Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 3, No. 2, (April 2012), pp. 305- 311
ISSN 2093–9310
http://www.afmi.or.kr

A note on fuzzy soft topological spaces

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Received 30 June 2011; Revised 9 August 2011; Accepted 10 October 2011

Abstract. The aim of this paper is to construct a topology on a fuzzy soft set. Also the concepts of fuzzy soft base, fuzzy soft subbase are introduced here and established some important theorems.

2010 AMS Classification: 54A40, 06D72

Keywords: Fuzzy soft set, Fuzzy soft topology, Fuzzy soft base, Fuzzy soft subbase.

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1. Introduction

In 1999, D. Molodtsov [7] introduced the soft set theory to solve complicated problems in economics, engineering, and environment. He has shown several applications of this theory in solving many practical problems. There are many theories like theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of rough sets etc. which can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. The theory of probabilities can deal only with possibility. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets, developed by Zadeh [9] in 1965. But there exists a difficulty, how to set the membership function in each particular cases. Also the theory of intuitionistic fuzzy set is more generalized concept than the theory of fuzzy set, but there have same difficulties. But the soft set theory is free from above difficulties. In 2001, P. K. Maji, A. R. Roy, R. Biswas [4], [5] also initiate the more generalized concept of fuzzy soft sets which is a combination of fuzzy set and soft set. Then many researcher have applied this concept on group theory [1], decision making problems ([3], [6]), relations [2] etc.

In this paper, we try to apply this concept on topological spaces. For this reason at first we redefine some definitions on fuzzy soft set in another form. Then we define a fuzzy soft topology, fuzzy soft base and fuzzy soft subbase and also here we established some important theorems related to this spaces.

2. Preliminaries

This section contains some basic definitions which will be needed in the sequel.

Definition 2.1 ([9]). A fuzzy set A of a non-empty set X is characterized by a membership function μ_A which associates each point of X to a real number in the interval $[0, 1]$. With the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

Definition 2.2 ([9]). Two fuzzy set A and B are equal, written as $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

Definition 2.3 ([9]). A fuzzy set A is contained in a fuzzy set B if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$.

Definition 2.4 ([9]). The union of two fuzzy sets A and B with respective membership functions μ_A and μ_B is a fuzzy set C, written as $C = A \cup B$ whose membership function is related to those of A and B by $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.

Definition 2.5 ([9]). The intersection of two fuzzy sets A and B with respective membership functions μ_A and μ_B is a fuzzy set C, written as $C = A \cap B$ whose membership function is related to those of A and B by
$\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.

Definition 2.6 ([7]). Let U be the initial universe set and E be the set of parameters. A pair $(F\,,\,A)$ is called a soft set over U where F is a mapping given by $F\,:\,A \rightarrow P(U)$ and $A \subseteq E$.

In other words, the soft set is a parameterized family of subsets of the set U. For $e \in E$, $F(e)$ may be considered as the set of $e-$elements of the soft set $(F\,,\,A)$, or as the set of $e-$approximate elements of the soft set.

Definition 2.7 ([4]). Let U be an initial universe set and let E be a set of parameters. Let I^U denotes the collection of all fuzzy subsets of U and $A \subseteq E$. A pair $(F\,,\,A)$ is called a fuzzy soft set over U, if F is a mapping given by $F : A \longrightarrow I^U$.

Definition 2.8 ([4]). A fuzzy soft set (F, A) over U is said to be null fuzzy soft set denoted by Φ, if for all $e \in E$, $F(e)$ is the null fuzzy set $\overline{0}$ of U, where $\overline{0}(x) = 0$ for all $x \in U$.

Definition 2.9 ([4]). A fuzzy soft set (F, A) over U is said to be absolute fuzzy soft set denoted by $\widetilde{E}$, if for all $e \in E$, $F(e)$ is the fuzzy set $\overline{1}$ of U where $\overline{1}(x) = 1$ for all $x \in U$.

Definition 2.10 ([4]). A fuzzy soft set (F, A) is said to be a fuzzy soft subset of a fuzzy soft set (G, B) over a common universe U if $A \subseteq B$ and $F(e) \subseteq G(e)$, for all $e \in A$.

