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Direct product of finite fuzzy subsets in LA-semigroups

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Abstract. In this paper, we introduce the notion of direct product of finite fuzzy subLA-semigroups of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$, and also introduce the notion of direct product of finite fuzzy (left, right and bi) ideals of $S_1 \times S_2 \times S_3 \times ... \times S_n$. In section 3 we introduce some more interesting results of fuzzy ideals with fuzzy points of LA-semigroups. In this respect we prove that if S is an LA-semigroup with left identity e and if f is a fuzzy left ideal of S, then $a_t \circ f$ is a fuzzy ideal of S, where a_t is fuzzy point of S.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [10]. Since its inception, the theory of fuzzy set has developed in many directions and find applications in a wide variety of fields. The concept of an LA-semigroup was first introduced by Kazim and Naseerudin [2]. Later, Q. Mushtaq and others have investigated the structure further and added many useful results to theory of LA-semigroups see [4, 5, 6, 7]. Its is a useful non associative algebraic structure, midway between a groupoid and a commutative semigroup. The direct product of LA-semigroups or Abel Grassmann's groupoids was first introduced by Q. Mushtaq and M. Khan [6]. If S_1 and S_2 are LA-semigroups, then $S_1 \times S_2 = \{(s_1, s_2) : s_1 \in S_1 \text{ and } s_2 \in S_2\}$ is an LA-semigroup under point-wise multiplication of order pairs. Q. Mushtaq and M. Khan defined the direct product of left (resp. right) ideals, prime ideals, minimal ideals and investigated the properties of such ideals. In [9], H. Sherwood initiated the concept of the product of fuzzy subgroups of 1983 and then A. K. Ray obtained more important results of the direct product of fuzzy subgroups in [8] of 1999. Recently, in [1], H. Aktas and N. Cagman generalized the concept of the product of fuzzy subgroups and obtained useful results.

In this paper, we introduce the notion of direct product of finite fuzzy subLA-semigroups of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$, and also introduce the notion of direct product of finite fuzzy (left, right and bi) ideals of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. In section 3 we obtain further interesting results of fuzzy ideals with fuzzy points of LA-semigroups. In this respect, we prove that: if f is a fuzzy right ideal of an LA-semigroup S with left identity, then $f \circ f$ is a fuzzy ideal of an LA-semigroup of S and if S is an LA-semigroup with left identity e and f is a fuzzy left ideal of S, then $a_t \circ f$ is a fuzzy ideal of S, where a_t is fuzzy point of S.

2. Preliminaries

We record here some basic concepts and clarify notions used in the sequel.

Let S be a non-empty set. Then S is called an LA-semigroup if $xy \in S$ and $(xy)z = (zy)x$ for all $x, y, z \in S$. A non-empty set S is called AG-groupoid if S is a groupoid and satisfies $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$. A non-empty set U of an LA-semigroup S is said to be a subLA-semigroup S if $UU \subseteq U$. A left (resp. right) ideal I of an LA-semigroup S is a non-empty subset I of S such that $SI \subseteq I$ (resp. $IS \subseteq I$). If I is both a left and a right ideal of an LA-semigroup S, then we say that I is an ideal of S. A subLA-semigroup B of an LA-semigroup S is called a bi-ideal of S, if $(BS)B \subseteq B$. For subsets A, B of an LA semigroup S, we denote $AB = \{ab \in S : a \in A, b \in B\}$. (See [2, 4, 5]).

Definition 2.1 ([6]). Let I_{S_1} and I_{S_2} be subsets of LA-semigroups S_1 and S_2, respectively. Then the direct product $I_{S_1} \times I_{S_2}$ is called left (resp. right) ideal of an LA-semigroup $S_1 \times S_2$, if $(S_1 \times S_2)(I_{S_1} \times I_{S_2}) \subseteq (I_{S_1} \times I_{S_2})$ (resp. $(I_{S_1} \times I_{S_2})(S_1 \times S_2) \subseteq (I_{S_1} \times I_{S_2})$)

Lemma 2.2 ([6]). *If I_{S_1} and I_{S_2} are ideals of LA-semigroups S_1 and S_2 respectively, then $I_{S_1} \times I_{S_2}$ is an ideal of an LA-semigroup $S_1 \times S_2$.*

Lemma 2.3 ([6]). *If $I_{S_1} \times I_{S_2}$ and $J_{S_1} \times J_{S_2}$ are ideals of an LA-semigroup $S_1 \times S_2$, then $(I_{S_1} \times I_{S_2}) \cap (J_{S_1} \times J_{S_2})$ is an ideal.*

Definition 2.4 ([3]). A fuzzy subset f of an LA-semigroup is called a fuzzy subLA-semigroup of an LA-semigroup S, if $f(xy) \geq \max\{f(x), f(y)\}$, for all $x, y \in S$

Definition 2.5 ([3]). A fuzzy subset of an LA-semigroup S is called a left (resp. right) ideal of an LA-semigroup S if $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for all $x, y \in S$. A fuzzy subset f is called a fuzzy ideal of an LA-semigroup S if it is both fuzzy left and right ideal of S.

