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The characterizations of some kinds of soft hemirings

Huajun Wu, Jianming Zhan

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Abstract. The characterizations of h-semisimple hemirings are first discussed based on soft set theory. Then notions of soft h-hemiregular hemirings, soft h-intra-hemiregular hemirings and soft h-semisimple hemirings are introduced and some properties of them are investigated, respectively.

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Corresponding Author: Jianming Zhan (zhanjianming@hotmail.com)

1. Introduction

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different areas of applied mathematics and information sciences because a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. They play an important role in studying optimization, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized automata theory, formal language theory, coding theory, analysis of computer programs, and so on (see [1, 2, 11, 13, 22, 24]). Indeed, many results in rings apparently have no analogues in semirings using only ideals. Henriksen [12] defined a more restricted class of ideals in semirings, which is called k-ideals, with the proprety that if the semiring S is a ring, then a complex in S is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [14]. However, a definition of ideal in any additively commutative semiring S can be given which coincides with Iizuka's definition provided S is a hemiring, and it is called h-ideal, La Torre [16] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems.

Molodtsov [21] initiated a novel concept called soft sets as a new mathematical tool for dealing with uncertainties. The soft set theory is free form many difficulties that have troubled the usual theoretical approaches and has potential applications in many different fields including the smoothness of functions, game theory, and measurement theory. Research works on soft sets are very active and progressing rapidly in these years. Maji et al. [20] discussed the application of soft set theory to decision-making program. Chen et al. [3] presented a new definition of soft set parametrization reduction, and compared it with attributes reduction in rough set theory. Maji et al. [19] defined and studied several operations on soft sets. Jun [15] introduced and investigated soft BCK/BCI-algebras. Furthermore, Feng et al. [8] applied soft set theory to the study of semirings (see [6, 7, 9]) and initiated the notions called soft semirings.

Recently, the notions of hemirings with special structure were introduced. Zhan et al. [27] gave the concept of h-hemiregularity of a hemiring and investigated some properties in terms of fuzzy theory. As a continuation of the paper [27]. Yin and Li [26] gave the concept of h-intra-hemiregularity of a hemiring and considered the characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings. Yin et al. [26] gave the notions of h-semisimple hemirings and invested the characterizations by fuzzy ideals. Dudek et al. [4, 5] investigated the characterizations of some types of fuzzy ideals of hemirings. Ma et al. [17, 18] introduced the concepts of $(\in, \in \vee q)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. In particular, they showed that the h-hemiregular hemirings and h-intra-hemiregular hemirings can be described by some of their generalized fuzzy h-ideals. Finally, the implication-based fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring were considered.

In this paper, we investigate characterizations of these special hemirings by soft theory, in which the notions of soft hemirings with special structures are introduced. Finally, we give the concepts of soft h-hemiregular hemirings, soft h-intra-hemiregular hemirings and soft h-semisimple hemirings, then consider their characterizations in terms of soft ideals.

2. Preliminaries

A semiring is an algebra system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws $a(b+c) = ab+ac$, and $(a+b)c = ac+bc$ are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$, we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a hemiring. Throughout this paper, H is a hemiring.

A subset A of H is called a subhemiring of H if A closed under the addition and multiplication. A subset B of H is called a left (right) ideal of H which is closed under addition such that $HA \subseteq A$ (resp., $AH \subseteq A$).

A subset B of H is called a bi-ideal of H if B is closed under addition and multiplication such that $BHB \subseteq B$. A subset Q of H is called a quasi-ideal of H if Q is closed under addition such that $HQ \cap QH \subseteq Q$.

A subset A of H is called an interior ideal of H if A is closed under addition and multiplication such that $HAH \subseteq A$.

A left ideal (right ideal, ideal, interior ideal and bi-ideal) A of H is called a left h-ideal (right h-ideal, h-ideal, h-interior-ideal and h-bi-ideal) of H if for any $x, y \in H$, and $a, b \in A$ from $x + a + z = b + z$ it follows $x \in A$.

The h-closure $\overline{A}$ of A in H is defined by $\overline{A} = \{x \in H \mid x + a_1 + z = a_2 + z\}$ for some $a_1, a_2 \in A, z \in H$. Clearly, if A is a left ideal of H, then $\overline{A}$ is the smallest left h-ideal of H containing A.

A quasi-ideal Q of H is called an h-quasi-ideal of H if $\overline{HQ} \cap \overline{QH} \subseteq Q$ and for any $x, z \in H$ and $a, b \in Q$ from $x + a + z = b + z$ it follows $x \in Q$.

