Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 3, No. 2, (April 2012), pp. 321- 333
ISSN 2093–9310
http://www.afmi.or.kr

More on left fuzzy topological rings

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Received 17 December 2010; Accepted 19 March 2011

Abstract. In this paper, a few properties of left fuzzy topological rings (left ftr) are established and a category FTR is formed with objects as left ftr and arrows as the fuzzy continuous ring homomorphisms. The category $TopRng$ is seen to be a full subcategory of FTR. It is also observed how ring homomorphisms induce left ftr structure on rings. Besides, a characterization of a fuzzy disconnected space Y is obtained via the number of idempotents in the ring $FC(Y, Z)$ of fuzzy continuous functions from Y to a left ftr Z.

2010 AMS Classification: 03E72, 54C35, 54A40, 54A99

Keywords: Left fuzzy topological ring (ftr), *q-nbds* of 0_α, Category FTR, Fuzzy disconnected space.

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1. Introduction

The concept of left fuzzy topological ring (in short, left ftr) was introduced and studied in [2]. In [3] we have observed that the collection of all left ftr-valued fuzzy continuous functions from a fuzzy topological space Y to a left ftr Z, denoted by $FC(Y, Z)$, induces a ring structure from that of Z.

In this paper, we continue the study of left ftr and have obtained several significant results. In Section 2 of this paper, we mainly observe that the collection of all left fuzzy topological rings and fuzzy continuous homomorphisms constitute a category, which we call FTR. Moreover, the category $TopRng$ of topological rings and continuous homomorphisms form a full subcategory of FTR. In Section 3, we show how a ring homomorphism $f : R_1 \to R_2$ helps in inducing a left ftr-structure on

(1) the domain, if its co-domain is a left ftr,
(2) the range, if its domain is a left ftr.

Finally, in the last section, we mainly show that fuzzy disconnectedness of a fuzzy topological space Y can be characterized by the number of idempotent elements in the ring $FC(Y,Z)$. In what follows, unless it is stated explicitly, the rings are noncommutative and without unity.

2. Preliminaries

We begin with a few basic definitions and results as our prerequisites.
Let X be a non empty set and I be the closed interval $[0,1]$. A fuzzy set μ [8] on X is a function on X into I and the collection of all fuzzy sets on X is denoted by I^X. The support of a fuzzy set μ, denoted by *suppμ*, is the crisp set $\{x \in X : \mu(x) > 0\}$. A fuzzy set with a singleton as its support is called a fuzzy point, denoted by x_α, and defined as,

$$x_\alpha(z) = \begin{cases} \alpha, & \text{for } \text{ z = x} \\ 0, & \text{otherwise.} \end{cases}$$

A collection $\tau \subset I^X$ is called a *fuzzy topology* [1] on X if

(i) $0, 1 \in \tau$,
(ii) $\forall \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \bigcap \mu_i \in \tau$,
(iii) $\mu_\alpha \in \tau$, $\forall \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \bigcup \mu_\alpha \in \tau$.

Then (X, τ) is called a fuzzy topological space (fts, for short).

A fuzzy point x_α is said to be *q-coincident* with a fuzzy set U (in notation, $x_\alpha q U$) if $U(x)+\alpha > 1$. A fuzzy set A in a fts X is called a *nbd* (respectively, *q-nbd*) of a fuzzy point x_α if and only if there exists a fuzzy open set V in X such that $x_\alpha \in V \leq A$ (respectively, $x_\alpha q V \leq A$). A *nbd* (or, *q-nbd*) A is said to be fuzzy open if and only if A itself is fuzzy open (see [6]).

Let X and Y be two nonempty sets and $f : X \to Y$. If A and B are fuzzy sets on X and Y respectively then $f(A)$ and $f^{-1}(B)$ are respective fuzzy sets on Y and X, given by [8],

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

and $f^{-1}(B)(x) = B(f(x))$. Let f be a function from a fts X to a fts Y. Then f is fuzzy continuous iff $f^{-1}(U)$ is fuzzy open in X for each fuzzy open set U in Y (see [5]). Let f be a function from a fts X to a fts Y. Then f is fuzzy continuous iff for each fuzzy point x_α in X and each nbd V of $f(x_\alpha)$ in Y, there exists a nbd U of x_α in X such that $f(U) \leq V$. A function $f : X \to Y$ is called a fuzzy homeomorphism if f is bijective, fuzzy continuous and f^{-1} is fuzzy continuous (see [6]).

Definition 2.1 ([2])**.** Let (X, τ) and (Y, σ) be two fuzzy topological spaces. A function $f : X \times X \to Y$ is said to be fuzzy left continuous if f is fuzzy continuous with respect to the fuzzy topology on the product $X \times X$ generated by the collection $\{U \times V : U, V \in \tau\}$ where $(U \times V)(s,t) = \begin{cases} V(t), & \text{if } U(s) > 0 \\ 0, & \text{otherwise.} \end{cases}$

Definition 2.2 ([2])**.** Let R be a ring and τ be a fuzzy topology on R such that, for all $x, y \in R$,

(i) $(x, y) \mapsto x + y$ is fuzzy left continuous.
(ii) $(x, y) \mapsto x \cdot y$ is fuzzy left continuous.
(iii) $x \mapsto -x$ is fuzzy continuous.

