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Characterizations of hemirings by the properties of their interval valued fuzzy ideals

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Abstract. In this paper we give some basic results for interval valued fuzzy ideals and characterize regular and weakly regular hemirings by the properties of their interval valued fuzzy ideals.

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1. Introduction

A set $R \neq \varnothing$ together with two associative binary operations, addition "+" and multiplication "·" such that "·" distributes over "+", is called semiring and was first introduced by Vandiver [18] in 1934. Additively commutative semirings with zero element are called hemirings. In more recent times semirings have been deeply studied, especially in relation with applications [8]. Semirings have also been used for studying optimization, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, theory of automata, formal language theory, coding theory, analysis of computer programmes [5, 6, 8, 9, 19]. Hemirings, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in computer sciences [10, 11, 12, 16]. Ideals of hemirings and semirings play an important role in the structure theory and are very useful for many purposes. Some important types of ideals are discussed in [15, 21]. In [3] J. Ahsan characterize weakly regular hemirings by the properties of their ideals. In 1965 Zadeh [20] introduced the concept of fuzzy sets. Since then fuzzy sets have been applied to many branches in Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [17] and he introduced the notion of fuzzy subgroups. In [2] J. Ahsan initiated the study of fuzzy semirings (See also [4]). The fuzzy algebraic structures play an important role in Mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [1, 10, 19]. The concept of interval valued fuzzy sets in algebra was initiated in [7] by Biswas and further this concept was investigated in [13]. In [14] Xueling defined and discussed interval valued fuzzy ideals in hemirings. In this paper we give some basic results for interval valued fuzzy ideals and characterize regular and weakly regular hemirings by the properties of their interval valued fuzzy ideals.

2. Preliminaries

A set $R \neq \phi$ together with two binary operations addition "+" and multiplication "·" is called semiring if $(R, +)$ and $(R, \cdot)$ are semigroups and multiplication is distributive over addition, that is for all $a, b, c \in R$,

$$a\,(b+c) = ab + ac \text{ and } (a+b)\,c = ac + bc$$

An element $0 \in R$ satisfying the conditions, $0x = x0 = 0$ and $0 + x = x + 0 = x$, for all $x \in R$, is called a zero of the semiring $(R, +, \cdot)$. And an element $1 \in R$ satisfying the condition, $1x = x1 = x$ for all $x \in R$, is called identity of the semiring $(R, +, \cdot)$. A semiring with commutative multiplication is called a commutative semiring. A semiring with commutative addition and zero element is called a hemiring. A non-empty $A \subseteq R$ is called a sub hemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. An element $a \in R$ is called multiplicatively idempotent if $a^2 = a$. A hemiring R is called multiplicatively idempotent if each element of R is multiplicatively idempotent. A non-empty $A \subseteq R$ is called a left (right) ideal of R if A is closed under addition and $RA \subseteq A$ ($AR \subseteq A$). If A and B are left (respectively right) ideals of a hemiring R then $A \cap B$ is a left (respectively right) ideal of R. If A and B are left (respectively right) ideals of a hemiring R then $A + B$ is the smallest left (respectively right) ideal of R containing both A and B. If A and B are ideals of a hemiring R then AB is an ideal of R contained in $A \cap B$.

If $A \subseteq R$, then characteristic function C_A of A is a function from X into $\{0, 1\}$, defined by

$$C_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

A fuzzy subset λ of a universe X is a function $\lambda : X \to [0, 1]$. For any two fuzzy subsets λ and μ of X, $\lambda \leq \mu$ means that, for all $x \in X$, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \wedge \mu$ and $\lambda \vee \mu$ will mean the following fuzzy subsets of X

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$$

for all $x \in X$. More generally, if $\{\lambda_i : i \in \Lambda\}$ is a family of fuzzy subsets of X, then $\wedge_{i\in\Lambda}\lambda_i$ and $\vee_{i\in\Lambda}\lambda_i$ are defined by

$$(\wedge_{i\in\Lambda}\lambda_i)(x) = \wedge_{i\in\Lambda}(\lambda_i(x))$$
$$(\vee_{i\in\Lambda}\lambda_i)(x) = \vee_{i\in\Lambda}(\lambda_i(x))$$

and are called the intersection and the union of the family $\{\lambda_i : i \in \Lambda\}$ of fuzzy subsets of X, respectively.