Lemma 2.1 ([27]). *For any hemiring H, then we have the following:*

(1) $A \subseteq \overline{A}$, $\forall \{0\} \subseteq A \subseteq H$;
(2) *If* $A \subseteq B \subseteq H$, *then* $\overline{A} \subseteq \overline{B}$;
(3) $\overline{\overline{A}} = \overline{A}$, $\forall A \subseteq H$;
(4) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$, $\forall A,\ B \subseteq H$.

It is clear that if A is a left (right) ideal of H, then $\overline{A}$ is the smallest left (right) h-ideal of H containing A.

Definition 2.2 ([21]). A pair (F, A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of the set U.

Definition 2.3 ([10]). Let (F, A) and (G, B) be two soft sets over U. Then (F, A) is said to be a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) $\forall x \in A,\ F(x) \subseteq G(x)$.

This relationship is denoted by $(F, A)\tilde{\subset}(G, B)$

Definition 2.4 ([23]). Let (F, A) and (G, A) be soft sets over H. (G, A) is called a soft h-closure of (F, A) if $G(x)$ is an h-closure of $F(x)$ for all $x \in A$, denoted by $\overline{(F, A)}$

Definition 2.5 ([19]). The product of two soft sets (F, A) and (G, B) over U is the soft set $(H, A \times B)$, where $H(x, y) = F(x)G(y)$, $(x, y) \in A \times B$. This is denoted by $(F, A) \times (G, B) = (H, A \times B)$.

Definition 2.6 ([19]). If (F, A) and (G, B) are soft sets over U, then (F, A) AND (G, B) is denoted by $(F, A) \wedge (G, B)$, where $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(x, y) = F(x) \cap G(y), (x, y) \in A \times B$.

Definition 2.7 ([19]). The intersection of two soft sets (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$,(as both are same set). We write $(F, A)\tilde{\cap}(G, B) = (H, C)$.

Definition 2.8 ([19]). The union of two soft sets of (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if} \quad e \in A - B, \\ G(e), & \text{if} \quad e \in B - A, \\ F(e) \cup G(e), & \text{if} \quad e \in A \cap B. \end{cases}$$

We write $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 2.9 ([19])**.** A soft set (F, A) over U is said to be an absolute soft set if $\forall x \in A$, $F(x) = U$, denoted by $\tilde{A}$.

Definition 2.10 ([9])**.** A soft set (F, A) over a semiring S is said to be a soft semiring over S if $\forall x \in Supp(F, A)$, $F(x)$ is a subsemiring of S.

Definition 2.11 ([9])**.** Let (η, A) be a soft semiring over a semiring S. A soft set (γ, I) over S is called a soft ideal of (η, A), denoted by $(\gamma, I)\tilde{\lhd}(\eta, A)$, if it satisfies:
(1) $I \subseteq A$;
(2) $\gamma(x)$ is an ideal of $\eta(x)$ for all $x \in Supp(\gamma, I)$.

Definition 2.12. Let (F, A) be a soft hemiring over H. A soft set (G, B) over H is called a soft h-ideal of (F, A), denoted by $(G, B)\tilde{\lhd}(F, A)$, if it satisfies:
(1) $I \subseteq A$;
(2) $G(x)$ is an h-ideal of $F(x)$ for all $x \in A$.

Definition 2.13 ([23])**.** (i) A soft set (F, A) over H is said to be a soft hemiring, if $\forall x \in A$, $F(x)$ is a subhemiring of H.

(ii)A soft set (F, A) over H is said to be an h-idealistic soft hemiring, if $\forall x \in A$, $F(x)$ is an h-ideal of H.

(iii) A soft set (F, A) over H is said to be an h-bi-idealistic soft hemiring, if $\forall x \in A$, $F(x)$ is an h-bi-ideal of H.

(iv) A soft set (F, A) over H is said to be an h-quasi-idealistic soft hemiring, if $\forall x \in A$, $F(x)$ is an h-quasi-ideal of H.

Definition 2.14. (i) A soft set (F, A) over H is said to be an h-interior-idealistic soft hemiring, if $\forall x \in A$, $F(x)$ is an h-interior-ideal of H.

(ii) An h-ideal M of H is said to be a principle h-ideal of H, if M is generated by an element x of H, that is, $M = \overline{xH + Hx + HxH + Nx}$.

(iii) A soft hemiring (F, A) over H is said to be a principle h-idealistic soft hemiring, if for every $x \in A$, $F(x)$ is a principle h-ideal.

3. h-semisimple hemirings

In this section, we investigate h-semisimple hemirings by soft set theory.

Definition 3.1 ([25])**.** A subset A of H is called weak idempotent if $A = \overline{A^2}$. H is called h-semisimple if every h-ideal A of H is weak idempotent.

