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Ideals and filters in CI-algebras based on bipolar-valued fuzzy sets

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Abstract. The notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras are introduced, and related properties are investigated. Characterizations of a bipolar fuzzy ideal and a (closed) bipolar fuzzy filter in CI-algebras are established. Relations between a bipolar fuzzy CI-subalgebra and a (closed) bipolar fuzzy filter are discussed, and conditions for a bipolar fuzzy CI-subalgebra to be a (closed) bipolar fuzzy filter are provided.

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1. Introduction

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 11]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Lee [4] applied the bipolar-valued fuzzy set theory to BCK/BCI-algebras, and introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCK/BCI-algebra. Kim et al. [2] discussed bipolar-valued fuzzy set theory in the ideal theory of semigroups. As a generalization of BE-algebras and $BCK/BCI/BCH$-algebras, Meng [7] introduced the notion of CI-algebras. Meng [7, 8], Kim [3] and Piekart et al. [10] discussed ideal and (closed) filter theory in CI-algebras. Mostafa et al. [9] considered the fuzzification of ideals in CI-algebras.

In this paper, we apply the bipolar-valued fuzzy set theory to CI-algebras, and introduce the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras. We establish characterizations of a bipolar fuzzy ideal and a (closed) bipolar fuzzy filter in CI-algebras. We discuss relations between a bipolar fuzzy CI-subalgebra and a (closed) bipolar fuzzy filter. We provide conditions for a bipolar fuzzy CI-subalgebra to be a (closed) bipolar fuzzy filter.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. An element $X \in K(\tau)$ is called a *CI-algebra* (see [7]) if it satisfies the following axioms:

(a1) $x * x = 1$,
(a2) $1 * x = x$,
(a3) $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$. If a CI-algebra X satisfies:

(a4) $x * 1 = 1$ for all $x \in X$,

then we say that X is a *BE-algebra*.

We can define a partial ordering $\leq$ on X by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1).$$

In a CI-algebra X, the following hold (see [7]):

(b1) $y * ((y * x) * x) = 1$,
(b2) $(x * 1) * (y * 1) = (x * y) * 1$,
(b3) $1 \leq x \;\Rightarrow\; x = 1$,

for all $x, y \in X$. A CI-algebra X is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A CI-algebra X is said to be *transitive* (see [7]) if it satisfies:

$$(\forall x, y, z \in X)\,((y * z) * ((x * y) * (x * z)) = 1)\,. \tag{2.1}$$

A nonempty subset S of a CI-algebra X is called a *CI-subalgebra* of X if $x*y \in S$ for all $x, y \in S$. A nonempty subset I of a CI-algebra X is called an *ideal* of X (see [7]) if it satisfies:

(I1) $(\forall x, a \in X)\,(a \in I \;\Rightarrow\; x * a \in I)$,
(I2) $(\forall x, a, b \in X)\,(a, b \in I \;\Rightarrow\; (a * (b * x)) * x \in I)$.

A subset F of a CI-algebra X is called a *filter* of X (see [7]) if it satisfies:

(F1) $1 \in F$,
(F2) $(\forall x, y \in X)\,(x * y \in F,\ x \in F \ \Rightarrow\ y \in F)$.

A filter F of a CI-algebra X is said to be *closed* (see [8]) if it satisfies:

$$(\forall x \in X)\,(x \in F \ \Rightarrow\ x * 1 \in F). \tag{2.2}$$

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Let X be a universe of discourse. A *bipolar-valued fuzzy set* f in X is an object having the form

$$f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$$

where $f_n : X \to [-1, 0]$ and $f_p : X \to [0, 1]$ are mappings. The positive membership degree $f_p(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$, and the negative membership degree $f_n(x)$ denotes the satisfaction degree of x to some implicit counter-property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) \neq 0$ and $f_n(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) = 0$ and $f_n(x) \neq 0$, it is the situation that x does not satisfy the property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. It is possible for an element x to be $f_p(x) \neq 0$ and $f_n(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [6]). For the sake of simplicity, we shall use the symbol $f = (X; f_n, f_p)$ for the bipolar-valued fuzzy set $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy CI-subalgebra of CI-algebras