Definition 2.1. Let λ and μ be any two fuzzy subsets of a hemiring R. Then the sum of λ and μ is defined as

$$(\lambda + \mu)(x) = \vee_{x=y+z} [\lambda(y) \wedge \mu(z)]$$

for all $x \in R$.

Definition 2.2. Let λ and μ be any two fuzzy subsets of a hemiring R. Then the product of λ and μ is defined as

$$(\lambda\mu)(x) = \vee_{x=\Sigma_{i=1}^{n} y_i z_i} [\wedge_i \{\lambda(y_i) \wedge \mu(z_i)\}]$$

for all $x \in R$.

Definition 2.3. A hemiring R is called von Neumann regular if for any $a \in R$ there exists $x \in R$ such that $a = axa$ or $a \in aRa$ for all $a \in R$.

Theorem 2.4. *A hemiring R is von Neumann regular if and only if for any right ideal A and any left ideal B of R, $A \cap B = AB$.*

Definition 2.5 ([3])**.** A hemiring R is called left (respectively right) weakly regular if we have $a \in RaRa$ (respectively $a \in aRaR$) for all $a \in R$.

Remark 2.6. Clearly if R is commutative then R is (right or left) weakly regular if and only if R is von-Neumann regular.

Theorem 2.7 ([4])**.** *The following assertions for a hemiring R are equivalent:*
(i) *R is right weakly regular.*
(ii) *All right ideals of R are idempotent.*
(iii) *$IJ = I \cap J$ for all right ideals I and two-sided ideas J of R.*

Theorem 2.8. *If R is commutative hemiring, then R is fully idempotent if and only if R is von Neumann regular.*

Theorem 2.9. *If R is commutative hemiring then R is fully idempotent iff R is weakly regular.*

Definition 2.10. Let λ be a fuzzy subset of a hemiring R. Then λ is said to be a fuzzy subhemiring of R if for all $x, y \in R$,
(i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$
(ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

Definition 2.11. A fuzzy subset λ of a hemiring R is said to be a fuzzy left (respectively right) ideal of the hemiring R if for all $x, y \in R$
(i) $\lambda(x + y) \geq \lambda(x) \wedge \lambda(y)$,
(ii) $\lambda(xy) \geq \lambda(y)$ (respectively $\lambda(xy) \geq \lambda(x)$).

A fuzzy subset λ of a hemiring R is called a fuzzy ideal of hemiring R if it is both fuzzy left and right ideal of R

Theorem 2.12. *If λ and μ are fuzzy left(respectively right) ideals of a hemiring R then $\lambda \cap \mu$ is also a fuzzy left (respectively right) ideal of R.*

Theorem 2.13. *If λ and μ are fuzzy left(respectively right) ideals of a hemiring R then their sum $\lambda + \mu$ is also a fuzzy left (respectively right) ideal of R.*

Theorem 2.14. *If λ and μ are fuzzy left (respectively right) ideals of a hemiring R then their product $\lambda\mu$ is also a fuzzy left (respectively right) ideal of R.*

3. Major Section

Let Ł be the family of all closed subintervals of $[0,1]$ with minimal element $\tilde{O} = [0,0]$ and maximal element $\tilde{I} = [1,1]$ according to the partial order $[\alpha, \alpha'] \leq [\beta, \beta']$ if and only if $\alpha \leq \beta, \alpha' \leq \beta'$ defined on Ł for all $[\alpha, \alpha'], [\beta, \beta'] \in$ Ł.

Definition 3.1. An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is a function $\tilde{\lambda} : R \to$ Ł.