Lemma 3.2 ([25])**.** *H is h-semisimple if and only if $A \subseteq \overline{HAHAH}$ for all $A \subseteq H$.*

Theorem 3.3. *The following statements are equivalent.*
(1) *H is h-semisimple.*
(2) *$A \cap B \subseteq \overline{HAHBH}$ for each pair of subsets of H.*
(3) *$A \cap B \subseteq \overline{AHB}$ for every left h-ideal A and every right h-ideal B of H.*

Proof. (1) $\Rightarrow$ (2). Assume that H is h-semisimple. Let A and B be two subsets of H, then $A \cap B \subseteq \overline{H(A \cap B)H(A \cap B)H} \subseteq \overline{HAHBH}$ by Lemma 3.2. (2) holds.

(2) $\Rightarrow$ (3). Assume that (2) holds. Let A and B be a left h-ideal and a right h-ideal of H, respectively, then $A \cap B \subseteq \overline{HAHBH} \subseteq \overline{AHB}$. (3) holds.

(3) $\Rightarrow$ (1). Assume that (3) holds. Let M be an h-ideal of H, then M is a both a right and a left h-ideal of H. $M = M \cap M \subseteq \overline{MHM} \subseteq \overline{MM} = \overline{M^2}$, it is clear that $\overline{M^2} \subseteq M$. So $M = \overline{M^2}$ and H is h-semisimple. (1) holds. □

Theorem 3.4. *H is h-semisimple if and only if*

$$(F,A) \wedge (G,B) \tilde{\subset} \overline{(N,C) * (F,A) * (N,C) * (G,B) * (N,C)}$$

for any two soft subsets (F,A), (G,B) and absolute soft set (N,C) over H.

Proof. Assume that H is h-semisimple. Let R and T be two subsets of H. $F(x) = R$, $G(y) = T$ and $N(z) = H$ for all $x \in A$, $y \in B$ and $z \in C$, then $R \cap T \subset \overline{HRHTH}$, i.e., $(F,A) \wedge (G,B) \tilde{\subset} \overline{(N,C) * (F,A) * (N,C) * (G,B) * (N,C)}$.

Conversely, let R and T be two subsets of H, $F(x) = R$, $G(y) = T$ and $N(z) = H$ for $x \in A$, $y \in B$ and $z \in C$. Then $F(x) \cap G(y) = \overline{N(z)F(x)N(z)G(y)N(z)}$, i.e., $R \cap T = \overline{SRSTS}$. By Theorem 3.3, H is h-semisimple. □

Theorem 3.5. *H is h-semisimple if and only if*

$$(F,A) \wedge (G,B) \tilde{\subset} \overline{(F,A) * (N,C) * (G,B)}$$

for every left h-idealistic soft hemiring (F,A), every absolute soft hemiring (N,C) and every right h-idealistic soft hemiring (G,B) over H.

Proof. Assume that H is h-semisimple. By the assumption, $F(x)$ and $G(y)$ are a left h-ideal and a right h-ideal of H for all $x \in A$ and $y \in B$, respectively. We have

$$F(x) \cap G(y) \subseteq \overline{F(x)HG(y)}$$

by Theorem 3.3. So $(F,A) \wedge (G,B) \tilde{\subset} \overline{(F,A) * (N,C) * (G,B)}$.

Conversely, let R and T be a left h-ideal and a right h-ideal of H, respectively, then $F(x) = R$ for all $x \in A$, $G(y) = T$ for all $y \in B$. By the assumption, we have $F(x) \cap G(y) \subseteq \overline{F(x)HG(y)}$, i.e., $R \cap T \subseteq \overline{RST}$. So H is h-semisimple. □

Theorem 3.6. *If H is h-semisimple, then every h-interior ideal is an h-ideal of H.*

Proof. Assume that H is an h-semisimple hemiring. Let A be an h-interior ideal of H. By Theorem 3.3, we have

$$AH \subseteq \overline{H(AH)H(AH)H} \subseteq \overline{(HAH)(HAH)} \subseteq \overline{AA} \subseteq \overline{A} = A.$$

This implies that A is a right h-ideal of H. In a similar way, we can get that A is a left h-ideal of H. Thus A is an h-ideal of H. □

Theorem 3.7. *H is h-semisimple if and only if $A \cap B = \overline{AB}$ for every pair h-ideals A and B of H.*

Proof. Assume that A and B are h-ideals of H, then $A \cap B$ is an h-ideal of H. Since H is h-semisimple, $A \cap B = \overline{(A \cap B)^2} \subseteq \overline{AB}$, on the other hand, $\overline{AB} \subseteq A \cap B$, so $A \cap B = \overline{AB}$.

Conversely, if $A \cap B = \overline{AB}$ for every pair h-ideals A and B of H, then $A = A \cap A = \overline{AA}$, so H is h-semisimple. □

Corollary 3.8. *H is h-semisimple if and only if $A \cap B = \overline{AB}$ for each pair h-interior ideals A and B of H.*

Proof. Assume that H is h-semisimple. Let A and B be h-interior ideals of H, then A and B are h-ideals of H. By Theorem 3.7, we have $A \cap B = \overline{AB}$.