In what follows let X denote a CI-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $f = (X; f_n, f_p)$ in X is called a *bipolar fuzzy CI-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in X) \begin{pmatrix} f_n(x * y) \leq \bigvee\{f_n(x), f_n(y)\} \\ f_p(x * y) \geq \bigwedge\{f_p(x), f_p(y)\} \end{pmatrix}. \tag{3.1}$$

Example 3.2. Let $X = \{1, a, b\}$ be a set with the following Cayley table:

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then $(X;*,1)$ is a proper CI-algebra (see [7]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ by

$$f = \{(1; -0.7, 0.8), (a; -0.5, 0.5), (b; -0.3, 0.2)\}.$$

It is easy to verify that $f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X.

Example 3.3. Let I be an ideal of a self-distributive CI-algebra X. For any $w \in X$, let $f = (X; f_n, f_p)$ be a bipolar fuzzy set in X defined by

$$f = \{(1; -0.9, 0.8), (x; -0.8, 0.7), (y; -0.5, 0.4) \mid x \in I_w,\ y \in X \setminus I_w\}$$

where $I_w = \{x \in X \mid w * x \in I\}$. Then $f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X.

Proposition 3.4. *Every bipolar fuzzy CI-subalgebra $f = (X; f_n, f_p)$ of X satisfies the following assertions:*

(1) $(\forall x \in X)\,(f_n(x*1) \geq f_n(x) \geq f_n(1),\ f_p(x*1) \leq f_p(x) \leq f_p(1))$,

Proof. Using (3.1) and (a1), we have

$$f_n(1) = f_n(x*x) \leq \bigvee \{f_n(x), f_n(x)\} = f_n(x)$$

and

$$f_p(1) = f_p(x*x) \geq \bigwedge \{f_p(x), f_p(x)\} = f_p(x).$$

Using (3.1) and (a1) again, we obtain

$$f_n(x*1) \leq \bigvee \{f_n(x), f_n(1)\} = \bigvee \{f_n(x), f_n(x*x)\} = f_n(x)$$

and

$$f_p(x*1) \geq \bigwedge \{f_p(x), f_p(1)\} = \bigwedge \{f_p(x), f_p(x*x)\} = f_p(x).$$

This completes the proof. □

For a bipolar fuzzy set $f = (X; f_n, f_p)$ in X, consider the following condition:

$$(3.2)\qquad (\forall x, y \in X)\,(f_n(x*y) \leq f_n(x),\ f_p(x*y) \geq f_p(x)).$$

Generally, any bipolar fuzzy CI-subalgebra $f = (X; f_n, f_p)$ of X does not satisfy the condition (3.2). In fact, in Example 3.2, $f_n(1*b) = f_n(b) = -0.3 > -0.7 = f_n(1)$.

Theorem 3.5. *If a bipolar fuzzy CI-subalgebra $f = (X; f_n, f_p)$ of X satisfies the condition (3.2), then $f = (X; f_n, f_p)$ is a bipolar constant, that is,*

$$(\forall x \in X)\,(f_n(x) = f_n(1),\ f_p(x) = f_p(1)).$$

Proof. Combining (a2) and (3.2) induces

$$f_n(x) = f_n(1*x) \leq f_n(1) \quad\text{and}\quad f_p(x) = f_p(1*x) \geq f_p(1).$$

It follows from Proposition 3.4 that $f_n(x) = f_n(1)$ and $f_p(x) = f_p(1)$ for all $x \in X$. □

For a bipolar fuzzy set $f = (X; f_n, f_p)$ in X and $(s,t) \in [-1,0] \times [0,1]$, we define

$$
\begin{aligned}
N(f_n; s) &= \{x \in X \mid f_n(x) \le s\}, \\
P(f_p; t) &= \{x \in X \mid f_p(x) \ge t\}
\end{aligned} \tag{3.3}
$$

which are called the *negative s-cut* of $f = (X; f_n, f_p)$ and the *positive t-cut* of $f = (X; f_n, f_p)$, respectively. The set

$$C(f; (s,t)) := N(f_n; s) \cap P(f_p; t)$$

is called the *(s,t)-cut* of $f = (X; f_n, f_p)$. For every $k \in [0,1]$, if $(s,t) = (-k,k)$ then the set

$$C(f; k) := N(f_n; -k) \cap P(f_p; k)$$

is called the *k-cut* of $f = (X; f_n, f_p)$.