Definition 2.6 ([3]). A fuzzy subLA-semigroup f of an LA-semigroup S is called a fuzzy bi-ideal of an LA-semigroup S if $f((xy)z) \geq \max\{f(x), f(z)\}$ for all $x, y, z \in S$.

3. Major section

Lemma 3.1. *Every LA-semigroup is a AG-groupoid but converse is not true in general.*

Proof. Direct part is straightforward. For converse see below example. □

Example 3.2. Let $S = \{0, 1, 2, 3\}$ be a non-empty set. Then, S is a AG-groupoid by the Cayley table:

$*$	0	1	2	3
0	3	2	3	3
1	0	0	0	0
2	2	3	3	3
3	3	2	3	3

But S is not an LA-semigroup, becouse $(1 * 2) * 3 \neq (3 * 2) * 1$.

Example 3.3. Let $S = \{0, 1, 2\}$ be a AG-groupoid with left identity by Cayley table but not an LA-semigrop.

$*$	0	1	2
0	0	1	2
1	0	2	2
2	0	2	2

Thus $(0 * 1) * 2 \neq (2 * 1) * 0$.

Lemma 3.4. *If f is a fuzzy right ideal of an LA-semigroup S with left identity e, then f is a fuzzy left ideal of S.*

Proof. Let f be a fuzzy right ideal of an LA-semigroup S with left identity e and let $x, y \in S$. Then

$$f(xy) = f((ex)y) = f((yx)e) \geq f(yx) \geq f(y)$$

for all $x, y \in S$. Hence f is a fuzzy left ideal of an LA-semigroup S. □

Lemma 3.5. *Every fuzzy right ideal of an LA-semigroup S with left identity is a fuzzy ideal of S.*

Proposition 3.6. *Let f be a fuzzy right ideal of an LA-semigroup of S with left identity. Then $f \circ f$ is a fuzzy ideal of an LA-semigroup S.*

Proof. Since f is fuzzy right ideal of S with left identity, Lemma 3.4 implies that f is a fuzzy left ideal of an LA-semigroup S. Let $a, b \in S$. If $a \neq yz$, then $(f \circ f)(a) = 0$, and so $(f \circ f)(ab) \geq 0 = (f \circ f)(a)$. If $a = yz$ for some $y, z \in S$, then

$$(f \circ f)(a) = \bigvee_{a=yz} \{f(y) \wedge f(z)\} = \bigvee_{a=yz} \{f(z) \wedge f(y)\}.$$

If $a = yz$, then $ab = (yz)b = (bz)y$ by left invertive law. Hence

$$\begin{aligned} (f \circ f)(a) &\leq \bigvee_{ab=(bz)y} \{f(bz) \wedge f(y)\} \\ &\leq \bigvee_{ab=cd} \{f(c) \wedge f(d)\} = (f \circ f)(ab) \\ (f \circ f)(ab) &\leq (f \circ f)(a) \end{aligned}$$

for all $a, b \in S$. Thus $f \circ f$ is a fuzzy right ideal of S. So by Lemma 3.4, $f \circ f$ is a fuzzy left ideal of S. Thus $f \circ f$ is a fuzzy ideal of S. □

Proposition 3.7. *Let f be a fuzzy left ideal and g be a fuzzy bi-ideal of an LA-semigroup S. Then $g \circ f$ and $(f \circ f) \circ g$ are fuzzy bi-ideal of LA-semigroup S.*

Definition 3.8. Let S be an LA-semigroup and $x \in S$. Then for $a \in S$ and $t \in (0, 1]$, we define

$$a_t : S \to [0,1]\,, \; x \to a_t(x) := \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

Then a_t is called fuzzy subset of S and called a fuzzy point with support t. By $a_t \in f$, we mean $f(a) \geq t$.

Theorem 3.9. *Let S be an LA-semigroup with left identity e. If f is a fuzzy left ideal of S, then $a_t \circ f$ is a fuzzy left ideal of S, where a_t is a fuzzy point of S.*

Proof. Let f be a fuzzy left ideal of S and let $x, y \in S$. If $y \neq pq$ for any $p, q \in S$, then $(a_t \circ f)(y) = 0$. Thus $(a_t \circ f)(xy) \geq 0 = (a_t \circ f)(y)$. If $y = pq$ for some$p, q \in S$, then

$$(a_t \circ f)(y) = \bigvee_{y=pq} \{(a_t)(p) \wedge f(q)\}.$$

Since $y = pq$, so $xy = x(pq) = (ex)(pq) = (ep)(xq) = p(xq)$. Thus

$$\begin{aligned} (a_t \circ f)(y) &= \bigvee_{y=pq} \{(a_t)(p) \wedge f(q)\} \\ &\leq \bigvee_{y=pq} \{(a_t)(p) \wedge f(xq)\}, \text{ since } f \text{ is a fuzzy left ideal of } S \\ &\leq \bigvee_{xy=cdq} \{(a_t)(c) \wedge f(d)\} \\ &= (a_t \circ f)(xy). \end{aligned}$$