Conversely, if $A \cap B = \overline{AB}$ for every pair h-interior ideals A and B of H, then $C \cap D = \overline{CD}$ for every pair h-ideals C and D of H since every h-ideal is an h-interior of H. So H is h-semisimple by Theorem 3.7. □

Corollary 3.9. *The following conditions are equivalent:*

(1) *H is h-semisimple.*

(2) *$A \cap B = \overline{AB}$ for every h-ideal A and every h-interior-ideal B of H.*

(3) *$A \cap B = \overline{AB}$ for every h-interior-ideal A and every h-ideal B of H.*

Proof. Straightforward. □

Theorem 3.10. *The following conditions are equivalent for H.*

(1) *H is h-semisimple.*

(2) *$(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$ for any two h-idealistic soft hemirings (F,A) and (G,B) over H.*

(3) *$(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$ for every h-idealistic soft hemiring (F,A) and every h-interior-idealistic soft hemiring (G,B) over H.*

(4) *$(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$ for every h-interior-idealistic soft hemiring (F,A) and every h-idealistic soft hemiring (G,B) over H.*

(5) *$(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$ for any two h-interior-idealistic soft hemirings (F,A) and (G,B) over H.*

Proof. We only give the proof of (1) $\Leftrightarrow$ (2). The others are similar.

(1) $\Rightarrow$ (2). Assume that (F,A) and (G,B) are two h-idealistic soft hemirings of H. Then $F(x)$ and $G(y)$ are h-ideals of H for all $x \in A$ and $y \in B$. Since H is h-semisimple, $F(x) \cap G(y) = \overline{F(x)G(y)}$, i.e., $(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$.

(2) $\Rightarrow$ (1) Assume that M and N are two h-ideals of H. Let $F(x) = M$ and $G(y) = N$ for all $x \in A$ and $y \in B$, then (F,A) and (G,B) are h-idealistic soft hemirings of H, so $(F,A) \cap (G,B) = \overline{(F,A) * (G,B)}$ and $F(x) \cap G(y) = \overline{F(x)G(y)}$, i.e., $M \cap N = \overline{MN}$. Hence H is h-semisimple. □

Theorem 3.11. *If H is h-semisimple,. A is an h-ideal of H and B is an h-ideal of A, then $\overline{B}$ is an h-ideal of H.*

Proof. Assume that H is h-semisimple. Let $C = \overline{B + BH + HB + HBH}$, then C is an h-ideal of H. It is clear that $B \subseteq C \subseteq A$, so

$$\begin{aligned} C &= \overline{CC} = \overline{\overline{C^2}C} = \overline{C^2C} = \overline{C^3} \subseteq \overline{ACA} \\ &= \overline{A\overline{(B + BH + HB + HBH)}A} = \overline{\overline{A}\ \overline{(B + BH + HB + HBH)}\ \overline{A}} \\ &= \overline{A(B + BH + HB + HBH)A} \subseteq \overline{ABA + ABHA + AHBA + AHBHA} \\ &\subseteq \overline{ABA + ABA + ABA + ABA} \subseteq \overline{ABA} \subseteq \overline{B}. \end{aligned}$$

Thus $\overline{B} = C$ and $\overline{B}$ is an h-ideal of H. □

Theorem 3.12. *If H is h-semisimple, (F,A) is an h-idealistic soft hemiring over H and (G,B) is a soft h-ideal of (F,A), then $\overline{(G,B)}$ is an h-idealistic soft hemiring of H.*

Proof. By the assumption, $\forall x \in B$, $G(x)$ is an h-ideal of $F(x)$. Since $\overline{F(x)}$ is an h-ideal of H and H is h-semisimple, by Theorem 3.11, we have $\overline{G(x)}$ is an h-ideal of H. So $\overline{(G, B)}$ is an h-idealistic soft hemiring. □

4. Some kinds of special hemirings related to soft set theory

We will divide this section into three parts. In the first part, we study soft h-hemiregular hemirings. In the second part, we consider soft h-intra-hemiregular hemirings. Finally, soft h-semisimple hemirings are investigated in the third part.

4.1. Soft h-hemiregular hemirings.

In this part, we introduce the concepts of soft h-hemiregular hemirings and investigate properties by soft set theory.

Definition 4.1 ([27])**.** *H is called h-hemiregular if for every element x, there exist $a, a', z \in H$ such that $x + xax + z = xa'x + z$.*

Definition 4.2. A soft hemiring (F, A) over H is called an h-hemiregular soft hemiring if for all $x \in A$, $F(x)$ is h-hemiregular.