We write $\tilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)] \subseteq [0,1]$ for all $x \in R$, where $\lambda^-, \lambda^+ : R \to [0,1]$ are lower and upper fuzzy sets of R, giving lower and upper limit of the image interval for each $x \in R$. Note that we have $0 \leq \lambda^-(x) \leq \lambda^+(x) \leq 1$ for all $x \in R$. For simplicity we write $\tilde{\lambda} = [\lambda^-, \lambda^+]$.

Definition 3.2. For any two interval valued fuzzy subsets $\tilde{\lambda}$ and $\tilde{\mu}$ of a hemiring R, union and intersection are defined as

$$\left(\tilde{\lambda} \cup \tilde{\mu}\right)(x) = \left[\lambda^-(x) \vee \mu^-(x), \lambda^+(x) \vee \mu^+(x)\right]$$

$$\left(\tilde{\lambda} \cap \tilde{\mu}\right)(x) = \left[\lambda^-(x) \wedge \mu^-(x), \lambda^+(x) \wedge \mu^+(x)\right],$$

for all $x \in R$.

More generally if $\left\{\tilde{\lambda}_i : i \in I\right\}$ is a family of interval valued fuzzy subsets of R then for all $x \in R$,

$$(\cup_i \tilde{\lambda}_i)(x) = \left[\vee_i \lambda_i^-(x), \vee_i \lambda_i^+(x)\right]$$
$$(\cap_i \lambda_i)(x) = \left[\wedge_i \lambda_i^-(x), \wedge_i \lambda_i^+(x)\right].$$

Definition 3.3. Let $\tilde{\lambda}$ and $\tilde{\mu}$ be interval valued fuzzy subsets of a hemiring R. Then their sum is defined as

$$(\tilde{\lambda} + \tilde{\mu})(x) = \vee_{x=y+z}\left[\lambda^-(y) \wedge \mu^-(z), \lambda^+(y) \wedge \mu^+(z)\right]$$

for all $x \in R$.

Definition 3.4. Let $\tilde{\lambda}$ and $\tilde{\mu}$ be interval valued fuzzy subsets of a hemiring R. Then their product is defined as

$$(\tilde{\lambda}\tilde{\mu})(x) = \vee_{x=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\}$$

if x can be expressed as $x = \sum_{i=1}^n y_i z_i$, otherwise $(\tilde{\lambda}\tilde{\mu})(x) = \tilde{O}$.

Definition 3.5. Let A be a subset of a hemiring R. Then the interval valued characteristic function $\tilde{C}_A$ of A is defined to be a function $\tilde{C}_A : R \to$ Ł such that for all $x \in R$

$$\tilde{C}_A(x) = \begin{cases} \tilde{I} = [1,1] & \text{if } x \in A \\ \tilde{O} = [0,0] & \text{if } x \notin A. \end{cases}$$

Clearly the interval valued characteristic function of any subset of R is also an interval valued fuzzy subset of R. The interval valued characteristic function can be used to indicate either membership or non-membership of any member of R in a subset A of R. Note that $\tilde{C}_R(x) = \tilde{I}$ for all $x \in R$.

Lemma 3.6. *Let $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\nu}$ be the interval valued fuzzy subsets of a hemiring R. If $\tilde{\lambda} \subseteq \tilde{\mu}$ then $\tilde{\lambda}\tilde{\nu} \subseteq \tilde{\mu}\tilde{\nu}$ and $\tilde{\nu}\tilde{\lambda} \subseteq \tilde{\nu}\tilde{\mu}$.*

Proof. Let $x \in R$. If x is not expressed as $x = \sum_{i=1}^{n} y_i z_i$, then

$$\left(\tilde{\lambda}\tilde{\nu}\right)(x) = \tilde{O} = (\tilde{\mu}\tilde{\nu})(x)$$