Thus $(a_t \circ f)(xy) \geq (a_t \circ f)(y)$. Hence $a_t \circ f$ is a fuzzy left ideal of S. □

Theorem 3.10. *Let S be an LA-semigroup with left identity e and a_t is a fuzzy point of S. Then the smallest fuzzy left ideal of S generated by a_t is l_{a_t} defined by*

$$l_{a_t} : S \to [0,1]\,, \; x \to l_{a_t}(x) := \begin{cases} t & \textit{if } x \in Sa \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $x, y \in S$, we have two case's: (1) $y \notin Sa$ and (2) $y \in Sa$.

Case (1) : If $y \notin Sa$, then $l_{a_t}(x) = 0$

Case (2) : If $y \in Sa$, then $y = sa$ for some $s \in S$. Hence

$$\begin{aligned} xy &= x(sa) = (ex)(sa) = ((sa)x)e = ((sa)(ex))e \\ &= ((se)(ax))e = (e(ax))(se) = ((se)x)a \in Sa. \end{aligned}$$

Hence $l_{a_t}(xy) = t = l_{a_t}(y)$. Thus l_{a_t} is a fuzzy left ideal of S. Also, by definition of l_{a_t}, we have $a_t \leq l_{a_t}$.

Now, let f be a fuzzy left ideal of S containing a_t.

Case (1) : If $x \notin Sa$, then $l_{a_t}(x) = 0 \leq f(x)$

Case (2) : If $x \in Sa$, then $x = sa$ for some $s \in S$ and so $l_{a_t}(x) = t$. Also,

$$\begin{aligned} t &= a_t(a) \leq f(a) \leq f(sa) = f(x) \\ f(x) &\geq t = l_{a_t}(x). \end{aligned}$$

Thus, $l_{a_t} \subseteq f$ in any case. This shows that l_{a_t} is a smallest fuzzy left ideal of S. □

Theorem 3.11. *Let S be an LA-semigroup with left identity e and a_t is a fuzzy point of S. Then the smallest fuzzy right ideal of S generated by a_t is f_{a_t} defined by*

$$f_{a_t} : S \to [0,1]\,,\ x \to f_{a_t}(x) := \begin{cases} t & \text{if } x \in aS \cup Sa \\ 0 & \text{otherwise} \end{cases}$$

Proof. It follows from Theorem 3.10. □

4. Direct product of finite fuzzy ideals

Definition 4.1. Let $f : S_1 \to [0,1]$ and $g : S_2 \to [0,1]$ be two fuzzy subsets of LA-semigroups S_1 and S_2, respectively. Then the direct product of fuzzy subset is denoted by $f \times g$ and define as $f \times g : S_1 \times S_2 \to [0,1]$

$$f \times g(s_1, s_2) = \min\{f(s_1), g(s_2)\}$$

Definition 4.2. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups $S_1, S_2, S_3, ..., S_n$, respectively, Then the direct product of finite fuzzy subsets of LA-semigroup is denoted by $f_1 \times f_2 \times f_3 \times ... \times f_n$ and is defined as

$$f_1 \times f_2 \times f_3 \times ... \times f_n : S_1 \times S_2 \times S_3 \times ... \times S_n \to [0,1]$$

by

$$(f_1 \times f_2 \times f_3 \times ... \times f_n)(s_1, s_2, s_3, , ..., s_n) = \min\{f_1(s_1), f_2(s_2), f_3(s_3), ..., f_n(s_n)\}$$

Definition 4.3. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is called fuzzy subLA-semigroup of $S_1 \times S_2 \times S_3 \times ... \times S_n$, if $(f_1 \times f_2 \times f_3 \times ... \times f_n)((s_1, s_2, s_3, , ..., s_n)(t_1, t_2, t_3, , ..., t_n)) \geq \min\{(f_1 \times f_2 \times f_3 \times ... \times f_n)(s_1, s_2, s_3, ..., s_n), (f_1 \times f_2 \times f_3 \times ... \times f_n)(t_1, t_2, t_3, , ..., t_n)\}$, for all $(s_1, s_2, s_3, , ..., s_n), (t_1, t_2, t_3, , ..., t_n) \in S_1 \times S_2 \times S_3 \times ... \times S_n$.