Theorem 3.6. *Let $f = (X; f_n, f_p)$ be a bipolar fuzzy CI-subalgebra of X. Then the following assertions are valid:*

(1) $(\forall s \in [-1,0])\,(N(f_n; s) \ne \varnothing \;\Rightarrow\; N(f_n; s)$ is a CI-subalgebra of $X)$.
(2) $(\forall t \in [0,1])\,(P(f_p; t) \ne \varnothing \;\Rightarrow\; P(f_p; t)$ is a CI-subalgebra of $X)$.

Proof. (1) Let $s \in [-1,0]$ be such that $N(f_n; s) \ne \varnothing$. If $x, y \in N(f_n; s)$, then $f_n(x) \le s$ and $f_n(y) \le s$. It follows from (3.1) that

$$f_n(x * y) \le \bigvee \{f_n(x), f_n(y)\} \le s$$

so that $x * y \in N(f_n; s)$. Hence $N(f_n; s)$ is a CI-subalgebra of X.

(2) Let $t \in [0,1]$ be such that $P(f_p; t) \ne \varnothing$. Let $x, y \in P(f_p; t)$. Then $f_p(x) \ge t$ and $f_p(y) \ge t$, which imply from (3.1) that

$$f_p(x * y) \ge \bigwedge \{f_p(x), f_p(y)\} \ge t.$$

Thus $x * y \in P(f_p; t)$, and so $P(f_p; t)$ is a CI-subalgebra of X. □

Corollary 3.7. *If $f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X, then the sets $N(f_n; f_n(1))$ and $P(f_p; f_p(1))$ are CI-subalgebras of X.*

4. Bipolar fuzzy ideals of CI-algebras

Definition 4.1. A bipolar fuzzy set $f = (X; f_n, f_p)$ in X is called a *bipolar fuzzy ideal* of X if it satisfies the following condition:

(1) $(\forall x, y \in X)\,(f_n(x * y) \le f_n(y),\; f_p(x * y) \ge f_p(y))$,
(2) $(\forall x, a, b \in X) \begin{pmatrix} f_n((a * (b * x)) * x) \le \bigvee \{f_n(a), f_n(b)\} \\ f_p((a * (b * x)) * x) \ge \bigwedge \{f_p(a), f_p(b)\} \end{pmatrix}$.

Example 4.2. Let $X = \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X, *, 1)$ is a CI-algebra (see [9]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ by

$$f = \{(1; -0.7, 0.6), (a; -0.7, 0.6), (b; -0.7, 0.6), (c; -0.3, 0.2), (d; -0.3, 0.2), (0; -0.3, 0.2)\}.$$

It is easy to verify that $f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X.

Lemma 4.3. *Every bipolar fuzzy ideal $f = (X; f_n, f_p)$ of X satisfies the following condition:*

$$(\forall x \in X)\,(f_n(x) \geq f_n(1),\ f_p(x) \leq f_p(1)) \tag{4.1}$$

Proof. Using (a1) and Definition 4.1(1) induces

$$f_n(1) = f_n(x * x) \leq f_n(x) \quad \text{and} \quad f_p(1) = f_p(x * x) \geq f_p(x)$$

for all $x \in X$. □

Proposition 4.4. *If $f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X, then*

$$(\forall x, y \in X)\,(f_n((x * y) * y) \leq f_n(x),\ f_p((x * y) * y) \geq f_p(x))\,. \tag{4.2}$$