The pair (R, τ) is called a left fuzzy topological ring, (in short, left ftr).

For non zero fuzzy sets U, V on R, the fuzzy sets $U+V$, UV and $-U$ are defined in [2] as follows :

$$(U+V)(x) = \sup_{U(s)>0} V(x-s),$$

$$(UV)(x) = \begin{cases} \sup\limits_{x=st,\ U(s)>0} V(t), & \text{if } \{(s,t) \in R \times R : \ st = x\} \neq \phi \\ 0, & \text{otherwise,} \end{cases}$$

$$(-U)(x) = U(-x)$$

for all $x \in R$.

Theorem 2.3 ([2])**.** *In a left ftr, for any fuzzy sets S_1, S_2, T_1 and T_2 with $S_1 \leq S_2$, $T_1 \leq T_2$, the following hold:*

(i) $S_1 + T_1 \leq S_2 + T_2$.
(ii) $S_1.T_1 \leq S_2.T_2$.
(iii) $x_\alpha S_1 \leq x_\alpha T_1$.
(iv) $S_1 x_\alpha \leq T_1 x_\alpha$ *for all* $x \in R$ *and* $\alpha \in (0, 1]$.

Theorem 2.4 ([2])**.** In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$,,

(i) V is fuzzy open if and only if $-V$ is fuzzy open.
(ii) V is a fuzzy *nbd* of 0_α if and only if $-V$ is a fuzzy *nbd* of 0_α.
(iii) V is fuzzy open (fuzzy closed) if and only if $x_\alpha + V$ is fuzzy open (respectively, fuzzy closed).

Definition 2.5 ([2])**.** A collection $\mathbb{B}$ of fuzzy *nbds* of x_α, for $0 < \alpha \leq 1$, is called a fundamental system of fuzzy nbds of x_α iff for any fuzzy *nbd* V of x_α, there exists $U \in \mathbb{B}$ such that $x_\alpha \leq U \leq V$.

Theorem 2.6 ([2])**.** *If R is a left fuzzy topological ring then there exists a fundamental system of fuzzy nbds $\mathbb{B}$ of 0_α $(0 < \alpha \leq 1)$, such that the following conditions hold:*

(i) *Each member of $\mathbb{B}$ is symmetric.*
(ii) $\forall\, U \in \mathbb{B}$, *there exists* $V \in \mathbb{B}$ *such that* $V + V \leq U$.
(iii) $\forall\, U \in \mathbb{B}$, *there exists* $V \in \mathbb{B}$ *such that* $V \cdot V \leq U$.
(iv) $\forall\, a \in R$, $\forall\, U \in \mathbb{B}$, *there exists* $V \in \mathbb{B}$ *such that* $a_\alpha V \leq U$ *and* $V a_\alpha \leq U$.

Conversely, given a ring R and a prefilter base $\mathbb{B}$ at 0_α satisfying conditions (i) – (iv)*, there exists a unique fuzzy topology τ on R such that (R, τ) forms a left fuzzy topological ring such that $\mathbb{B}$ forms a fundamental system of fuzzy nbds of 0_α.*

3. Left fuzzy topological rings and the category FTR

The following is a description of *q-nbd* of any fuzzy point x_α in terms of *q-nbd* of 0_α.

Theorem 3.1. *In a left ftr R, V is a fuzzy q-nbd of* 0_α *iff* $-V$ *is a fuzzy q-nbd of* 0_α.

Proof. Let V be a fuzzy *q-nbd* of 0_α. There exists fuzzy open set A such that $0_\alpha q A \leq V$, i.e., $\alpha + A(0) > 1$ and $A \leq V$. For all $x \in R$, $A(-x) \leq V(-x) \Rightarrow -A \leq -V$. Now, $0_\alpha(0) + (-A)(0) = \alpha + A(-0) > 1$. Hence, $0_\alpha q(-A)$ and $-A \leq -V$. Using Theorem 2.4, $-V$ is a fuzzy *q-nbd* of 0_α.

Conversely, let $-V$ be a fuzzy *q-nbd* of 0_α. There exist fuzzy open set A such that $0_\alpha q A \leq -V$. As above, $-A \leq V$ and $0_\alpha q(-A)$. i.e., V is a fuzzy *q-nbd* of 0_α. □

Theorem 3.2. *In a left ftr (R, τ), for each α with $0 < \alpha \leq 1$ and $x \in R$, if V is a fuzzy q-nbd (fuzzy open q-nbd or fuzzy closed q-nbd) of 0_α, then $x_\alpha + V$ is a fuzzy q-nbd (fuzzy open q-nbd or fuzzy closed q-nbd) of x_α. Moreover, any fuzzy q-nbd of x_α is precisely of the form $x_\alpha + V$, where V is a fuzzy q-nbd of 0_α.*