Definition 4.4. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy right ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$ if

$$\begin{aligned}&(f_1 \times f_2 \times f_3 \times ... \times f_n)((s_1, s_2, s_3, , ..., s_n)(t_1, t_2, t_3, , ..., t_n))\\ &\geq (f_1 \times f_2 \times f_3 \times ... \times f_n)(s_1, s_2, s_3, , ..., s_n)\end{aligned}$$

for all $(s_1, s_2, s_3, , ..., s_n), (t_1, t_2, t_3, , ..., t_n) \in S_1 \times S_2 \times S_3 \times ... \times S_n$.

Definition 4.5. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy left ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$ if

$$\begin{aligned}&(f_1 \times f_2 \times f_3 \times ... \times f_n)((s_1, s_2, s_3, , ..., s_n)(t_1, t_2, t_3, , ..., t_n))\\ &\geq (f_1 \times f_2 \times f_3 \times ... \times f_n)(t_1, t_2, t_3, , ..., t_n),\end{aligned}$$

for all $(s_1, s_2, s_3, , ..., s_n), (t_1, t_2, t_3, , ..., t_n) \in S_1 \times S_2 \times S_3 \times ... \times S_n$.

If $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups $S_1, S_2, S_3, ..., S_n$, Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$. If $f_1 \times f_2 \times f_3 \times ... \times f_n$ is both fuzzy left and fuzzy right ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$.

Example 4.6. Let $S_1 = \{0, 1, 2\}$ and $S_2 = \{a, b, c\}$ be two LA-semigroups by given tables

$*$	0	1	2
0	1	2	1
1	1	1	1
2	1	1	1

$\bullet$	a	b	c
a	c	b	c
b	b	b	b
c	b	b	b

Then $S_1 \times S_2 = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$ is an LA-semigroup with pointwise multiplication. Define $f_1 : S_1 \to [0, 1]$ by $f_1(0) = 0.3$, $f_1(1) = f_1(2) = 0.4$. Then f_1 is a fuzzy left ideal of S_1. Define $f_2 : S_1 \to [0, 1]$ by $f_1(a) = 0.2, f_1(b) = f_1(c) = 0.3$ is fuzzy left ideal of S_2. Define $f_1 \times f_2 : S_1 \times S_2 \to [0, 1]$ by

$$
\begin{aligned}
(f_1 \times f_2)\,(0, a) &= (f_1 \times f_2)\,(1, a) = (f_1 \times f_2)\,(2, a) = 0.2, \\
(f_1 \times f_2)\,(0, b) &= (f_1 \times f_2)\,(0, c) = (f_1 \times f_2)\,(1, b) = 0.3, \\
(f_1 \times f_2)\,(1, c) &= (f_1 \times f_2)\,(2, b) = (f_1 \times f_2)\,(2, c) = 0.3.
\end{aligned}
$$

Then clearly $f_1 \times f_2$ is fuzzy ideal of $S_1 \times S_2$.

Proposition 4.7. *Let $f : S_1 \to [0, 1]$ and $g : S_2 \to [0, 1]$ be two fuzzy subLA-semigroup of S_1 and S_2, respectively. Then $f \times g$ is fuzzy subLA-semigroup of $S_1 \times S_2$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$. Then

$$
\begin{aligned}
(f \times g)((x_1, y_1)(x_2, y_2)) &= (f \times g)(x_1x_2, y_1y_2) \\
&= \min\{f(x_1x_2), g(y_1y_2)\} \\
&\geq \min\{\min\{f(x_1), f(x_2)\}, \min\{g(y_1), g(y_2)\}\} \\
&= \min\{\min\{f(x_1), g(y_1)\}, \min\{f(x_2), g(y_2)\}\} \\
&= \min\{(f \times g)(x_1, y_1), (f \times g)(x_2, y_2)\},
\end{aligned}
$$

and so

$$(f \times g)((x_1, y_1)(x_2, y_2)) \geq \min\{(f \times g)(x_1, y_1), (f \times g)(x_2, y_2)\}$$

for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$. Hence $f \times g$ is fuzzy subLA-semigroup of $S_1 \times S_2$. □

Proposition 4.8. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy sub LA-semigroups of LA-semigroups $S_1, S_2, S_3, ..., S_n$ respectively, then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy sub LA-semigroup of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proposition 4.9. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy left (resp, right) ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$, respectively. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy left (resp, right) ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Let $f_1, f_2, ..., f_n$ be fuzzy left ideals of LA-semigroups $S_1, S_2, ..., S_n$ respectively. Let $(s_1, s_2, ..., s_n), (t_1, t_2, ..., t_n) \in S_1 \times S_2 \times ... \times S_n$. Then