Otherwise,

$$\begin{aligned}(\tilde{\lambda}\tilde{\nu})(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \nu^-(z_i), \lambda^+(y_i) \wedge \nu^+(z_i)\right]\right\} \\ &\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\mu^-(y_i) \wedge \nu^-(z_i), \mu^+(y_i) \wedge \nu^+(z_i)\right]\right\} \quad \text{since } \lambda \subseteq \mu \\ &= (\tilde{\mu}\tilde{\nu})(x).\end{aligned}$$

Hence $\tilde{\lambda}\tilde{\nu} \subseteq \tilde{\mu}\tilde{\nu}$. Similarly we can prove that $\tilde{\nu}\tilde{\lambda} \subseteq \tilde{\nu}\tilde{\mu}$. □

Definition 3.7. Let $\tilde{\lambda}$ be an interval valued fuzzy subset of a hemiring R. Then $\tilde{\lambda}$ is said to be an interval valued fuzzy subhemiring of R if for all $x, y \in R$,

$$\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y) \text{ and } \tilde{\lambda}(xy) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y).$$

Definition 3.8 ([14])**.** Let $\tilde{\lambda}$ be an interval valued fuzzy subset of a hemiring R. Then $\tilde{\lambda}$ is said to be an interval valued fuzzy left (resp. right) ideal of R if and only if for all $x, y \in R$
(i) $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$
(ii) $\tilde{\lambda}(xy) \geq \tilde{\lambda}(y)$ $\left(\text{respectively } \tilde{\lambda}(xy) \geq \tilde{\lambda}(x)\right)$.

An interval valued fuzzy subset of R is called an interval valued fuzzy ideal of hemiring R if it is both interval valued fuzzy left and right ideal of R.

Remark 3.9. An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is an interval valued fuzzy two sided ideal of R if for all $x, y \in R$
(i) $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$
(ii) $\tilde{\lambda}(xy) \geq \tilde{\lambda}(x) \vee \tilde{\lambda}(y)$.

Remark 3.10. Every interval valued fuzzy ideal of a hemiring R is also an interval valued fuzzy subhemiring of R but the converse is not true.

Remark 3.11. Note that if $\tilde{\lambda} = [\lambda^-, \lambda^+]$ is an interval valued fuzzy left ideal of R then for all $x, y \in R$

$$\begin{aligned}&\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y) \\ &\Rightarrow \left[\lambda^-(x+y), \lambda^+(x+y)\right] \geq \left[\lambda^-(x), \lambda^+(x)\right] \wedge \left[\lambda^-(y), \lambda^+(y)\right] \\ &\Rightarrow \left[\lambda^-(x+y), \lambda^+(x+y)\right] \geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(y) \wedge \lambda^+(y)\right] \\ &\Rightarrow \lambda^-(x+y) \geq \lambda^-(x) \wedge \lambda^-(y) \text{ and } \lambda^+(x+y) \geq \lambda^+(x) \wedge \lambda^+(y)\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}(xy) \geq \tilde{\lambda}(y) &\Rightarrow \left[\lambda^-(xy), \lambda^+(xy)\right] \geq \left[\lambda^-(y), \lambda^+(y)\right] \\ &\Rightarrow \lambda^-(xy) \geq \lambda^-(y) \text{ and } \lambda^+(xy) \geq \lambda^+(y).\end{aligned}$$

This shows that $\lambda^-$and $\lambda^+$are fuzzy left ideals of R. It's converse is also true and can be proved by reversing the above process. Similarly $\tilde{\lambda} = [\lambda^-, \lambda^+]$ is an interval valued fuzzy right (two sided) ideal of R if and only if $\lambda^-$and λ^+ are fuzzy right (two sided) ideals of R. Similarly $\tilde{\lambda} = [\lambda^-, \lambda^+]$ is an interval valued fuzzy subhemiring of R if and only if $\lambda^-$and λ^+ are fuzzy subhemirings of R.