Example 4.3. Let $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in N^+ \right\}$, (F, A) is a soft set over H, $A = N^+$. Let $F(x) = xH$ for all $x \in A$. It is easy to check that $F(x)$ is h-hemiregular for all $x \in A$. Therefore, (F, A) is an h-hemiregular soft hemiring.

Definition 4.4. H is called soft h-hemiregular if every soft hemiring (F, A) over H is an h-hemiregular soft hemiring.

Example 4.5. Let $H = \{0, a, b\}, 0 < a < b, x + y = \max\{x, y\}, xx = x$ and $xy = 0$ for $x \neq y$. It is easy to check that H is an h-hemiregular hemiring. Moreover, every soft hemiring of H is h-hemiregular soft hemiring, so H is soft h-hemiregular.

Theorem 4.6. *If H is soft h-hemiregular, then H is h-hemiregular.*

Proof. Straightforward. □

Proposition 4.7. *Let A and B be two subhemirings of H. Then $A \cap B$ is subhemiring of H.*

Proof. Straightforward. □

Theorem 4.8. *If (F, A) is an h-hemiregular soft hemiring and $B \subseteq A$, then $(F|_B, B)$ is an h-hemiregular soft hemiring.*

Proof. Straightforward. □

Theorem 4.9. *If H is soft h-hemiregular, (F, A) and (G, B) are soft hemirings, then $(F, A) \wedge (G, B)$ is an h-hemiregular soft hemiring.*

Proof. Let $(N, C) = (F, A) \wedge (G, B)$, then $N(x, y) = F(x) \cap G(y)$ for any $(x, y) \in A \times B$. Since (F, A) and (G, B) are soft hemirings over H, $F(x)$ and $G(y)$ are subhemirings of H for all $x \in A$, $y \in B$. By Proposition 4.7, $F(x) \cap G(y)$ is a subhemiring of H. So (N, C) is a soft hemiring over H. Since H is soft h-hemiregular, we have (N, C) is an h-hemiregular soft hemiring. □

Theorem 4.10. *If H is soft h-hemiregular. (F, A) and (G, B) are soft hemirings. Then $(F, A)\tilde{\cap}(G, B)$ is an h-hemiregular soft hemiring.*

Proof. Let $(N, C) = (F, A)\tilde{\cap}(G, B)$, where $C = A \cap B$, $N(e) = F(e)$ or $G(e)$. Since (F, A) is a soft hemiring over H and H is a soft h-hemiregular. $F(e')$ is a subhemiring of H for all $e' \in A$, and (N, C) is a soft hemiring over H. Therefore (N, C) is an h-hemiregular soft hemiring. □

Theorem 4.11. *If H is soft h-hemiregular, (F, A) and (G, B) are soft hemirings over H, where A and B are disjoint. Then $(F, A)\tilde{\cup}(G, B)$ is an h-hemiregular soft hemiring.*

Proof. Let $(N, C) = (F, A)\tilde{\cup}(G, B)$, where $C = A \cup B$,

$$N(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

Since (F, A) and (G, B) are soft hemirings over H, $F(x)$ and $G(y)$ are subhemirings of H for all $x \in A$ and $y \in B$. As $A \cap B = \phi$, either $e \in A - B$ or $e \in B - A$. If $e \in A - B$, $N(e)$ is a subhemiring of H because $F(e)$ is a subhemiring of H. If $e \in B - A$, $N(e)$ is a subhemiring of H because $G(e)$ is a subhemiring of H, so (N, C) is a soft hemiring of H, and (N, C) is an h-hemiregular soft hemiring over H because H is soft h-hemiregular. □

Theorem 4.12. *H is soft h-hemiregular if and only if every subhemiring of H is h-hemiregular.*

Proof. Assume that M is a subhemiring of H. Let (F, A) be a soft set over H such that $F(x) = M$ for all $x \in A$. Since H is soft h-hemiregular, (F, A) is an h-hemiregular soft hemiring. So $F(x)$ is h-hemiregular, i.e., M is h-hemiregular.

Conversely, assume that (F, A) is any soft hemiring over H, then $F(x)$ is a subhemiring of H. By assumption, $F(x)$ is h-hemiregular, and so H is soft h-hemiregular. □

Theorem 4.13. *H is soft h-hemiregular if and only if*

$$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$$

for every right h-idealistic soft hemiring (F, A) and every left h-idealistic soft hemiring (G, B) over M which is a subhemiring of H.