$$\begin{aligned}
&(f_1 \times f_2 \times ... \times f_n)((s_1, s_2, ..., s_n), (t_1, t_2, ..., t_n)) \\
&= (f_1 \times f_2 \times ... \times f_n)(s_1t_1, s_2t_2, ..., s_nt_n) \\
&= \min\{f_1(s_1t_1), f_2(s_2t_2), ..., f_n(s_nt_n)\} \\
&\geq \min\{f_1(t_1), f_2(t_2), ..., f_n(t_n)\} \\
&= (f_1 \times f_2 \times ... \times f_n)(t_1, t_2, ..., t_n)
\end{aligned}$$

since each f_i is a fuzzy left ideal of S_i for $i = 1, 2, ...n$, respectively. Hence

$$\begin{aligned}
&(f_1 \times f_2 \times ... \times f_n)((s_1, s_2, ..., s_n), (t_1, t_2, ..., t_n)) \\
&\geq (f_1 \times f_2 \times ... \times f_n)(t_1, t_2, ..., t_n)
\end{aligned}$$

for all $(s_1, s_2, ..., s_n), (t_1, t_2, ..., t_n) \in S_1 \times S_2 \times ... \times S_n$. Therefore $f_1 \times f_2 \times ... \times f_n$ is a fuzzy left ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. Similarly for fuzzy right ideal. □

Proposition 4.10. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy ideals of LA-semigroups S_1, S_2, $S_3, ..., S_n$, respectively. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Same as Proposition 4.9. □

Proposition 4.11. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy right ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$, with left identities $e_1, e_2, e_3, ..., e_n$ respectively, then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy left ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy right ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$ with left identities $e_1, e_2, e_3, ..., e_n$ respectively, Then by Lemma 3.4, f_1, f_2, f_3, $f_4, ..., f_n$ are fuzzy left ideals of $S_1, S_2, S_3, S_4, ..., S_n$ respectively. So by Proposition 4.9, $f_1 \times f_2 \times f_3 \times ... \times f_n$ is fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. □

Proposition 4.12. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy right ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$ with identities $e_1, e_2, e_3, ..., e_n$, respectively. Then*

$$(f_1 \circ f_1) \times (f_2 \circ f_2) \times (f_3 \circ f_3) \times ... \times (f_n \circ f_n)$$

is a fuzzy ideal of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.

Proof. Since each $f_1, f_2, f_3, ..., f_n$ is fuzzy right ideal of LA-semigroups $S_1, S_2, S_3, ..., S_n$ with left identities respectively, then by Lemma 3.4, $(f_1 \circ f_1)$, $(f_2 \circ f_2)$, $(f_3 \circ f_3)$, ..., $(f_n \circ f_n)$ are fuzzy ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$, respectively. Let $(s_1, s_2, ..., s_n), (t_1, t_2, ..., t_n) \in S_1 \times S_2 \times ... \times S_n$. Then we have

$$\begin{aligned}
&((f_1 \circ f_1) \times (f_2 \circ f_2) \times ... \times (f_n \circ f_n))(s_1, s_2, ..., s_n)(t_1, t_2, ..., t_n) \\
&= ((f_1 \circ f_1) \times (f_2 \circ f_2) \times ... \times (f_n \circ f_n))(s_1t_1, s_2t_2, ..., s_nt_n) \\
&= \min\{(f_1 \circ f_1)(s_1t_1), (f_2 \circ f_2)(s_2t_2), ..., (f_n \circ f_n)(s_nt_n)\} \\
&\geq \min\{(f_1 \circ f_1)(t_1), (f_2 \circ f_2)(t_2), ..., (f_n \circ f_n)(t_n)\} \\
&= ((f_1 \circ f_1) \times (f_2 \circ f_2) \times ... \times (f_n \circ f_n))(t_1, t_2, , ..., t_n).
\end{aligned}$$

Hence

$$((f_1 \circ f_1) \times (f_2 \circ f_2) \times ... \times (f_n \circ f_n))(s_1, s_2, ..., s_n)(t_1, t_2, ..., t_n)$$
$$\geq ((f_1 \circ f_1) \times (f_2 \circ f_2) \times ... \times (f_n \circ f_n))(t_1, t_2, ..., t_n)$$

for all $(s_1, s_2, s_3, ..., s_n), (t_1, t_2, ..., t_n) \in S_1 \times S_2 \times ... \times S_n$. Thus

$$(f_1 \circ f_1) \times (f_2 \circ f_2) \times (f_3 \circ f_3) \times ... \times (f_n \circ f_n)$$

is fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times S_3 \times S_4 \times ... \times S_n$. Then by Lemma 3.4, $(f_1 \circ f_1) \times (f_2 \circ f_2) \times (f_3 \circ f_3) \times ... \times (f_n \circ f_n)$ is fuzzy right ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. This completes the proof. □