Proof. If V is a fuzzy *q-nbd* of 0_α, there is a fuzzy open set A such that $0_\alpha q A \leq V$, i.e., $\alpha + A(0) > 1$ and $A \leq V$. By Theorem 2.4, $x_\alpha + A$ is a fuzzy open set. By Theorem 2.3, $x_\alpha + A \leq x_\alpha + V$. We verify that $x_\alpha q(x_\alpha + A)$. Now,

$$\alpha + (x_\alpha + A)(x) = \alpha + \sup_{x_\alpha(s)>0} A(x - s) = \alpha + A(0) > 1.$$

This shows $(x_\alpha + A)$ is fuzzy open, such that $x_\alpha q(x_\alpha + A) \leq x_\alpha + V$. Hence, $x_\alpha + V$ is a fuzzy *q-nbd* of x_α. Suppose, V^* is any fuzzy *q-nbd* of x_α. Then there is a fuzzy open set U^* such that $x_\alpha q U^* \leq V^*$. i.e., $\alpha + U^*(x) > 1$ and $U^*(y) \leq V^*(y), \forall\ y$. Consider $U = (-x)_\alpha + U^*$ and $V = (-x)_\alpha + V^*$. Then U is a fuzzy open set. To show $0_\alpha q U \leq V$. Now,

$$0_\alpha(0) + U(0) = \alpha + [(-x)_\alpha + U^*](0) = \alpha + U^*(0) > 1.$$

So, $0_\alpha q U$. As $(-x)_\alpha \leq (-x)_\alpha$ and $U^* \leq V^*$, $U \leq V$. Hence, $0_\alpha q U$ and $U \leq V$. Again, $x_\alpha + U = x_\alpha + (-x)_\alpha + V^* = 0_\alpha + V^* = V^*$. This completes the proof. □

It is well known that a topological ring is "homogeneous", i.e., a function defined on it is continuous throughout its domain of definition whenever it is continuous at 0. The following theorem reflects a similar behaviour of left ftr.

Theorem 3.3. *Let (R, τ) and (S, σ) be left fuzzy topological rings and $f : R \to S$ be a ring homomorphism. Then $f : (R, \tau) \to (S, \sigma)$ is fuzzy continuous iff f is fuzzy continuous at 0_α, where $0 < \alpha \leq 1$.*

Proof. Let $f : (R, \tau) \to (S, \sigma)$ be fuzzy continuous. In particular f is fuzzy continuous at 0_α.

Conversely, let $f : (R, \tau) \to (S, \sigma)$ be fuzzy continuous at $0_\alpha, \forall\ \alpha \in (0, 1]$. For any fuzzy open set U containing $(f(0))_\alpha = f(0_\alpha)$ in S, there exist fuzzy open set V containing 0_α on R such that $f(V) \leq U$. Let x_α be fuzzy point on R and B be any fuzzy open set on S containing the fuzzy point $(f(x))_\alpha$ on S. Now, $x_\alpha + V$ is fuzzy open set containing x_α. As B is a fuzzy open set on S containing $(f(x))_\alpha$, we have

$B = (f(x))_\alpha + U$. To show $f((x)_\alpha + V) \leq (f(x))_\alpha + U$,

$$\begin{aligned}
(f((x)_\alpha + V))(z) &= \sup_{f(t)=z} (x_\alpha + V)(t) \\
&= \sup_{f(t)=z} V(t-x) \\
&= \sup_{f(x+p)=z} V(p) \\
&= \sup_{f(x)+f(p)=z} V(p) \\
&= \sup_{f(p)=z-f(x)} V(p) \\
&= f(V)(z - f(x)) \\
&\leq U(z - f(x)) \\
&= (f(x))_\alpha + U)(z).
\end{aligned}$$

Hence, $f((x)_\alpha + V) \leq (f(x))_\alpha + U$. □

Using the language of categories, we obtain the following:

Theorem 3.4. *The collection of all left ftr and fuzzy continuous homomorphisms form a category.*

Proof. Consider the collection of all left ftr as objects and for each pair of objects X,Y, the set of all arrows as the collection of fuzzy continuous homomorphisms from X to Y. Then it is easy to observe that taking composition of arrows as the usual composition of functions, one gets:

(i) composition of arrows is associative and
(ii) for each object X, $id : X \to X$ given by $id(x) = x$ is the identity arrow.

Consequently, it forms a category. □

Remark 3.5. The category mentioned in Theorem 3.4 will henceforth be referred to as FTR.

Remark 3.6. It is well known that corresponding to any topological space (X, τ), one can obtain the characteristic fuzzy topological space (X, τ_f).

Theorem 3.7. *If (X, τ) is a topological ring then (X, τ_f) is a left ftr.*

Proof. For a topological space (X, τ), it is known that (X, τ_f) is a fuzzy topological space. We have to show the following :

(i) $\forall\, x, y \in Z, (x, y) \mapsto x + y$ is left fuzzy continuous.
(ii) $\forall\, x, y \in Z, (x, y) \mapsto x \cdot y$ is left fuzzy continuous.
(iii) $\forall\, x \in Z, x \mapsto -x$ is fuzzy continuous.