Proof. Let M be a subhemiring of H. Since H is soft h-hemiregular, M is h-hemiregular. Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring over M, respectively. Then, $F(x)$ and $G(y)$ are a right h-ideal and a left h-ideal of M for all $x \in A$ and $y \in B$ and $F(x) \cap G(y) = \overline{F(x)G(y)}$, so $(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$

Conversely, let M be a subhemiring of H, P and Q any right h-ideal and any left h-ideal of M. Let (F, A) and (G, B) be soft sets over M such that $F(x) = P$ and $G(y) = Q$ for all $x \in A$ and $y \in B$, then (F, A) and (G, B) are a right h-idealistic soft hemiring and a left h-idealistic soft hemiring. By assumption, $(F, A) \wedge (G, B) =$

$\overline{(F,A)*(G,B)}$ and $F(x) \cap G(y) = \overline{F(x)G(y)}$, i.e., $P \cap Q = \overline{PQ}$, so M is h-hemiregular and H is soft h-hemiregular. □

Theorem 4.14. *If every principle h-idealistic soft hemiring is an h-hemiregular soft hemiring, then H is soft h-hemiregular.*

Proof. Let (F,A) be a soft hemiring over H. Then $\forall x \in A$, $F(x)$ is a subhemiring of H. $\forall y \in F(x)$, $\overline{Hy + yH + HyH + Ny}$ is a principle h-ideal of H generated by y, denoted by M. According to the assumption, M is h-hemiregular. So, there exist $a, a', z \in M$ such that $y + yay + z = ya'y + z$. It is clear that $M \subseteq F(x)$, so $a, a', z \in F(x)$. Then $F(x)$ is h-hemiregular and so (F,A) is an h-hemiregular soft hemiring over H. Therefore H is soft h-hemiregular. □

Theorem 4.15. *Let P and Q be two soft h-hemiregular subhemirings of H. Then $P \cap Q$ is soft h-hemiregular.*

Proof. Let M be a subhemiring of $P \cap Q$. Then M is a subhemiring of P or Q and M is h-hemiregular by the assumption. Therefore, by Theorem 4.12, $P \cap Q$ is soft h-hemiregular. □

Theorem 4.16. *If H is soft h-hemiregular, then every subhemiring of H is soft h-hemiregular.*

Proof. Let M and N be a subhemiring of H and M, respectively. Then N is a subhemiring of H. Since H is soft h-hemiregular, then N is h-hemiregular. Hence M is soft h-hemiregular. □

4.2. **Soft h-intra-hemiregular hemirings.** In this part, we give the notions of soft h-intra-hemiregular hemirings and study some characterizations by soft set theory.

Definition 4.17. H is h-intra-hemiregular if and only if for each $x \in H$, there exist $a_i, a'_i, b_j, b'_j, z \in H$ such that $x + \sum_{i=1}^{m} a_i x^2 a'_i + z = \sum_{j=1}^{n} b_j x^2 b'_j + z$.

Definition 4.18. A soft hemiring (F,A) over H is called an h-intra-hemiregular soft hemiring if for all $x \in A$, $F(x)$ is h-intra-hemiregular.

Example 4.19. Let $H = N^+ \cup \{0\}$ with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows: $a + b = \max\{a,b\}$, $a \cdot b = \min\{a,b\}$. Let (F,A) be a soft set over H, $A = N^+$, $F(x) = \{0, x, 2x\}$ for all $x \in A$. Then $F(x)$ is h-intra-hemiregular and (F,A) is an h-intra-hemiregular soft hemiring.

Definition 4.20. H is called soft h-hemiregular if every soft hemiring (F,A) over H is an h-intra-hemiregular soft hemiring.

Example 4.21. Let $H = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follow:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	0
b	0	0	a

Then H is an h-intra-hemiregular hemiring. Moreover, it is easy to check that H is a soft h-intra-hemiregular hemiring.

Theorem 4.22. *If H is soft h-intra-hemiregular, then H is h-intra-hemiregular.*

Proof. Straightforward. □

Theorem 4.23. *If (F, A) is an h-intra-hemiregular soft hemiring and $B \subseteq A$, then $(F|_B, B)$ is an h-intra-hemiregular soft hemiring.*

Proof. Straightforward. □

Theorem 4.24. *If H is soft h-intra-hemiregular, (F, A) and (G, B) are soft hemirings over H. Then $(F, A) \wedge (G, B)$ is an h-intra-hemiregular soft hemiring over H.*

Proof. The proof is similar to the proof of Theorem 4.9. □

Theorem 4.25. *If H is soft h-intra-hemiregular, (F, A) and (G, B) are soft hemirings over H. Then $(F, A)\tilde{\cap}(G, B)$ is an h-intra-hemiregular soft hemiring over H.*

Proof. The proof is similar to the proof of Theorem 4.10. □

Theorem 4.26. *If H is soft h-intra-hemiregular, (F, A) and (G, B) are soft hemirings over H in which A and B are disjoint. Then $(F, A)\tilde{\cup}(G, B)$ is an h-intra-hemiregular soft hemiring over H.*

Proof. The proof is similar to the proof of Theorem 4.11. □

Lemma 4.27 ([26])**.** *Let H be a hemiring, then the following conditions are equivalent.*

(1) *H is h-intra-hemiregular.*
(2) *$L \cap R \subseteq \overline{LR}$ for every left h-ideal L and every right h-ideal R of H.*

Theorem 4.28. *H is soft h-intra-hemiregular if and only if every subhemiring of H is h-intra-hemiregular.*

Proof. Assume that M is a subhemiring of H. Let $F(x) = M$ for all $x \in A$. Since H is soft h-intra-hemiregular, $F(x)$ is h-intra-hemiregular, so M is h-intra-hemiregular.