Corollary 4.13. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy right ideals of LA-semigroups S_1, S_2, $S_3, ..., S_n$ with identities $e_1, e_2, e_3, ..., e_n$ respectively. Then $f_1^n \times f_2^n \times f_3^n \times ... \times f_n^n$ is a fuzzy ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$, where $f_i^n = \underbrace{(f_i \circ f_i \circ f_i \circ ... \circ f_i)}_{upto\ n\ time}$ for $i = 1, 2, 3, ...n$*

Definition 4.14. Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is called a fuzzy bi-ideal of LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. If $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy subLA-semigroup of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$ and satisfies the following condition

$$(f_1 \times f_2 \times f_3 \times ... \times f_n)\{[(s_1, s_2, s_3, ..., s_n)(b_1, b_2, b_3, ..., s_n)](t_1, t_2, t_3, ..., t_n)\}$$
$$\geq \min\{(f_1 \times f_2 \times f_3 \times ... \times f_n)(s_1, s_2, s_3, ..., s_n),$$
$$(f_1 \times f_2 \times f_3 \times ... \times f_n)(t_1, t_2, t_3, ..., t_n)\}$$

for all $(s_1, s_2, s_3, ..., s_n), (b_1, b_2, b_3, ..., s_n), (t_1, t_2, t_3, ..., t_n) \in S_1 \times S_2 \times S_3 \times ... \times S_n$.

Definition 4.15. Let f be a fuzzy subset of S. Then for $t \in [0, 1]$, $t-$level set denoted by f_t and define as $f_t = \{x \in S : f(x) \geq t\}$.

Theorem 4.16. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$ and let $t \in [0, 1]$, then $(f_1 \times f_2 \times f_3 \times ... \times f_n)_t = f_{1_t} \times f_{2_t} \times f_{3_t} \times ... \times f_{n_t}$.*

Proof. Let $t \in [0, 1]$ and for any element

$$\begin{aligned} (s_1, s_2, s_3, ..., s_n) &\in (f_1 \times f_2 \times f_3 \times ... \times f_n)_t \\ (f_1 \times f_2 \times f_3 \times ... \times f_n)(s_1, s_2, s_3, ..., s_n) &\geq t \\ \min\{f_1(s_1), f_2(s_2), f_3(s_3), ..., f_n(s_n)\} &\geq t \end{aligned}$$

$$\begin{aligned} \text{which implies that } f_1(s_1) &\geq t,\ f_2(s_2) \geq t, f_3(s_3) \geq t, ..., f_n(s_n) \geq t \\ (s_1, s_2, s_3, ..., s_n) &\in (f_{1_t} \times f_{2_t} \times f_{3_t} \times ... \times f_{n_t}) \end{aligned}$$

Conversely, let $(s_1, s_2, ..., s_n) \in (f_{1_t} \times f_{2_t} \times ... \times f_{n_t})$. Then for each $s_i \in f_{i_t}$, $i = 1, 2, ...n$

$$f_1(s_1) \geq t, f_2(s_2) \geq t, f_3(s_3) \geq t, ..., f_n(s_n) \geq t$$

Thus

$$\begin{aligned} \min\{f_1(s_1), f_2(s_2), ..., f_n(s_n)\} &\geq t \\ (f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n) &= \{f_1(s_1), f_2(s_2), ..., f_n(s_n)\} \geq t \\ (s_1, s_2, s_3, ..., s_n) &\in (f_1 \times f_2 \times f_3 \times ... \times f_n)_t \end{aligned}$$

Thus

$$(f_1 \times f_2 \times f_3 \times ... \times f_n)_t = f_{1_t} \times f_{2_t} \times f_{3_t} \times ... \times f_{n_t}$$

This completes the proof. □

Proposition 4.17. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy bi-ideals of LA-semigroups S_1, S_2, $S_3, ..., S_n$ respectively. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy bi-ideal of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Let $f_1, f_2, f_3, ..., f_n$ be fuzzy bi-ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$ respectively. Then by Corollary 4.8, $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy sub LA-semigroup of $S_1 \times S_2 \times S_3 \times ... \times S_n$. Now, let $(s_1, s_2, s_3, ..., s_n), (b_1, b_2, b_3, ..., s_n), (t_1, t_2, t_3, ..., t_n) \in S_1 \times S_2 \times S_3 \times ... \times S_n$. Then

$$\begin{aligned}
&(f_1 \times f_2 \times f_3 \times ... \times f_n)\{[(s_1, s_2, s_3, ..., s_n)(b_1, b_2, b_3, ..., b_n)](t_1, t_2, t_3, ..., t_n)\} \\
&= (f_1 \times f_2 \times f_3 \times ... \times f_n)((s_1 b_1)t_1, s_2(b_2)t_2, (s_3 b_3)t_3, ..., (s_n b_n)t_n) \\
&= \min\{f_1((s_1 b_1)t_1), f_2(s_2(b_2)t_2), f_3((s_3 b_3)t_3), ..., f_n((s_n b_n)t_n)) \\
&\geq \min\{\min\{f_1(s_1), f_1(t_1)\}, \min\{f_2(s_2), f_2(t_2)\}, ..., \min\{f_n(s_n), f_n(t_n)\} \\
&= \min\{\min\{f_1(s_1), f_2(s_2), ...f_n(s_n)\}, \min\{f_1(t_2), f_2(t_2), ..., f_n(t_n)\}\} \\
&= \min\{(f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n), (f_1 \times f_2 \times ... \times f_n)(t_1, t_2, ..., t_n)\}.
\end{aligned}$$