We show that '+' is left fuzzy continuous. Let μ be a fuzzy open set on (X, τ_f) with $(x + y)_\alpha \leq \mu$. Then $\mu = \chi_A$ for some $A \in \tau$. Hence,

$$(x + y)_\alpha \leq \chi_A \Rightarrow \alpha \leq \chi_A(x + y) \Rightarrow x + y \in A.$$

Since (X, τ) is a topological ring, there exist open sets B, $C \in \tau$ such that $x \in B$, $y \in C$ and $B + C \subseteq A$. Then $x_\alpha \leq \chi_B$ and $y_\alpha \leq \chi_C$, where χ_B, $\chi_C \in \tau_f$. Now to complete the proof, we show $\chi_B + \chi_C \leq \chi_A = \mu$. Now, $\forall\ z \in X$,

$$\begin{aligned}(\chi_B + \chi_C)(z) &= \sup_{\chi_B(t)>0} \chi_C(z-t) \\ &= \sup_{t\in B} \chi_C(z-t) \\ &= \begin{cases} 1, & \text{for } \ t \in B \text{ and } z-t \in C \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{for } z \in B + C \\ 0, & \text{otherwise} \end{cases} \\ &= \chi_{B+C}(z) \leq \chi_A(z).\end{aligned}$$

Hence, '+' is left fuzzy continuous. Proceeding similarly, (ii) and (iii) can be obtained. Hence, (X, τ_f) is a left ftr. □

Theorem 3.8. *If f is a continuous homomorphism from a topological ring (X, τ) to a topological ring (Y, σ) then $f : (X, \tau_f) \to (Y, \sigma_f)$ is a fuzzy continuous homomorphism between the corresponding left ftr.*

Proof. Straightforward. □

We express the above findings in terms of categories as follows:

Theorem 3.9. *If $TopRng$ is the category of topological rings and continuous homomorphisms, then $TopRng$ is a full subcategory of FTR.*

Proof. In the light of Theorems 3.7 and 3.8, any object of $TopRng$ can be viewed as an object of FTR and any morphism between two objects of $TopRng$ is a morphism between the corresponding objects of FTR. Hence, $TopRng$ is a subcategory of FTR. Now, consider the inclusion functor $i : TopRng \to FTR$ that sends (X, τ) to its characteristic fts (X, τ_f) and $f : (X, \tau) \to (Y, \sigma)$ to $f : (X, \tau_f) \to (Y, \sigma_f)$. To show that the functor i is full. Let (X, τ) and (Y, σ) be two objects in $TopRng$ and $f^* : (X, \tau_f) \to (Y, \sigma_f)$ a morphism in FTR. If $U \in \sigma$ then $\chi_U \in \sigma_f$ and so, ${f^*}^{-1}(\chi_U) = \chi_{{f^*}^{-1}(U)} \in \tau_f$ which in turn gives ${f^*}^{-1}(U) \in \tau$. Hence, there exist $f^* : (X, \tau) \to (Y, \sigma)$ a morphism in $TopRng$ such that $i(f^*) = f^*$. i.e., i is full. Consequently, $TopRng$ is a full subcategory of FTR. □

Theorem 3.10. *Let (Z, σ) be a left ftr then $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring.*

Proof. We need to show the following:

(i) $\forall\ x, y \in Z, (x, y) \mapsto x + y$ is continuous.
(ii) $\forall\ x, y \in Z, (x, y) \mapsto x \cdot y$ is continuous.
(iii) $\forall\ x \in Z, x \mapsto -x$ is continuous.

Let $x, y \in Z$ and A be any open set in $(Z, i_\alpha(\sigma))$ containing $x+y$. There exist fuzzy open set μ in (Z, σ) such that $\mu^\alpha = A$. So, $(x + y)_\alpha < \mu$. As, (Z, σ) is a left ftr, there exist fuzzy open sets U and V such that $x_\alpha < U$, $y_\alpha < V$ and $U + V \leq \mu$.

Then $x \in U^\alpha$ and $y \in V^\alpha$. We shall show that $U^\alpha + V^\alpha \subseteq A$. Let $z \in U^\alpha + V^\alpha$. Then $z = s + t$ where $s \in U^\alpha$ and $t \in V^\alpha$. i.e., $U(s) > \alpha$ and $V(t) > \alpha$. Now,

$$\begin{aligned}(U+V)(z) &= \sup_{U(p)>0} V(z-p) \\ &\geq V(t), \text{ where } U(s) > 0 \text{ and } z = s+t \\ &> \alpha\end{aligned}$$