Conversely, assume that (F, A) is any soft hemiring over H, then $F(x)$ is a subhemiring of H for all $x \in A$. By the assumption, $F(x)$ is h-intra-hemiregular, so H is soft h-intra-hemiregular. □

Theorem 4.29. *H is soft h-intra-hemiregular if and only if*

$$(F, A) \wedge (G, B)\tilde{\subset}\overline{(F, A) * (G, B)}$$

for every left h-idealistic soft hemiring (F, A) and every right h-idealistic soft hemiring (G, B) over any subhemiring M of H.

Proof. Let M be a subhemiring of H. Since H is soft h-intra-hemiregular, M is h-intra-hemiregular. Let (F, A) and (G, B) be a left h-idealistic soft hemiring and a right h-idealistic soft hemiring over M, respectively. Then, $F(x)$ and $G(y)$ are a left h-ideal and a right h-ideal of M for all $x \in A$ and $y \in B$ and $F(x) \cap G(y) \subseteq \overline{F(x)G(y)}$, so $(F, A) \wedge (G, B)\tilde{\subset}\overline{(F, A) * (G, B)}$

Conversely, let M be a subhemiring of H, P and Q any left h-ideal and any right h-ideal of M. Let (F, A) and (G, B) be soft sets over M such that $F(x) = P$ and $G(y) = Q$ for all $x \in A$ and $y \in B$, then (F, A) and (G, B) are a left h-idealistic soft hemiring and a right h-idealistic soft hemiring. By the assumption, $(F, A) \wedge (G, B) \tilde{\subset} \overline{(F, A) * (G, B)}$ and $F(x) \cap G(y) \subseteq \overline{F(x)G(y)}$, i.e., $P \cap Q \subseteq \overline{PQ}$, so M is h-intra-hemiregular and H is soft h-intra-hemiregular. □

Theorem 4.30. *If every principle h-idealistic soft hemiring is an h-intra-hemiregular soft hemiring, then H is soft h-intra-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.14. □

Theorem 4.31. *Let P and Q be two soft h-intra-hemiregular subhemirings of H, then $P \cap Q$ is soft h-intra-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.15. □

Theorem 4.32. *If H is soft h-intra-hemiregular, then every subhemiring of H is soft h-intra-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.16. □

Lemma 4.33 ([26])**.** *The following conditions are equivalent for H.*
(1) *H is both h-hemiregular and h-intra-hemiregular.*
(2) *$B = B^2$ for every h-bi-ideal B of H.*
(3) *$Q = Q^2$ for every h-quasi-ideal of H.*

Theorem 4.34. *If one of the following conditions holds for H, then H is both soft h-hemiregular and soft h-intra-hemiregular.*
(1) *$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$ for any two h-bi-idealistic soft hemirings (F, A) and (G, B) over any subhemirng M of H.*
(2) *$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$ for any two h-quasi-idealistic soft hemirings (F, A) and (G, B) over any subhemirng M of H.*
(3) *$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$ for every h-bi-idealistic soft hemiring (F, A) and every h-quasi-idealistic soft hemiring (G, B) over any subhemirng M of H.*
(4) *$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$ for every h-quasi-idealistic soft hemiring (F, A) and every h-bi-idealistic soft hemiring (G, B) over any subhemirng M of H.*

Proof. We only show that (1) implies H is both soft h-hemiregular and soft h-intra-hemiregular, the others are similar.

Let M be any subhemiring of H, P an h-bi-ideal of M and (F, A) a soft set over M. Moreover, assume that $F(x) = M$ for all $x \in A$, then (F, A) is an h-bi-idealistic soft hemiring over M. By the assumption, $(F, A) = \overline{(F, A) * (F, A)}$, i.e., $P = \overline{P^2}$. Therefore, M is both h-hemiregular and h-intra-hemiregular and H is both soft h-hemiregular and soft h-intra-hemiregular. □

4.3. **Soft h-semisimple hemirings.** In this part, we give the concepts of soft h-semisimple hemirings and consider characterizations by soft set theory.

Definition 4.35. A soft hemiring (F, A) over H is called h-semisimple soft hemiring if for all $x \in A$, $F(x)$ is h-semisimple.