Proof. Taking $a = x$, $b = 1$ and $x = y$ in Definition 4.1(2) implies that

$$f_n((x * y) * y) = f_n((x * (1 * y)) * y) \leq \bigvee \{f_n(x), f_n(1)\} = f_n(x)$$

and

$$f_p((x * y) * y) = f_p((x * (1 * y)) * y) \geq \bigwedge \{f_p(x), f_p(1)\} = f_p(x)$$

for all $x, y \in X$ by using (a2) and Lemma 4.3. □

Corollary 4.5. *Every bipolar fuzzy ideal $f = (X; f_n, f_p)$ of X satisfies the following assertion:*

$$(\forall x, y \in X)\,(x \leq y\ \Rightarrow f_n(x) \geq f_n(y),\ f_p(x) \leq f_p(y))\,. \tag{4.3}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f_n(y) = f_n(1 * y) = f_n((x * y) * y) \leq f_n(x)$$

and

$$f_p(y) = f_p(1 * y) = f_p((x * y) * y) \geq f_p(x)$$

by (a2) and (4.2). □

Proposition 4.6. *If X is transitive, then every bipolar fuzzy ideal $f = (X; f_n, f_p)$ of X satisfies the following assertion:*

$$(\forall x, y, z \in X) \left(\begin{array}{l} f_n(x * z) \leq \bigvee \{f_n(x * (y * z)), f_n(y)\} \\ f_p(x * z) \geq \bigwedge \{f_p(x * (y * z)), f_p(y)\} \end{array} \right). \tag{4.4}$$

Proof. If X is transitive, then $((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$ for all $x, y, z \in X$. Using (a2), Definition 4.1(2) and Proposition 4.4, we have

$$\begin{aligned} f_n(x * z) &= f_n(1 * (x * z)) \\ &= f_n((((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z)) \\ &\leq \bigvee \{f_n((y * z) * z), f_n(x * (y * z))\} \\ &\leq \bigvee \{f_n(y), f_n(x * (y * z))\} \end{aligned}$$

and

$$\begin{aligned} f_p(x * z) &= f_p(1 * (x * z)) \\ &= f_p((((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z)) \\ &\geq \bigwedge \{f_p((y * z) * z), f_p(x * (y * z))\} \\ &\geq \bigwedge \{f_p(y), f_p(x * (y * z))\}. \end{aligned}$$

This completes the proof. □

Theorem 4.7. *For a bipolar fuzzy set $f = (X; f_n, f_p)$ in X, the following are equivalent:*

(1) *$f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X.*
(2) *$f = (X; f_n, f_p)$ satisfies the following assertions:*
 (i) $(\forall s \in [-1, 0]) \, (N(f_n; s) \neq \varnothing \;\Rightarrow\; N(f_n; s)$ is an ideal of $X)$.
 (ii) $(\forall t \in [0, 1]) \, (P(f_p; t) \neq \varnothing \;\Rightarrow\; P(f_p; t)$ is an ideal of $X)$.

Proof. (1) $\Rightarrow$ (2). Let $s \in [-1, 0]$ be such that $N(f_n; s) \neq \varnothing$. Then there exists $y \in N(f_n; s)$, and so $f_n(y) \leq s$. It follows from Definition 4.1(1) that $f_n(x * y) \leq f_n(y) \leq s$ so that $x * y \in N(f_n; s)$. Let $x \in X$ and $a, b \in N(f_n; s)$. Then $f_n(a) \leq s$ and $f_n(b) \leq s$. Using Definition 4.1(2), we have

$$f_n((a * (b * x)) * x) \leq \bigvee \{f_n(a), f_n(b)\} \leq s$$

which implies that $(a * (b * x)) * x \in N(f_n; s)$. Therefore $N(f_n; s)$ is an ideal of X. Assume that $P(f_p; t) \neq \varnothing$ for $t \in [0, 1]$, and let $a \in P(f_p; t)$. Then $f_p(a) \geq t$, and so $f_p(x * a) \geq f_p(a) \geq t$ for all $x \in X$ by Definition 4.1(1). Thus $x * a \in P(f_p; t)$ for all $x \in X$. Let $x \in X$ and $a, b \in P(f_p; t)$. Then $f_p(a) \geq t$ and $f_p(b) \geq t$. It follows from Definition 4.1(2) that

$$f_p((a * (b * x)) * x) \geq \bigwedge \{f_p(a), f_p(b)\} \geq t$$

so that $(a * (b * x)) * x \in P(f_p; t)$. Hence $P(f_p; t)$ is an ideal of X.