So, $\mu(z) > \alpha$, $z \in \mu^\alpha = A$. Hence, $U^\alpha + V^\alpha \subseteq A$. This proves '+' is continuous. The proof for '$\cdot$' is continuous is similar and hence omitted. Now, we shall prove that $x \mapsto -x$ is continuous. Let $x \in Z$ and A be an open set on $(Z, i_\alpha(\sigma))$ containing $-x$. There is a fuzzy open set μ on (Z, σ) such that $\mu^\alpha = A$. So, $(-x) \in \mu^\alpha \Rightarrow (-x)_\alpha < \mu$. As (Z, σ) is left ftr, there exist fuzzy open set U containing x_α such that $x_\alpha < U$ and $-U \leq \mu$. We shall show $(-U)^\alpha \subseteq A$. Let $z \in (-U)^\alpha \Rightarrow \mu(z) \geq -U(z) > \alpha$. So, $z \in \mu^\alpha$. Hence, $(-U)^\alpha \subseteq A$. □

Theorem 3.11 ([7]). *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous for each $\alpha \in I_1$, where (X, τ), (Y, σ) are fts.*

Theorem 3.12. *A function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous homomorphism iff $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is continuous homomorphism for each $\alpha \in I_1$, where $(X, \tau), (Y, \sigma)$ are left ftr.*

Proof. Immediate from Theorem 3.11. □

In view of Theorems 3.10 and 3.12, we get:

Theorem 3.13. *For each $\alpha \in I_1$, $i_\alpha : FTR \to TopRng$ is a covariant functor.*

4. Left fuzzy topological rings induced by ring homomomorphisms

Definition 4.1. A fuzzy topology that makes a ring left ftr is called a fuzzy topology *compatible* with the ring structure.

Theorem 4.2. *Let (R, τ) be a fuzzy topological ring. If $h : H \to R$ is a ring homomorphism from any ring H to R then h induces a unique compatible fuzzy topology on H that makes h fuzzy continuous.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy nbds of 0_α in R. Then it is enough to show that $h^{-1}(\mathbb{B})$ determines a unique fuzzy topology on H such that $h^{-1}(\mathbb{B})$ forms a fundamental system of fuzzy nbds of 0_α in H. It is clear that $h^{-1}(\mathbb{B})$ is a prefilterbase at 0_α in H. In view of Theorem 2.6, it is now to verify that $h^{-1}(\mathbb{B})$ satisfies the conditions (i) - (iv) of Theorem 2.6.

(i) Any element of $h^{-1}(\mathbb{B})$ is of the form $h^{-1}(V)$, for some $V \in \mathbb{B}$. Now, $\forall x \in H$

$$\begin{aligned}(-h^{-1}(V))(x) &= h^{-1}(V)(-x) = V(h(-x)) \\ &= V(-h(x)) \text{ (since } h \text{ is a homomorphism)} \\ &= (-V)(h(x)) \\ &= V(h(x)) \text{ (since } V = -V) \\ &= h^{-1}(V)(x)\end{aligned}$$

Hence, $-h^{-1}(V) = h^{-1}(V)$, showing that each member of $h^{-1}(\mathbb{B})$ is symmetric.

(ii) Let $h^{-1}(U) \in h^{-1}(\mathbb{B})$, for some $U \in \mathbb{B}$. Then as $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$, such that $V + V \leq U$. For any $z \in H$,

$$\begin{aligned}(h^{-1}(V) + h^{-1}(V))(z) &= \sup_{h^{-1}(V)(x)>0} h^{-1}(V)(z-x) = \sup_{V(h(x))>0} h^{-1}(V)(z-x) \\ &= \sup_{V(h(x))>0} V(h(z-x)) = \sup_{V(h(x))>0} V(h(z)-h(x)) \\ &\leq \sup_{V(y)>0} V(h(z)-y) = (V+V)(h(z)) \\ &\leq U(h(z)) = h^{-1}(U)(z).\end{aligned}$$

Hence, there exists $h^{-1}(V) \in h^{-1}(\mathbb{B})$, such that $h^{-1}(V) + h^{-1}(V) \leq h^{-1}(U)$.

(iii) Let $h^{-1}(U) \in h^{-1}(\mathbb{B})$, for some $U \in \mathbb{B}$. Then as $U \in \mathbb{B}$, there exists $V \in \mathbb{B}$, such that $V \cdot V \leq U$. For any $z \in H$,

$$\begin{aligned}(h^{-1}(V) \cdot h^{-1}(V))(z) &= \sup_{\{z=xy,\ h^{-1}(V)(x)>0\}} h^{-1}(V)(y) \\ &= \sup_{\{z=xy,\ V(h(x))>0\}} h^{-1}(V)(y) \\ &= \sup_{\{z=xy,\ V(h(x))>0\}} V(h(y)) \\ &= \sup_{\{h(z)=h(x)h(y),\ V(h(x))>0\}} V(h(y)) \\ &\leq \sup_{\{h(z)=st,\ V(s)>0\}} V(t) \\ &= (V \cdot V)(h(z)) \leq U(h(z)) = h^{-1}(U)(z).\end{aligned}$$

So, there exists $h^{-1}(V) \in h^{-1}(\mathbb{B})$, such that $h^{-1}(V) \cdot h^{-1}(V) \leq h^{-1}(U)$.