Definition 2.11 ([4]). Two fuzzy soft sets (F, A) and (G, B) over a common universe U are said to be fuzzy soft equal if (F, A) is a fuzzy soft subset of (G, B) and (G, B) is a fuzzy soft subset of (F, A).

Definition 2.12 ([4]). The union of two fuzzy soft sets (F, A) and (G, B) over the common universe U is the fuzzy soft set $(H, C) = (F, A) \sqcup (G, B)$, where $C = A \cup B$ and for all $e \in E$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Definition 2.13 ([4]). If (F, A) and (G, B) be two fuzzy soft set, then the intersection of (F, A) and (G, B) is a fuzzy soft set $(F, A) \sqcap (G, B) = (H, C)$, Where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

3. Fuzzy soft topological space

In the paper [8], the definition of fuzzy soft topology was introduced over a subset of the initial universe set. Thereafter the authors introduced the neighborhood and interior of a fuzzy soft set in a fuzzy soft topological space and a few properties, one of them, open sets are characterized by the interior of a fuzzy soft set. Though the fuzzy soft basis for such spaces was introduced here, but open sets are not characterized by basic open sets and there are no necessary and sufficient conditions for a subfamily to be a fuzzy soft basis for such space. But, in our paper, we have introduced the fuzzy soft topology over the initial universe set and then mainly we deal with bases and subbases for such spaces. Here, different conditions are given for a subfamily of fuzzy soft sets to be a fuzzy soft base or fuzzy soft subbase.

In this section, for the sake of simplicity, we restate a few basic definitions e.g., Definitions 2.7 and 2.8 etc. in the following form to study a few results of fuzzy soft topological spaces properly. Throughout this paper, U refers to an initial universe, E is the set of all parameters for U and I^U denotes the set of all fuzzy subset of U and also by "(U, E)", we mean "the universal set U and the parameter set E".

Definition 3.1. Let $A \subseteq E$. Then the mapping $F_A : E \to I^U$, defined by $F_A(e) = \mu^e_{F_A}$(a fuzzy subset of U), is called fuzzy soft set over (U, E), where $\mu^e_{F_A} = \overline{0}$ if $e \in E \setminus A$ and $\mu^e_{F_A} \neq \overline{0}$ if $e \in A$.

The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.

Definition 3.2. The fuzzy soft set $F_\phi \in FS(U, E)$ is called null fuzzy soft set and it is denoted by Φ. Here $F_\phi(e) = \overline{0}$ for every $e \in E$.

Definition 3.3. Let $F_E \in FS(U, E)$ and $F_E(e) = \overline{1}$ for all $e \in E$. Then F_E is called absolute fuzzy soft set. It is denoted by $\widetilde{E}$.

Definition 3.4. Let $F_A, G_B \in FS(U, E)$. If $F_A(e) \subseteq G_B(e)$ for all $e \in E$, i.e., if $\mu^e_{F_A} \subseteq \mu^e_{G_B}$ for all $e \in E$, i.e., if $\mu^e_{F_A}(x) \leq \mu^e_{G_B}(x)$ for all $x \in U$ and for all $e \in E$, then F_A is said to be fuzzy soft subset of G_B, denoted by $F_A \sqsubseteq G_B$.

Definition 3.5. Let $F_A, G_B \in FS(U, E)$. Then the union of F_A and G_B is also a fuzzy soft set H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cup \mu^e_{G_B}$ for all $e \in E$, where $C = A \cup B$. Here we write $H_C = F_A \sqcup G_B$.

Following the arbitrary union of fuzzy subsets and the union of two fuzzy soft sets, the definition of arbitrary union of fuzzy soft sets can be described in the similarly fashion.

Definition 3.6. Let F_A, $G_B \in FS(U, E)$. Then the intersection of F_A and G_B is also a fuzzy soft set H_C, defined by $H_C(e) = \mu^e_{H_C} = \mu^e_{F_A} \cap \mu^e_{G_B}$ for all $e \in E$, where $C = A \cap B$. Here we write $H_C = F_A \sqcap G_B$.