(2) $\Rightarrow$ (1). Assume that there exist $a, b \in X$ such that $f_n(a * b) > f_n(b)$. Taking

$$s_0 := \tfrac{1}{2} (f_n(a * b) + f_n(b))$$

implies $f_n(b) < s_0 < f_n(a * b)$. Thus $b \in N(f_n; s_0)$ and $a * b \notin N(f_n; s_0)$. This is a contradiction, and thus $f_n(x * y) \leq f_n(y)$ for all $x, y \in X$. Suppose that

$$f_n((a * (b * x)) * x) > \bigvee \{f_n(a), f_n(b)\}$$

for some $a, b, x \in X$ and let

$$s_1 := \tfrac{1}{2}\left(f_n((a * (b * x)) * x) + \bigvee \{f_n(a), f_n(b)\}\right).$$

Then $\bigvee \{f_n(a), f_n(b)\} < s_1 < f_n((a * (b * x)) * x)$, and so $a, b \in N(f_n; s_1)$ but $(a * (b * x)) * x \notin N(f_n; s_1)$. This is a contradiction. Therefore

$$f_n((a * (b * x)) * x) \leq \bigvee \{f_n(a), f_n(b)\}$$

for all $x, a, b \in X$. Now, if $f_p(a * b) < f_p(b)$ for some $a, b \in X$, then $f_p(a * b) < t_0 \leq f_p(b)$ for some $t_0 \in (0, 1]$. Thus $b \in P(f_p; t_0)$ but $a * b \in P(f_p; t_0)$, which is a contradiction. Thus $f_p(x * y) \geq f_p(y)$ for all $x, y \in X$. If

$$f_p((a * (b * c)) * c) < \bigwedge \{f_n(a), f_n(b)\}$$

for some $a, b, c \in X$, then there exists $t_1 \in (0, 1]$ such that

$$f_p((a * (b * c)) * c) < t_1 \leq \bigwedge \{f_n(a), f_n(b)\}.$$

It follows that $a, b \in P(f_p; t_1)$ but $(a * (b * c)) * c \notin P(f_p; t_0)$, a contradiction. Consequently, $f_p((a * (b * x)) * x) \geq \bigwedge \{f_n(a), f_n(b)\}$ for all $a, b, x \in X$. Therefore $f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X. □

For any $a, b \in X$, the set

$$A(a, b) := \{x \in X \mid a * (b * x) = 1\}$$

is called an *upper set* of a and b. Obviously $1, a, b \in A(a, b)$ (see [3]).

Theorem 4.8. *A bipolar fuzzy set $f = (X; f_n, f_p)$ in X is a bipolar fuzzy ideal of X if and only if it satisfies:*

$$(\forall a, b \in X)\, (\forall (s, t) \in [-1, 0] \times [0, 1]) \begin{pmatrix} a, b \in C(f; (s, t)) \Rightarrow \\ A(a, b) \subseteq C(f; (s, t)) \end{pmatrix}. \tag{4.5}$$

Proof. Assume that $f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X. Let $a, b \in X$ and $(s, t) \in [-1, 0] \times [0, 1]$ be such that $a, b \in C(f; (s, t))$. Then $f_n(a) \leq s$, $f_n(b) \leq s$, $f_p(a) \geq t$ and $f_p(b) \geq t$. If $x \in A(a, b)$, then $a * (b * x) = 1$. Hence

$$f_n(x) = f_n(1 * x) = f_n\left((a * (b * x)) * x\right) \leq \bigvee \{f_n(a), f_n(b)\} \leq s$$

and

$$f_p(x) = f_p(1 * x) = f_p\left((a * (b * x)) * x\right) \geq \bigwedge \{f_p(a), f_p(b)\} \geq t$$

which imply that $x \in C(f; (s, t))$. Therefore $A(a, b) \subseteq C(f; (s, t))$.