(iv) Let $a \in H$ and $h^{-1}(U) \in h^{-1}(\mathbb{B})$. Then $h(a) \in R$ and $U \in \mathbb{B}$ so that there exists $V \in \mathbb{B}$ such that $h(a)_\alpha \cdot V \leq U$ and $V \cdot h(a)_\alpha \leq U$.

$$\begin{aligned}(a_\alpha \cdot h^{-1}(V))(z) &= \begin{cases} \sup_{z=at} h^{-1}(V)(t), & if\ \exists t\ s.t.\ z = at \\ 0, & otherwise \end{cases} \\ &= \begin{cases} \sup_{h(z)=h(a)h(t)} V(h(t)), & if\ \exists t\ s.t.\ z = at \\ 0, & otherwise \end{cases} \\ &\leq \begin{cases} \sup_{h(z)=h(a)x} V(x), & if\ \exists x\ s.t.\ h(z) = h(a)x \\ 0, & otherwise \end{cases} \\ &= h(a)_\alpha \cdot V(h(z)) \\ &\leq U(h(z)) \\ &= h^{-1}(U)(z).\end{aligned}$$

Similarly, one can verify that $(h^{-1}(V) \cdot a_\alpha)(z) \leq h^{-1}(U)(z)$, for all $z \in H$. Consequently, $h^{-1}(\mathbb{B})$ satisfies the conditions (i) - (iv) of Theorem 2.6 and the result follows. In view of Theorem 3.3, the fuzzy continuity of h is immediate. $\square$

Corollary 4.3. *Any subring of a left ftr is a left ftr.*

Proof. Let (R, τ) be a left fuzzy topological ring and H be a subring of R. If $\mathbb{B}$ is a fundamental system of fuzzy nbds of 0_α in R, and $h : H \to R$ is the inclusion homomorphism (i.e., $h(x) = x$, $\forall x \in H$) then by the above theorem $h^{-1}(\mathbb{B})$ determines a unique compatible fuzzy topology on H such that $h^{-1}(\mathbb{B})$ forms a fundamental system of fuzzy nbds of 0_α in H. □

Corollary 4.4. *Let R be a left ftr and $h : R \to S$ a ring homomorphism onto S. If I is an ideal of R containing $ker(h)$, then $\overline{h} : R/I \to S$ induces a compatible fuzzy topology on R/I that makes $\overline{h}$ fuzzy continuous.*

Proof. Follows from the theorem as $\overline{h} : R/I \to S$ is a ring homomorphism. □

The following theorem shows that the homomorphic image of a left ftr is a left ftr.

Theorem 4.5. *Let (H, τ) be a left ftr. If $h : H \to R$ is a ring homomorphism from H onto any ring R, then h induces a fuzzy topology compatible with the ring structure on R that makes h fuzzy continuous.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy nbds of 0_α in H. In view of Theorem 2.6, it is enough to show that $h(\mathbb{B})$ is a fundamental system of fuzzy nbds of 0_α in R.

(i) For any $U \in \mathbb{B}$, $U = -U$ and so, $h(U) = h(-U)$. Now, $\forall z \in R$

$$\begin{aligned} h(-U)(z) &= sup\{(-U)(t) \ : \ h(t) = z\} \\ &= sup\{U(-t) \ : \ h(t) = z\} \\ &= sup\{U(-t) \ : \ h(-t) = -z\} \\ &= h(U)(-z) \\ &= -h(U)(z). \end{aligned}$$

Consequently, $h(U) = -h(U)$.

(ii) If $h(U) \in h(\mathbb{B})$, then $U \in \mathbb{B}$ and so, there exists $V \in \mathbb{B}$, such that $V + V \leq U$. For any $z \in R$,

$$\begin{aligned} (h(V) + h(V))(z) &= \sup_{h(V)(x)>0} h(V)(z - x) \\ &= \sup_{h(V)(x)>0} \sup_{h(y)=z-x} V(y) \\ &= \sup_{\{\sup_{h(x')=x} V(x')>0\}} \sup_{h(y)=z-x} V(y) \\ &= \sup_{h(y)=z-h(x')} \sup_{\{x=h(x'),\ V(x')>0\}} V(y + x' - x') \\ &= \sup_{h(y+x')=z} \sup_{x=h(x'),\ V(x')>0} V(y + x' - x') \\ &\leq \sup_{h(t)=z} \sup_{\{x':V(x')>0\}} V(t - x') \\ &= \sup_{h(t)=z} (V + V)(t) \\ &= h(V + V)(z) \\ &\leq h(U)(z). \end{aligned}$$

Proceeding similarly, one can show that condition (iii) also holds for $h(\mathbb{B})$.