Example 4.36. Let $H = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b
0	0	a	b
a	a	0	0
b	b	0	b

$\cdot$	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

Then H is an h-semisimple hemiring. Let (F, A) be a soft set over H, in which $A = \{1, 2, \}$ and $F(1) = \{0, a\}$, $F(2) = \{0, b\}$. It is easy to check that $F(x)$ is h-semisimple for every $x \in A$. So (F, A) is a soft h-sempsimple hemiring.

Definition 4.37. A hemiring H is called a soft h-semisimple hemiring if every soft hemiring (F, A) over H is an h-semisimple soft hemiring.

Example 4.38. In the above Example 4.36, H is a soft h-semisimple hemiring.

Theorem 4.39. *If H is soft h-semisimple, then H is h-semisimple.*

Proof. Straightforward. □

Theorem 4.40. *If (F, A) is an h-semisimple soft hemiring and $B \subseteq A$, then $(F|_B, B)$ is an h-hemiregular soft hemiring.*

Proof. Straightforward. □

Theorem 4.41. *If H is soft h-semisimple, (F, A) and (G, B) are soft hemirings, then $(F, A) \wedge (G, B)$ is an h-semisimple soft hemiring.*

Proof. The proof is similar to the proof of Theorem 4.8. □

Theorem 4.42. *If H is soft h-semisimple, (F, A) and (G, B) are soft hemirings, then $(F, A)\tilde{\cap}(G, B)$ is an h-semisimple soft hemiring.*

Proof. The proof is similar to the proof of Theorem 4.9. □

Theorem 4.43. *If H is soft h-semisimple, (F, A) and (G, B) are soft hemirings over H in which A and B are disjoint. Then $(F, A)\tilde{\cup}(G, B)$ is an h-semisimple soft hemiring.*

Proof. The proof is similar to the proof of Theorem 4.10. □

Theorem 4.44. *H is soft h-semisimple if and only if every subhemiring of H is h-semisimple.*

Proof. The proof is similar to the proof of Theorem 4.11. □

Theorem 4.45. *H is soft h-semisimple if and only if $(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$ for every pair h-idealistic soft hemirings (F, A) and (G, B) over any subhemiring M of H.*

Proof. Let M be a subhemiring of H, (F, A) and (G, B) h-idealistic soft hemirings over H. Then $F(x)$ and $G(y)$ are h-ideals of M for all $x \in A$ and $y \in B$. Since H is soft h-semisimple, then M is h-semisimple. By Theorem 3.7, $F(x) \cap G(y) = \overline{F(x)G(y)}$, i.e., $(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$.

Conversely, let M be any subhemiring of H, P and Q h-ideals of M. Assume that (F, A) and (G, B) are soft sets over M such that $F(x) = P$ and $G(y) = Q$ for all $x \in A$ and $y \in B$. Thus (F, A) and (G, B) are h-idealistic soft hemirings over M. By the assumption, we have $F(x) \cap G(y) = \overline{F(x)G(y)}$ for all $x \in A$ and $y \in B$, i.e., $P \cap Q = \overline{PQ}$. So M is h-semisimple and H is soft h-semisimple. □

Theorem 4.46. *H is soft h-semisimple if and only if*

$$(F, A) \wedge (G, B) = \overline{(F, A) * (G, B)}$$

for every pair h-interior-idealistic soft hemirings (F, A) and (G, B) over any subhemiring M of H.

Proof. The proof is similar to the proof of Theorem 4.45. □

Theorem 4.47. *If every principle h-idealistic soft hemiring is an h-semisimple soft hemiring, then H is soft h-semisimple.*

Proof. The proof is similar to the proof of Theorem 4.14. □

Theorem 4.48. *Let P and Q be two soft h-semisimple subhemirings of H, then $P \cap Q$ is soft h-semisimple.*

Proof. The proof is similar to the proof of Theorem 4.15. □

Theorem 4.49. *If H is soft h-semisimple, then every subhemiring of H is soft h-semisimple.*

Proof. The proof is similar to the proof of Theorem 4.16. □

Theorem 4.50. *If H is either soft h-hemiregular or soft h-intra-hemiregular, then H is soft h-semisimple.*

Proof. We only show that from H is soft h-hemiregular implies H is soft h-semisimple. The other is similar.

Let M be an h-ideal of H. Since H is soft h-hemiregular, then M is h-hemiregular. Suppose that P and Q are h-ideals of M, we can regard M and N as a right h-ideal and a left h-ideal of M, respectively. Thus $P \cap Q = \overline{PQ}$ and M is h-semisimple. Therefore H is soft h-semisimple. □

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Huajun Wu (whj7480@126.com)
Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province 445000, China

Jianming Zhan (zhanjianming@hotmail.com)
Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province 445000, China