Conversely, suppose that $f = (X; f_n, f_p)$ satisfies (4.5). Let $(s, t) \in [-1, 0] \times [0, 1]$ be such that $N(f_n; s) \neq \varnothing$ and $P(f_p; t) \neq \varnothing$. Obviously, $1 \in A(a, b) \subseteq C(f; (s, t))$ for all $a, b \in X$. Let $x, y, z \in X$ be such that $x * (y * z) \in C(f; (s, t))$ and $y \in C(f; (s, t))$. Using (a1) and (a3), we get

$$(x * (y * z)) * (y * (x * z)) = (x * (y * z)) * (x * (y * z)) = 1$$

for all $x, y, z \in X$. Hence $x * z \in A(x * (y * z), y) \subseteq C(f; (s, t))$. This shows that $N(f_n; s)$ and $P(f_p; t)$ are ideals of X. It follows from Theorem 4.7 that $f = (X; f_n, f_p)$ is a bipolar fuzzy ideal of X. □

5. Bipolar fuzzy filters of CI-algebras

Definition 5.1. A bipolar fuzzy set $f = (X; f_n, f_p)$ in X is called a *bipolar fuzzy filter* of X if it satisfies the following condition:

$$(5.1) \qquad (\forall x, y \in X) \left(\begin{array}{l} f_n(1) \leq f_n(y) \leq \bigvee \{f_n(x * y), f_n(x)\} \\ f_p(1) \geq f_p(y) \geq \bigwedge \{f_p(x * y), f_p(x)\} \end{array} \right).$$

Definition 5.2. A bipolar fuzzy filter $f = (X; f_n, f_p)$ of X is said to be *closed* if it satisfies the following assertion:

$$(5.2) \qquad (\forall x \in X) \left(f_n(x * 1) \leq f_n(x), \ f_p(x * 1) \geq f_p(x) \right).$$

Example 5.3. Let X be the set of all positive real numbers. Define a binary operation $*$ in X by

$$(\forall x, y \in X) \left(x * y = \tfrac{y}{x} \right).$$

Then $(X, *, 1)$ is a CI-algebra (see [8]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ of X by

$$f = \{(1; -0.9, 0.8), (2^n; -0.8, 0.7), (a; -0.5, 0.4) \mid n \in \mathbb{N}, \ a \in X \setminus \{2^n \mid n \in \mathbb{N}\}\}.$$

Then $f = (X; f_n, f_p)$ is a bipolar fuzzy filter of X, but it is not closed since

$$f_n\left(2^3 * 1\right) = f_n\left(\tfrac{1}{8}\right) = -0.5 > -0.8 = f_n\left(2^3\right)$$

and/or

$$f_p\left(2^5 * 1\right) = f_p\left(\tfrac{1}{2^5}\right) = 0.4 < 0.7 = f_p\left(2^5\right).$$

Define a bipolar fuzzy set $g = (X; g_n, g_p)$ of X by

$$g = \{(1; -0.9, 0.8), (2^n; -0.8, 0.7), (a; -0.5, 0.3) \mid n \in \mathbb{Z} \setminus \{0\}, \ a \in X \setminus \{2^n \mid n \in \mathbb{Z}\}\}.$$

Then $f = (X; f_n, f_p)$ is a closed bipolar fuzzy filter of X.

Example 5.4. The set $B(X) := \{x \in X \mid x * 1 = 1\}$ is called the *BE-part* of X (see [8]). Using the notion of the BE-part of X, we make a closed bipolar fuzzy filter. The bipolar fuzzy set $f = (X; f_n, f_p)$ in X given by

$$f = \left\{ (1; -0.99, 0.88), (x; -0.87, 0.76), (y; -0.55, 0.44) \;\middle|\; \begin{array}{l} x \in B(X) \setminus \{1\}, \\ y \in X \setminus B(X) \end{array} \right\}$$

is a closed bipolar fuzzy filter of X.