(iv) Let $b \in R$ and $h(U) \in h(\mathbb{B})$. Then there exists $a \in H$ with $h(a) = b$. So, there exists $V \in \mathbb{B}$ such that $a_\alpha V \leq U$ and $V a_\alpha \leq U$. It is to show that $b_\alpha h(V) \leq h(U)$ and $h(V) b_\alpha \leq h(U)$. For any $z \in R$, it is easy to see that

$$b_\alpha h(V)(z) = h(a_\alpha)h(V)(z) \leq h(a_\alpha V)(z) \leq h(U)(z),$$

and also,

$$(h(V)b_\alpha)(z) = h(V)h(a_\alpha)(z) \leq h(V a_\alpha)(z) \leq h(U)(z).$$

Hence, all the conditions (i) - (iv) are satisfied proving $h(\mathbb{B})$ to be a fundamental system of fuzzy nbds of 0_α in R. Let U be any fuzzy nbd of 0_α in R. By definition of fundamental system of fuzzy nbds, there exists some $B \in \mathbb{B}$, such that $0_\alpha \leq h(B) \leq U$. As $B \leq h^{-1}(h(B)) \leq h^{-1}(U)$ and $0_\alpha \in B$ it follows that $h^{-1}(U)$ is a fuzzy nbd of 0_α in R. Hence, in view of Theorem 3.3 h is fuzzy continuous. □

Corollary 4.6. *Let R be a ring and I be an ideal of R. The ring epimorphism $k : R \to R/I$ given by $k(x) = x + I$ induces a fuzzy topology compatible with the ring R/I that makes k fuzzy continuous.*

Proof. Follows from the theorem. □

Theorem 4.7. *Let R be a left ftr. If S is a left ftr induced from a ring homomorphism $h : R \to S$ from R onto S and $I = ker(h)$, then the fuzzy topology compatible with R/I induced from $\overline{h} : R/I \to S$ and the fuzzy topology compatible with R/I induced from $k : R \to R/I$ are same.*

Proof. Let $\mathbb{B}$ be a fundamental system of fuzzy nbds of 0_α in R. It follows from the previous theorems that $\{k(V) : V \in \mathbb{B}\}$ is a fundamental system of fuzzy nbds of 0_α for the compatible fuzzy topology on R/I induced by k and $\{\overline{h}^{-1}(h(V)) : V \in \mathbb{B}\}$ is a fundamental system of fuzzy nbds of 0_α for the compatible fuzzy topology on R/I induced by $\overline{h}$. Since, $\overline{h} \circ k = h$, we have

$$\begin{aligned}
\overline{h}^{-1}(h(V))(x+I) &= h(V)(\overline{h}(x+I)) \\
&= h(V)(\overline{h} \circ k)(x) \\
&= h(V)(h(x)) \\
&= \sup_{h(y)=h(x)} V(y) \\
&= \sup_{h(y-x)=0} V(y) \\
&= \sup_{y-x \in Ker h = I} V(y) \\
&= \sup_{y+I=x+I} V(y) \\
&= \sup_{k(y)=x+I} V(y) \\
&= k(V)(x+I).
\end{aligned}$$

Hence, both the fundamental systems are identical leading to the same compatible topology on R/I. □

5. Application

Definition 5.1 ([4]). A fts (X, τ) is fuzzy disconnected if there exist fuzzy sets U and V such that $U \vee V = 1, U \not{q} \overline{V}$ and $V \not{q} \overline{U}$.

Lemma 5.2. *If (X, τ) is fuzzy disconnected fts then there exist fuzzy closed sets C and D such that $C \vee D = 1$ and $C \not{q} D$.*

Proof. Let (X, τ) be fuzzy disconnected. There exist fuzzy sets A and B such that $A \vee B = 1, A \not{q} \overline{B}$ and $B \not{q} \overline{A}$, i.e., $\forall\ y \in Y,\ A(y) \vee B(y) = 1,\ A(y) + \overline{B}(y) \leq 1$ and $\overline{A}(y) + B(y) \leq 1$. Hence, for each $y \in Y$ we have either $[A(y) = 1$ and $B(y) = 0]$ or $[A(y) = 0$ and $B(y) = 1]$. We shall prove that the fuzzy closed sets $1 - int(clA)$ and $1 - int(clB)$ are the required sets. Now, $\forall\ y \in Y$ if $A(y) = 1$ then $A(y) \leq 1 - \overline{B}(y) \leq 1 - int(clB)(y)$, i.e., $1 - int(clB) = 1$ and if $B(y) = 1$ then similarly, we have $1 - int(clA) = 1$, showing $(1 - int(clA)) \vee (1 - int(clB)) = 1$ and $(1 - int(clA)) \not{q} (1 - int(clB))$. □

Theorem 5.3. *If a fts (X, τ) is fuzzy disconnected then $\forall\ \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected.*

Proof. Let (X, τ) be fuzzy disconnected. By Lemma 5.2, there exist fuzzy open sets A and B on X such that $A \vee B = 1$ and $A \not{q} B$. Hence, for each $x \in X$ we have either $[A(x) = 1$ and $B(x) = 0]$ or $[A(x) = 0$ and $B(x) = 1]$. Now, $\forall\ \alpha \in I_1,\ A^\alpha$ and B^α are open in $(X, i_\alpha(\tau))$ with $A^\alpha \cup B^\alpha = (A \vee B)^\alpha = X$. If possible let $z \in A^\alpha \cap B^\alpha$. Then, $A(z) > \alpha$ and $B(z) > \alpha$, which is not possible. Hence, $A^\alpha \cap B^\alpha = \Phi$ and so, $(X, i_\alpha(\tau))$ is disconnected. □