Lemma 3.12. *An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is an interval valued fuzzy subhemiring of R if and only if $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$ and $\tilde{\lambda}^2 \subseteq \tilde{\lambda}$.*

Proof. Let $\tilde{\lambda}$ be an an interval valued fuzzy subhemiring of R. Then for all $x \in R$

$$\begin{aligned}\left(\tilde{\lambda} + \tilde{\lambda}\right)(x) &= \vee_{x=y+z}\left[\lambda^-(y) \wedge \lambda^-(z), \lambda^+(y) \wedge \lambda^+(z)\right] \\ &\leq \vee_{x=y+z}\left[\lambda^-(y+z), \lambda^+(y+z)\right] \\ &= \vee_{x=y+z}\left[\lambda^-(x), \lambda^+(x)\right] \\ &= \tilde{\lambda}(x).\end{aligned}$$

Thus $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$. And $\tilde{\lambda}^2(x) = \tilde{O} \leq \tilde{\lambda}(x)$ if x cannot be expressed as $x = \sum_{i=1}^n y_i z_i$. Otherwise,

$$\begin{aligned}\tilde{\lambda}^2(x) &= \left(\tilde{\lambda}\tilde{\lambda}\right)(x) \\ &= \vee_{x=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right\} \\ &\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\wedge_i \lambda^-(y_i z_i), \wedge_i \lambda^+(y_i z_i)\right] \\ &\leq \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\lambda^-(\sum_{i=1}^n y_i z_i), \lambda^+(\sum_{i=1}^n y_i z_i)\right] \\ &= \vee_{x=\Sigma_{i=1}^n y_i z_i}\left[\lambda^-(x), \lambda^+(x)\right] \\ &= \tilde{\lambda}(x).\end{aligned}$$

Thus $\tilde{\lambda}^2 \subseteq \tilde{\lambda}$.

Conversely, let λ be an interval valued fuzzy subset of R such that $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$ and $\tilde{\lambda}^2 \subseteq \tilde{\lambda}$. Then for all $x, y \in R$

$$\begin{aligned}\tilde{\lambda}(x+y) &\geq \left(\tilde{\lambda} + \tilde{\lambda}\right)(x+y) \\ &= \vee_{x+y=a+b}\left[\lambda^-(a) \wedge \lambda^-(b), \lambda^+(a) \wedge \lambda^+(b)\right] \\ &\geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(x) \wedge \lambda^+(y)\right] \\ &= \left[\lambda^-(x), \lambda^+(x)\right] \wedge \left[\lambda^-(y), \lambda^+(y)\right] \\ &= \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}(xy) &\geq \tilde{\lambda}^2(xy) = \left(\tilde{\lambda}\tilde{\lambda}\right)(xy) \\ &= \vee_{xy=\Sigma_{i=1}^n y_i z_i}\left\{\wedge_i\left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right\} \\ &\geq \left[\lambda^-(x) \wedge \lambda^-(y), \lambda^+(x) \wedge \lambda^+(y)\right] \\ &= \left[\lambda^-(x), \lambda^+(x)\right] \wedge \left[\lambda^-(y), \lambda^+(y)\right] \\ &= \tilde{\lambda}(x) \wedge \tilde{\lambda}(y).\end{aligned}$$

Thus $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ and $\tilde{\lambda}(xy) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ for all $x, y \in R$. Hence $\tilde{\lambda}$ is an interval valued fuzzy subhemiring of R. □

Lemma 3.13. *An interval valued fuzzy subset $\tilde{\lambda}$ of a hemiring R is an interval valued fuzzy left (respectively right) ideal of R if and only if $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$ and $C_R\tilde{\lambda} \subseteq \tilde{\lambda}$ (respectively $\tilde{\lambda}C_R \subseteq \tilde{\lambda}$).*

Proof. Let $\tilde{\lambda}$ be an interval valued fuzzy left ideal of R. Then for all $x \in R$

$$\begin{aligned}
\left(\tilde{\lambda} + \tilde{\lambda}\right)(x) &= \vee_{x=y+z}\left[\lambda^{-}(y) \wedge \lambda^{-}(z), \lambda^{+}(y) \wedge \lambda^{+}(z)\right] \\
&\leq \vee_{x=y+z}\left[\lambda^{-}(y+z), \lambda^{+}(y+z)\right] \\
&= \vee_{x=y+z}\left[\lambda^{-}(x), \lambda^{+}(x)\right] \\
&= \tilde{\lambda}(x).
\end{aligned}$$