Definition 3.7. A fuzzy soft topology $\mathcal{T}$ on (U, E) is a family of fuzzy soft sets over (U, E) satisfying the following properties

1. $\Phi, \widetilde{E} \in \mathcal{T}$.
2. If F_A, $G_B \in \mathcal{T}$ then $F_A \sqcap G_B \in \mathcal{T}$.
3. If $F^{\alpha}_{A_\alpha} \in \mathcal{T}$ for all $\alpha \in \Lambda$, an index set, then $\sqcup_{\alpha \in \Lambda} F^{\alpha}_{A_\alpha} \in \mathcal{T}$.

Definition 3.8. If τ is a fuzzy soft topology on (U, E), the triple (U, E, τ) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (U, E, τ).

Definition 3.9. A collection β of some fuzzy soft sets over (U, E) is called a fuzzy soft open base or simply a base for some fuzzy soft topology on (U, E) if the following conditions hold:

1. $\Phi \in \beta$.
2. $\sqcup \beta = \widetilde{E}$ i.e. for each $e \in E$ and $x \in U$, there exists $F_A \in \beta$ such that $\mu^e_{F_A}(x) = 1$.
3. If F_A, $G_B \in \beta$ then for each $e \in E$ and $x \in U$, there exists $H_C \in \beta$ such that $H_C \sqsubseteq F_A \sqcap G_B$ and $\mu^e_{H_C}(x) = \min\{\mu^e_{F_A}(x), \mu^e_{G_B}(x)\}$, where $C \subseteq A \cap B$.

Theorem 3.10. *Let β be a fuzzy soft base for a fuzzy soft topology on (U, E). Suppose $\mathcal{T}_\beta$ consists of those fuzzy soft set G_A over (U, E) for which corresponding to each $e \in E$ and $x \in U$, there exists $F_B \in \beta$ such that $F_B \sqsubseteq G_A$ and $\mu^e_{F_B}(x) = \mu^e_{G_A}(x)$, where $B \subseteq A$. Then $\mathcal{T}_\beta$ is a fuzzy soft topology on (U, E).*

Proof. We have $\Phi \in \mathcal{T}_\beta$ by default. Again since for each $e \in E$ and $x \in U$, there exists $F_A \in \beta$ such that $\mu^e_{F_A}(x) = 1$, $\widetilde{E} \in \mathcal{T}_\beta$. Now let F_A, $G_B \in \mathcal{T}_\beta$. Then for each $e \in E$ and $x \in U$, there exists H_C, $I_D \in \beta$, where $C \subseteq A$ and $D \subseteq B$ such that $H_C \sqsubseteq F_A$, $I_D \sqsubseteq G_B$ and also $\mu^e_{H_C}(x) = \mu^e_{F_A}(x)$, $\mu^e_{I_D}(x) = \mu^e_{G_B}(x)$. Let $F_A \sqcap G_B = J_{A\cap B}$. Since H_C, $I_D \in \beta$ and $e \in E$, $x \in U$, there exists $K_P \in \beta$ such that $K_P \sqsubseteq H_C \sqcap I_D$ and $\mu^e_{K_P}(x) = \min\{\mu^e_{H_C}(x), \mu^e_{I_D}(x)\}$. Let $a \in E$. Then

$$K_P(a) \subseteq H_C(a) \cap I_D(a) \subseteq F_A(a) \cap G_B(a) = J_{A\cap B}(a).$$

Therefore $K_P \sqsubseteq J_{A\cap B}$. Now

$$\mu^e_{K_p}(x) = \min\{\mu^e_{H_C}(x), \mu^e_{I_D}(x)\} = \min\{\mu^e_{F_A}(x), \mu^e_{G_B}(x)\} = \mu^e_{J_{A\cap B}}(x).$$

So $J_{A\cap B} \in \mathcal{T}_\beta$ i.e., $F_A \sqcap G_B \in \mathcal{T}_\beta$. Again let $F^{\alpha}_{A_\alpha} \in \mathcal{T}_\beta$, for all $\alpha \in \Lambda$, an index set and $e \in E$, $x \in U$. Let $J_C = \sqcup_{\alpha\in\Lambda} F^{\alpha}_{A_\alpha}$, where $C = \cup_{\alpha\in\Lambda} A_\alpha$. Therefore

$$\mu^e_{J_C}(x) = \max\{\mu^e_{F^{\alpha}_{A_\alpha}}(x) : \alpha \in \Lambda\}.$$

