fuzzy ideal f of S, then it is a (λ, μ)-fuzzy interior ideal of S. So we have $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$. Thus S is (λ, μ)-fuzzy simple by its definition. And from the previous theorem, we conclude that S is simple. □

As a consequence we have

Theorem 4.12. *For an ordered semigroup S, the following are equivalent:*

(1) *S is simple.*

(2) *$S = (SaS]$ for every $a \in S$.*

(3) *S is (λ, μ)-fuzzy simple.*

(4) *For every (λ, μ)-fuzzy interior ideal f of S, we have $f(a) \vee \lambda \geq f(b) \wedge \mu$, for all $a, b \in S$.*

5. Conclusion and further research

In this paper, we generalized Kehayopulu and Tsingelis' results. We introduced (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals of an ordered semigroup and studied them. When $\lambda = 0$ and $\mu = 1$, we meet ordinary fuzzy ideals and fuzzy interior ideals. From this view, we say that (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy interior ideals are more general concepts than fuzzy ones.

In [8], Yao gave the definition of (λ, μ)-fuzzy bi-ideals in semigroups. One can study (λ, μ)-fuzzy bi-ideals in ordered semigroups. For example, one can research the relationship among (λ, μ)-fuzzy ideals, (λ, μ)-fuzzy interior ideals and (λ, μ)-fuzzy bi-ideals. We would like to explore this in next papers.

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