Thus $\tilde{\lambda}+\tilde{\lambda} \subseteq \tilde{\lambda}$. And $C_R\tilde{\lambda}(x) = \tilde{O} \leq \tilde{\lambda}(x)$ if x cannot be expressed as $x = \sum_{i=1}^{n} y_i z_i$, otherwise

$$\begin{aligned}
C_R\tilde{\lambda}(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[C_R^{-}(y_i) \wedge \lambda^{-}(z_i), C_R^{+}(y_i) \wedge \lambda^{+}(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[1 \wedge \lambda^{-}(z_i), 1 \wedge \lambda^{+}(z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[\lambda^{-}(z_i), \lambda^{+}(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[\lambda^{-}(y_i z_i), \lambda^{+}(y_i z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left[\lambda^{-}(\sum_{i=1}^{n} y_i z_i), \lambda^{+}(\sum_{i=1}^{n} y_i z_i)\right] \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\left[\lambda^{-}(x), \lambda^{+}(x)\right] \\
&= \tilde{\lambda}(x).
\end{aligned}$$

Thus $C_R\tilde{\lambda} \subseteq \tilde{\lambda}$.

Conversely, let $\tilde{\lambda}$ be an interval valued fuzzy subset of R such that $\tilde{\lambda} + \tilde{\lambda} \subseteq \tilde{\lambda}$ and $C_R\tilde{\lambda} \subseteq \tilde{\lambda}$. Then for all $x, y \in R$

$$\begin{aligned}
\tilde{\lambda}(x+y) &\geq \left(\tilde{\lambda} + \tilde{\lambda}\right)(x+y) \\
&= \vee_{x+y=a+b}\left[\lambda^{-}(a) \wedge \lambda^{-}(b), \lambda^{+}(a) \wedge \lambda^{+}(b)\right] \\
&\geq \left[\lambda^{-}(x) \wedge \lambda^{-}(y), \lambda^{+}(x) \wedge \lambda^{+}(y)\right] \\
&= \left[\lambda^{-}(x), \lambda^{+}(x)\right] \wedge \left[\lambda^{-}(y), \lambda^{+}(y)\right] \\
&= \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\lambda}(xy) &\geq C_R\tilde{\lambda}(xy) \\
&= \vee_{xy=\Sigma_{i=1}^{n} y_i z_i}\left\{\wedge_i\left[C_R^{-}(y_i) \wedge \lambda^{-}(z_i), C_R^{+}(y_i) \wedge \lambda^{+}(z_i)\right]\right\} \\
&\geq \left[C_R^{-}(x) \wedge \lambda^{-}(y), C_R^{+}(x) \wedge \lambda^{+}(y)\right] \\
&= \left[1 \wedge \lambda^{-}(y), 1 \wedge \lambda^{+}(y)\right] \\
&= \left[\lambda^{-}(y), \lambda^{+}(y)\right] \\
&= \tilde{\lambda}(y).
\end{aligned}$$

Thus $\tilde{\lambda}(x+y) \geq \tilde{\lambda}(x) \wedge \tilde{\lambda}(y)$ and $\tilde{\lambda}(xy) \geq \tilde{\lambda}(y)$ for all $x, y \in R$. Hence $\tilde{\lambda}$ is an interval valued fuzzy left ideal of R. □

Theorem 3.14. *A subset A of a hemiring R is a subhemiring of R if and only if the interval valued characteristic function $\tilde{C}_A$ is an interval valued fuzzy subhemiring of R.*

Proof. Suppose that A is a subhemiring of R and $x, y \in R$.