Thus $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy bi-ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$. □

Proposition 4.18. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy left(right) ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy bi-ideal of $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Since $f_1, f_2, f_3, ..., f_n$ are fuzzy left ideals of LA-semigroups $S_1, S_2, S_3, ..., S_n$. Then by Lemma 4.10, $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times S_3 \times ... \times S_n$. Let $(s_1, s_2, , ..., s_n), (b_1, b_2, ..., b_n), (t_1, t_2, ..., t_n) \in S_1 \times S_2 \times ... \times S_n$. Then

$$\begin{aligned}
&(f_1 \times f_2 \times ... \times f_n)([(s_1, s_2, ..., s_n)(b_1, b_2, ..., b_n)](t_1, t_2, ..., t_n)) \\
&\geq (f_1 \times f_2 \times f_3 \times ... \times f_n)(t_1, t_2, t_3, ..., t_n) \\
&\geq \min\{(f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n), (f_1 \times f_2 \times ... \times f_n)(t_1, t_2, ..., t_n)\}.
\end{aligned}$$

Hence $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy bi-ideal of an LA-semigroup $S_1 \times S_2 \times ... \times S_n$. □

Proposition 4.19. *Let $f_1, f_2, f_3, ..., f_n$ be n fuzzy subsets of LA-semigroups S_1, S_2, $S_3, ..., S_n$ respectively. Then $f_1 \times f_2 \times f_3 \times ... \times f_n$ is a fuzzy subLA-semigroup of $S_1 \times S_2 \times S_3 \times ... \times S_n$ if and only if for $t \in [0, 1]$ $(f_1 \times f_2 \times f_3 \times ... \times f_n)_t$ is a subLA-semigroup of $S_1 \times S_2 \times S_3 \times ... \times S_n$.*

Proof. Let $f_1 \times f_2 \times ... \times f_n$ be a fuzzy sub LA-semigroup of $S_1 \times S_2 \times ... \times S_n$. Let $(s_1, s_2, , ..., s_n), (b_1, b_2, ..., b_n) \in (f_1 \times f_2 \times f_3 \times ... \times f_n)_t$. Then

$$(f_1 \times f_2 \times ... \times f_n)(a_1, a_2, ..., a_n) \geq t$$

and

$$(f_1 \times f_2 \times ... \times f_n)(b_1, b_2, ..., b_n) \geq t.$$

Thus

$$\begin{aligned} & (f_1 \times f_2 \times f_3 \times ... \times f_n)(a_1 b_1, a_2 b_2, ..., a_n b_n) \\ = \; & \min\{(f_1 \times f_2 \times ... \times f_n)(a_1, a_2, ..., a_n), (f_1 \times f_2 \times ... \times f_n)(b_1, b_2, ..., b_n) \geq t, \end{aligned}$$

and so $(a_1 b_1, a_2 b_2, ..., a_n b_n) \in (f_1 \times f_2 \times ... \times f_n)_t$. Let $(f_1 \times f_2 \times ... \times f_n)_t$ be subLA-semigroup of $S_1 \times S_2 \times ... \times S_n$. Suppose $f_1 \times f_2 \times ... \times f_n$ is not fuzzy subLA-semigroup of $S_1 \times S_2 \times ... \times S_n$. Then there exist $(s_1, s_2, ..., s_n), (r_1, r_2, ..., r_n) \in S_1 \times S_2 \times ... \times S_n$ such that

$$\begin{aligned} & (f_1 \times f_2 \times ... \times f_n)[(s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n)] \\ < \; & \min\{(f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n), (f_1 \times f_2 \times ... \times f_n)(r_1, r_2, ..., r_n). \end{aligned}$$

Let

$$\begin{aligned} t_\circ = \; & \frac{1}{2}\{f_1 \times f_2 \times ... \times f_n)[(s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n)] \\ & + \min\{(f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n), (f_1 \times f_2 \times ... \times f_n)(r_1, r_2, ..., r_n)\} \end{aligned}$$