Proposition 5.5. *Every bipolar fuzzy filter $f = (X; f_n, f_p)$ of X satisfies the condition (4.3).*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and thus

$$f_n(y) \leq \bigvee \{f_n(x * y), f_n(x)\} = \bigvee \{f_n(1), f_n(x)\} = f_n(x)$$

and

$$f_p(y) \geq \bigwedge \{f_p(x * y), f_p(x)\} = \bigwedge \{f_p(1), f_p(x)\} = f_p(x).$$

This completes the proof. □

Proposition 5.6. *Every bipolar fuzzy filter* $f = (X; f_n, f_p)$ *of* X *satisfies the following condition:*

$$(5.3) \qquad (\forall x, y, z \in X) \left(x \leq y * z \ \Rightarrow \left\{ \begin{array}{l} f_n(z) \leq \bigvee \{f_n(x), f_n(y)\}, \\ f_p(z) \geq \bigwedge \{f_p(x), f_p(y)\} \end{array} \right\} \right).$$

Proof. Let $x, y, z \in X$ be such that $x \leq y * z$. Then

$$f_n(y * z) \leq \bigvee \{f_n(x * (y * z)), f_n(x)\} = \bigvee \{f_n(1), f_n(x)\} = f_n(x)$$

and

$$f_p(y * z) \geq \bigwedge \{f_p(x * (y * z)), f_p(x)\} = \bigwedge \{f_p(1), f_p(x)\} = f_p(x).$$

Hence

$$f_n(z) \leq \bigvee \{f_n(y * z), f_n(y)\} \leq \bigvee \{f_n(x), f_n(y)\}$$

and

$$f_p(z) \geq \bigwedge \{f_p(y * z), f_p(y)\} \geq \bigwedge \{f_p(x), f_p(y)\}.$$

This completes the proof. □

The notions of a bipolar fuzzy filter and a bipolar fuzzy CI-subalgebra are mutually independent as seen in the following example.

Example 5.7. (1) The bipolar fuzzy filter $f = (X; f_n, f_p)$ in Example 5.3 is not a bipolar fuzzy CI-subalgebra of X since

$$f_n\left(2^4 * 2^2\right) = f_n\left(\tfrac{1}{4}\right) = -0.5 \nleq -0.8 = \bigvee \left\{f_n\left(2^4\right), f_n\left(2^4\right)\right\}$$

and/or

$$f_p\left(7 * 2^3\right) = f_p\left(\tfrac{8}{7}\right) = 0.4 \ngeq 0.7 = \bigvee \left\{f_n\left(7\right), f_n\left(2^3\right)\right\}.$$

(2) Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then $(X, *, 1)$ is a CI-algebra (see [3]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ by

$$f = \{(1; -0.9, 0.8), (a; -0.7, 0.6), (b; -0.6, 0.6), (c; -0.3, 0.2), (d; -0.3, 0.2)\}.$$

It is easy to verify that $f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X. Since

$$f_n(c) = -0.3 > -0.6 = \bigvee \{f_n(b * c), f_n(b)\}$$

and/or

$$f_p(c) = 0.2 < 0.6 = \bigvee \{f_n(a * c), f_n(a)\},$$

we know that $f = (X; f_n, f_p)$ is not a bipolar fuzzy filter of X.

We provide conditions for a bipolar fuzzy CI-subalgebra to be a bipolar fuzzy filter.

Theorem 5.8. *If a bipolar fuzzy CI-subalgebra $f = (X; f_n, f_p)$ of X satisfies the following condition:*

$$(5.4)\quad (\forall x, y \in X)\left(f_n(y) \le \bigvee \{f_n(x * y), f_n(x)\},\ f_p(y) \ge \bigwedge \{f_p(x * y), f_p(x)\}\right),$$

then $f = (X; f_n, f_p)$ is a bipolar fuzzy filter of X.