So $\mu^e_{J_C}(x) = \mu^e_{F^\alpha_{A_\alpha}}(x)$ for some $\alpha \in \Lambda$. Now since $F^\alpha_{A_\alpha} \in \mathcal{T}$, there exists $G_B \in \beta$ such that $G_B \sqsubseteq F^\alpha_{A_\alpha}$ and $\mu^e_{G_B}(x) = \mu^e_{F^\alpha_{A_\alpha}}(x)$. Therefore $G_B \sqsubseteq J_C$ and $\mu^e_{G_B}(x) = \mu^e_{J_C}(x)$. Therefore $J_C \in \mathcal{T}_\beta$. Thus $\mathcal{T}_\beta$ is a fuzzy soft topology on (U, E). □

Definition 3.11. Suppose β is a fuzzy soft base for a fuzzy soft topology on (U, E). Then $\mathcal{T}_\beta$, described in above theorem, is called a fuzzy soft topology generated by β and β is called a fuzzy soft base for $\mathcal{T}_\beta$.

Theorem 3.12. *Let β be a fuzzy soft base for a fuzzy soft topology $\mathcal{T}_\beta$ on (U, E). Then $F_A \in \mathcal{T}_\beta$ if and only if $F_A = \sqcup_{\alpha\in\Lambda} B^\alpha_{A_\alpha}$, where $B^\alpha_{A_\alpha} \in \beta$ for each $\alpha \in \Lambda$, Λ an index set.*

Proof. Since every member of β is also a member of $\mathcal{T}_\beta$, any union of members of β is a member of $\mathcal{T}_\beta$.

Conversely, let $F_A \in \mathcal{T}_\beta$, where F_A is not equal to Φ. Then for each $e \in E$ and $x \in U$, there exists $X^e_{B^x_e} \in \beta$ such that $X^e_{B^x_e} \sqsubseteq F_A$ and $\mu^e_{X^e_{B^x_e}}(x) = \mu^e_{F_A}(x)$, where $B^x_e \subseteq A$. Let $B = \bigcup_{e\in E, x\in U} B^x_e$ and $G_B = \sqcup_{e\in E,x\in U} X^e_{B^x_e}$. We now show that $G_B = F_A$. Obviously, $G_B \sqsubseteq F_A$. Let $a \in E$ and $y \in U$. Then

$\mu^a_{G_B}(y) = \max\{\mu^a_{X^e_{B^x_e}}(y) : e \in E \text{ and } x \in U\}$.

$\geq \mu^a_{Y^a_{B^y_a}}(y)$ [where corresponding to each $a \in E$ and $y \in U$, we have $Y^a_{B^y_a} \in \beta$]

$= \mu^a_{F_A}(y)$.

Therefore $\mu^a_{G_B}(y) \geq \mu^a_{F_A}(y)$ for all $a \in E$ and $y \in U$. So, $F_A \sqsubseteq G_B$. Hence $G_B = F_A$, i.e., F_A is expressible as the union of some members of β. The case $F_A = \Phi$ is obvious as $\Phi \in \beta$. □

Theorem 3.13. *Let $(U, E, \mathcal{T})$ be a fuzzy soft topological space and β be a sub collection of $\mathcal{T}$ such that every member of $\mathcal{T}$ is a union of some members of β. Then β is a fuzzy soft base for the fuzzy soft topology $\mathcal{T}$ on (U, E).*

Proof. Since $\Phi \in \mathcal{T}$, $\Phi \in \beta$. Again since $\widetilde{E} \in \mathcal{T}$, $\widetilde{E} = \sqcup\beta$. Let $F_{A_1}, G_{A_2} \in \beta$. Then $F_{A_1}, G_{A_2} \in \mathcal{T}$ and so $F_{A_1} \sqcap G_{A_2} \in \mathcal{T}$. Then there exists $B^\alpha_{C_\alpha} \in \beta, \alpha \in \Lambda$ such that

$$F_{A_1} \sqcap G_{A_2} = \sqcup\{B^\alpha_{C_\alpha} : \alpha \in \Lambda\}.$$

Therefore $F_{A_1}(e) \cap G_{A_2}(e) = \cup\{B^\alpha_{C_\alpha}(e) : \alpha \in \Lambda\}$, for $e \in E$. That is

$$\min\{\mu^e_{F_{A_1}}(x), \mu^e_{G_{A_2}}(x)\} = \max\{\mu^e_{B^\alpha_{C_\alpha}}(x) : \alpha \in \Lambda\}$$

for $e \in E$ and $x \in U$. Therefore there exists $\alpha \in \Lambda$ such that

$$\min\{\mu^e_{F_{A_1}}(x),\, \mu^e_{G_{A_2}}(x)\} = \mu^e_{B^\alpha_{C_\alpha}}(x).$$