Lemma 5.4. *Let $C(Y, Z)$ denote the ring of continuous functions from a topological space $(Y, i_\alpha(\tau))$ to a topological ring $(Z, i_\alpha(\sigma))$, for each $\alpha \in I_1$. If Y is disconnected then there exist $f \in C(Y, Z)$ such that $f \neq 0, 1$ and $f^2 = f$.*

Proof. If Y is disconnected, there exist nonempty disjoint closed sets $A,\ B$ such that $Y = A \cup B$. Defining $f : Y \to Z$ by $f(y) = \begin{cases} 1, & if\ y \in A \\ 0 & if\ y \in B, \end{cases}$ we get the desired non trivial idempotent. □

Theorem 5.5. *Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0, 1]$. If Y is any fully stratified fts such that the ring $FC(Y, Z)$ has some nontrivial idempotent element then Y is fuzzy disconnected.*

Proof. Let $f \in FC(Y, Z)$ be such that $f^2 = f$ and $f \neq 0, 1$. To show Y is fuzzy disconnected. $\forall\, y \in Y, f^2(y) = f(y) \Rightarrow f(y)(1 - f(y)) = 0$. As Z has no zero divisor, for each $y \in Y$ we have, $f(y) = 0$ or $f(y) = 1$. 0_α for all $\alpha \in I_1$ is fuzzy closed in Z. Consider, $\alpha = 1$. As $1_\alpha = 0_\alpha + 1_\alpha$ and 0_α is fuzzy closed, using Theorem 2.4, 1_α is fuzzy closed in Z. f being fuzzy continuous, $f^{-1}(0_\alpha)$ and $f^{-1}(1_\alpha)$ are fuzzy closed in Y. Now,

$$\begin{aligned} (f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha))(y) &= \sup\{f^{-1}(0_\alpha)(y), f^{-1}(1_\alpha)(y)\} \\ &= \sup\{(0_\alpha)(f(y)), (1_\alpha)f((y))\} \\ &= \alpha \\ &= 1 \end{aligned}$$

Hence, $f^{-1}(0_\alpha) \vee f^{-1}(1_\alpha) = 1_Y$. Clearly, for all $x \in Y$, $f^{-1}(0_\alpha)(x) + f^{-1}(1_\alpha)(x) = \alpha = 1$, i.e., $f^{-1}(0_\alpha) \not{q} f^{-1}(1_\alpha)$. This shows that Y is disconnected. □

Theorem 5.6. *Let Z be any left ftr and Y be any fuzzy disconnected fts then the ring $FC(Y, Z)$ has some nontrivial idempotent element.*

Proof. Let Y be fuzzy disconnected. By Theorem 5.3, $\forall\ \alpha \in I_1, (X, i_\alpha(\tau))$ is disconnected. By Theorem 3.10, $\forall\ \alpha \in I_1, (Z, i_\alpha(\sigma))$ is a topological ring. Now, by Lemma 5.4, there exist a continuous function $f : (Y, i_\alpha(\tau)) \to (Z, i_\alpha(\sigma))$ such that $f \neq 0, 1$ and $f^2 = f$. By Theorem 3.11, $f : (Y, \tau) \to (Z, \sigma)$ is fuzzy continuous such that $f \neq 0, 1$ and $f^2 = f$. □

Combining Theorems 5.5 and 5.6, we have:

Theorem 5.7. *Let Z be a left ftr with 1 and without zero divisor such that 0_α is fuzzy closed for each $\alpha \in (0, 1]$. If Y is any fully stratified fts, then the ring $FC(Y, Z)$ has exactly two idempotents iff Y is fuzzy connected.*

Theorem 5.8. *Let X and Y be two fully stratified fts and $f : X \to Y$ be a fuzzy continuous function. For any left ftr Z, $f^* : FC(Y, Z) \to FC(X, Z)$ given by $f^*(g) = g \circ f$ is a ring homomorphism.*

Proof. Straightforward. □

Theorem 5.9. *If X is any fully stratified fts and Z_1, Z_2 are left ftr, then every fuzzy continuous ring homomorphism $\phi : Z_1 \to Z_2$ induces a ring homomorphism $\hat{\phi} : FC(X, Z_1) \to FC(X, Z_2)$ given by $\hat{\phi}(f) = \phi \circ f$.*

Proof. Immediate. □

Reframing the results discussed above in the language of categories, we obtain the following functors:

Theorem 5.10. *If FTS is the category of fully stratified fuzzy topological spaces and fuzzy continuous functions; Rng is the category of all rings and ring homomorphisms, then*

(i) *$FC(-, Z) : FTS \to Rng$ given by $Y \to FC(Y, Z)$ is a contravariant functor, for each left ftr Z.*
(ii) *$FC(X, -) : Rng \to Rng$ given by $Z \to FC(X, Z)$ is a covariant functor, for each fully stratified fuzzy topological space X.*

Acknowledgements. The authors are grateful to the referees for their valuable comments.

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