Proof. Combining (5.4) and Proposition 3.4, it is straightforward. □

Theorem 5.9. *If a bipolar fuzzy CI-subalgebra $f = (X; f_n, f_p)$ of X satisfies the following condition:*

$$(5.5)\quad (\forall x, y \in X)\,(f_n(x * y) \le f_n(y * x),\ f_p(x * y) \ge f_p(y * x)),$$

then $f = (X; f_n, f_p)$ is a bipolar fuzzy filter of X.

Proof. For any $x, y \in X$, we obtain

$$\begin{aligned} f_n(y) &= f_n(1 * y) \le f_n(y * 1) = f_n(y * (x * x)) = f_n(x * (y * x)) \\ &\le \bigvee \{f_n(x), f_n(y * x)\} \le \bigvee \{f_n(x), f_n(x * y)\} \end{aligned}$$

and

$$\begin{aligned} f_p(y) &= f_p(1 * y) \ge f_p(y * 1) = f_p(y * (x * x)) = f_p(x * (y * x)) \\ &\ge \bigwedge \{f_p(x), f_p(y * x)\} \ge \bigwedge \{f_p(x), f_p(x * y)\} \end{aligned}$$

by (a2), (5.5), (a1), (a3), (3.1), Combining Proposition 3.4, we conclude that $f = (X; f_n, f_p)$ is a bipolar fuzzy filter of X. □

Theorem 5.10. *For a bipolar fuzzy filter $f = (X; f_n, f_p)$ of X, the following are equivalent:*

(1) *$f = (X; f_n, f_p)$ is closed.*
(2) *$f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X.*

Proof. Let $f = (X; f_n, f_p)$ be a closed bipolar fuzzy filter of X. Using (5.1), (a3), (a1) and (5.2), we obtain

$$\begin{aligned} f_n(x * y) &\le \bigvee \{f_n(y * (x * y)), f_n(y)\} \\ &= \bigvee \{f_n(x * (y * y)), f_n(y)\} \\ &= \bigvee \{f_n(x * 1), f_n(y)\} \\ &\le \bigvee \{f_n(x), f_n(y)\} \end{aligned}$$

and

$$\begin{aligned} f_p(x * y) &\ge \bigwedge \{f_p(y * (x * y)), f_p(y)\} \\ &= \bigwedge \{f_p(x * (y * y)), f_p(y)\} \\ &= \bigwedge \{f_p(x * 1), f_p(y)\} \\ &\ge \bigwedge \{f_p(x), f_p(y)\} \end{aligned}$$

for all $x, y \in X$. Hence $f = (X; f_n, f_p)$ is a bipolar fuzzy CI-subalgebra of X.

Conversely, assume that $f = (X; f_n, f_p)$ is a bipolar fuzzy filter which is also a bipolar fuzzy CI-subalgebra of X. Using (3.1) and (5.1), we have

$$f_n(x * 1) \leq \bigvee \{f_n(x), f_n(1)\} = f_n(x)$$

and

$$f_p(x * 1) \geq \bigwedge \{f_p(x), f_p(1)\} = f_p(x)$$

for all $x \in X$. Therefore $f = (X; f_n, f_p)$ is a closed bipolar fuzzy filter of X. □

By the similar method to the proof of Theorem 4.7 we have the following theorem.

Theorem 5.11. *For a bipolar fuzzy set $f = (X; f_n, f_p)$ in X, the following are equivalent:*

(1) *$f = (X; f_n, f_p)$ is a (closed) bipolar fuzzy filter of X.*

(2) *$f = (X; f_n, f_p)$ satisfies the following assertions:*

(i) $(\forall s \in [-1, 0])\,(N(f_n; s) \neq \varnothing \;\Rightarrow\; N(f_n; s)$ is a (closed) filter of $X)$.

(ii) $(\forall t \in [0, 1])\,(P(f_p; t) \neq \varnothing \;\Rightarrow\; P(f_p; t)$ is a (closed) filter of $X)$.

Corollary 5.12. *If $f = (X; f_n, f_p)$ is a (closed) bipolar fuzzy filter of X, then the sets $N(f_n; f_n(a))$ and $P(f_p; f_p(a))$ are (closed) filters of X for all $a \in X$.*

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