Then

$$\begin{aligned} & (f_1 \times f_2 \times ... \times f_n)[(s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n)] < t_\circ \\ < \; & \min\{(f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n), (f_1 \times f_2 \times ... \times f_n)(r_1, r_2, ..., r_n), \end{aligned}$$

$$\begin{aligned} (s_1, s_2, ..., s_n) & \in (f_1 \times f_2 \times ... \times f_n)_t \text{ and} \\ (r_1, r_2, ..., r_n) & \in (f_1 \times f_2 \times ... \times f_n)_t, \text{but} \\ (s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n) & \in (f_1 \times f_2 \times ... \times f_n)_t, \end{aligned}$$

which is a contradiction. Hence $f_1 \times f_2 \times ... \times f_n$ is a fuzzy subLA-semigroup of $S_1 \times S_2 \times ... \times S_n$. □

Proposition 4.20. *Let $f_1 \times f_2 \times ... \times f_n$ be a fuzzy subset of LA-semigroup $S_1 \times S_2 \times ... \times S_n$. Then $(f_1 \times f_2 \times ... \times f_n)$ is a fuzzy left (resp. right) ideal of an LA-semigroups $S_1 \times S_2 \times ... \times S_n$ if and only if for any $t \in [0,1]$, $(f_1 \times f_2 \times ... \times f_n)_t$ is a left (resp. right) ideal of an LA-semigroup of $S_1 \times S_2 \times ... \times S_n$.*

Proof. Let $f_1 \times f_2 \times ... \times f_n$ be a fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times ... \times S_n$. Let $(f_1 \times f_2 \times ... \times f_n)_t$ be a subset of $S_1 \times S_2 \times ... \times S_n$ and for any $(s_1, s_2, ..., s_n) \in S_1 \times S_2 \times ... \times S_n$ and $(r_1, r_2, ..., r_n) \in (f_1 \times f_2 \times ... \times f_n)_t$. Then

$$(f_1 \times f_2 \times ... \times f_n)(r_1, r_2, ..., r_n) \geq t,$$

and so

$$\begin{aligned} & (f_1 \times f_2 \times ... \times f_n)(s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n) \\ & \geq (f_1 \times f_2 \times ... \times f_n)(r_1, r_2, ..., r_n) \geq t. \end{aligned}$$

and so $(s_1, s_2, ..., s_n)(r_1, r_2, ..., r_n) \in (f_1 \times f_2 \times ... \times f_n)_t$. Let $(f_1 \times f_2 \times ... \times f_n)_t$ be a left ideal of $S_1 \times S_2 \times ... \times S_n$. Suppose $(f_1 \times f_2 \times ... \times f_n)$ is not a fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times ... \times S_n$. Then there exist $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \in S_1 \times S_2 \times ... \times S_n$ such that

$$(f_1 \times f_2 \times ... \times f_n)[(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)] < (f_1 \times f_2 \times ... \times f_n)(b_1, b_2, ..., b_n)$$

Let

$$t_\circ = \frac{1}{2}\{(f_1 \times f_2 \times ... \times f_n)[(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)] \\ + (f_1 \times f_2 \times ... \times f_n)(b_1, b_2, ..., b_n)\}.$$

Then

$$(f_1 \times f_2 \times ... \times f_n)[(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)] < t_\circ \\ < (f_1 \times f_2 \times ... \times f_n)(b_1, b_2, ..., b_n),$$

which implies $(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n) \notin (f_1 \times f_2 \times ... \times f_n)_t$ but

$(b_1, b_2, ..., b_n) \in (f_1 \times f_2 \times ... \times f_n)_t$ and $(a_1, a_2, ..., a_n) \in S_1 \times S_2 \times ... \times S_n$.

This is a contradiction. Hence $(f_1 \times f_2 \times ... \times f_n)$ is a fuzzy left ideal of an LA-semigroup $S_1 \times S_2 \times ... \times S_n$. □

Proposition 4.21. *Let $f_1 \times f_2 \times ... \times f_n$ be fuzzy subset of an LA-semigroup $S_1 \times S_2 \times ... \times S_n$. Then $(f_1 \times f_2 \times ... \times f_n)$ is a fuzzy bi ideal of LA-semigroups $S_1 \times S_2 \times ... \times S_n$ if and only if for any $t \in [0, 1]$ $(f_1 \times f_2 \times ... \times f_n)_t$ is a bi ideal of LA-semigroup of $S_1 \times S_2 \times ... \times S_n$.*

Proof. Straightforward. □

5. Conclusions

In the study of the structure of a fuzzy algebraic system, we notice that fuzzy ideals with special properties always play an important role. In this paper, we showed every LA-semigroups is a AG-groupoid but converse is not true in general and also give some results of fuzzy ideals in LA-semigroups. Moreover, we have defined direct product of finite fuzzy subLA-semigroups (left, right bi) ideals in LA-semigroups and different properties are discussed. In our future work, we shall focuss on define direct product of finite fuzzy ideals in other algebraic structure .i.e., semigroups, semiring etc., and hopefully we will obtain useful results.

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