Thus for $e \in E$ and $x \in U$, we get $B^\alpha_{C_\alpha} \in \beta$ such that $B^\alpha_{C_\alpha} \sqsubseteq F_{A_1} \sqcap G_{A_2}$ and

$$\min\{\mu^e_{F_{A_1}}(x),\, \mu^e_{G_{A_2}}(x)\} = \mu^e_{B^\alpha_{C_\alpha}}(x).$$

Therefore β is a fuzzy soft base for the fuzzy soft topology $\mathcal{T}$ on (U, E). □

Definition 3.14. A collection Ω of some members of a fuzzy soft topological space $(U, E, \mathcal{T})$ is said to be subbase for $\mathcal{T}$ if and only if the collection of all finite intersections of members of Ω is a base for $\mathcal{T}$.

Theorem 3.15. *A collection Ω of fuzzy soft sets over (U, E) is a subbase for a suitable fuzzy soft topology $\mathcal{T}$ if and only if*

(1) $\Phi \in \Omega$ *or* Φ *is the intersection of a finite number of members of* Ω.
(2) $\widetilde{E} = \sqcup\Omega$.

Proof. First let Ω be a subbase for $\mathcal{T}$ and β be a base generated by Ω. Since $\Phi \in \beta$, either $\Phi \in \Omega$ or Φ is expressible as an intersection of finitely many members of Ω. Now let $x \in U$ and $e \in E$. Since $\sqcup\beta = \widetilde{E}$, there exists $B_A \in \beta$ such that $\mu^e_{B_A}(x) = 1$. Since $B_A \in \beta$, there exists $S^i_{A_i} \in \Omega$, $i = 1, 2, \cdots, n$ such that $B_A = \sqcap^n_{i=1} S^i_{A_i}$ Therefore $\mu^e_{B_A}(x) = \min^n_{i=1} \mu^e_{S^i_{A_i}}(x)$ and so $\mu^e_{B_A}(x) = \mu^e_{S^i_{A_i}}(x)$, for some $i \in \{1, 2, \cdots, n\}$. Thus $\mu^e_{S^i_{A_i}}(x) = 1$. Hence $\widetilde{E} = \sqcup\Omega$.

Conversely let Ω be a collection of some fuzzy soft sets over (U, E) satisfying the conditions (1) and (2). Let β be the collection of all finite intersections of members of Ω. Now it enough to show that β forms base for suitable fuzzy soft topology. Since β is the collection of all finite intersections of members of Ω, by assumption (1) we get $\Phi \in \beta$ and by (2) we get $\sqcup\beta = \widetilde{E}$. Again let $F_A, G_B \in \beta$ and $x \in U, e \in E$. Since $F_A \in \beta$, there exists $F^i_{A_i} \in \Omega$, for $i = 1, 2, \cdots, n$ such that $F_A = \sqcap^n_{i=1} F^i_{A_i}$, where $A = \cap^n_{i=1} A_i$. Again since $G_B \in \beta$, there exists $G^j_{B_j} \in \Omega$, for $j = 1, 2, \cdots, m$ such that $G_B = \sqcap^m_{j=1} G^j_{B_j}$, where $B = \cap^m_{j=1} B_j$. Therefore

$$F_A \sqcap G_B = (\sqcap^n_{i=1} F^i_{A_i}) \sqcap (\sqcap^m_{j=1} G^j_{B_j}) \in \beta.$$

That is, $F_A \sqcap G_B \in \beta$. This completes the proof. □

4. Conclusion

In this paper, we have introduced fuzzy soft topology, fuzzy soft open base and subbase. So, one can try to introduce some special properties like connectedness, paths etc. on fuzzy soft topological spaces.

Acknowledgements. The authors are very grateful to the editor and the reviewers for their valuable suggestions.

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