Case I: If $x, y \in A$ then $x+y, xy \in A$. Thus

$$\tilde{C}_A(x+y) = \tilde{I} = \tilde{I} \wedge \tilde{I} = \tilde{C}_A(x) \wedge \tilde{C}_A(y)$$

and $\tilde{C}_A(xy) = \tilde{I} = \tilde{I} \wedge \tilde{I} = \tilde{C}_A(x) \wedge \tilde{C}_A(y)$.

Case II: If at least one, say $y \notin A$ then $\tilde{C}_A(y) = \tilde{O}$. Then

$$\tilde{C}_A(x+y) \geq \tilde{O} = \tilde{C}_A(x) \wedge \tilde{O} = \tilde{C}_A(x) \wedge \tilde{C}_A(y)$$

and

$$\tilde{C}_A(xy) \geq \tilde{O} = \tilde{C}_A(x) \wedge \tilde{O} = \tilde{C}_A(x) \wedge \tilde{C}_A(y).$$

Thus in both cases $\tilde{C}_A(x+y) \geq \tilde{C}_A(x) \wedge \tilde{C}_A(y)$, and$\tilde{C}_A(xy) \geq \tilde{C}_A(x) \wedge \tilde{C}_A(y)$. Hence $\tilde{C}_A$ is an interval valued fuzzy subhemiring of R.

Conversely, suppose that $\tilde{C}_A$ is an interval valued fuzzy subhemiring of R and $x, y \in A$. Then $\tilde{C}_A(x+y) \geq \tilde{C}_A(x) \wedge \tilde{C}_A(y) = \tilde{I} \wedge \tilde{I} = \tilde{I}$ and

$$\tilde{C}_A(xy) \geq \tilde{C}_A(x) \wedge \tilde{C}_A(y) = \tilde{I} \wedge \tilde{I} = \tilde{I}.$$

Thus $x+y, xy \in A$ for all $x, y \in A$. This shows that A is a subhemiring of R. □

Theorem 3.15. *A subset A of a hemiring R is a left (respectively right) ideal of R if and only if the interval valued characteristic function $\tilde{C}_A$ is an interval valued fuzzy left (respectively right) ideal of R.*

Theorem 3.16. *If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of R then their sum $\tilde{\lambda} + \tilde{\mu}$ is also an interval valued fuzzy left (respectively right) ideal of R.*

Proof. To show that $\tilde{\lambda} + \tilde{\mu}$ is an interval valued fuzzy left ideal of R, we will prove that

(i) $\left(\tilde{\lambda} + \tilde{\mu}\right)(x+y) \geq \left(\tilde{\lambda} + \tilde{\mu}\right)(x) \wedge \left(\tilde{\lambda} + \tilde{\mu}\right)(y)$

(ii) $\left(\tilde{\lambda} + \tilde{\mu}\right)(yx) \geq \left(\tilde{\lambda} + \tilde{\mu}\right)(x)$

for all $x, y \in R$. Recall that if $\tilde{\lambda} = [\lambda^-, \lambda^+]$ and $\tilde{\mu} = [\mu^-, \mu^+]$ are interval valued fuzzy left ideals of R then $\lambda^-, \lambda^+, \mu^-$ and μ^+ are fuzzy left ideals of R. Let $x, y \in R$.

Then

$$\begin{aligned}
&\left(\tilde{\lambda}+\tilde{\mu}\right)(x)\wedge\left(\tilde{\lambda}+\tilde{\mu}\right)(y)\\
&=\left\{\vee_{x=a+b}\left[\lambda^-(a)\wedge\mu^-(b),\lambda^+(a)\wedge\mu^+(b)\right]\right\}\\
&\qquad\qquad\qquad\wedge\left\{\vee_{y=u+v}\left[\lambda^-(u)\wedge\mu^-(v),\lambda^+(u)\wedge\mu^+(v)\right]\right\}\\
&=\vee_{x=a+b}\left[\lambda^-(a)\wedge\mu^-(b),\lambda^+(a)\wedge\mu^+(b)\right]\wedge\\
&\qquad\qquad\qquad\left\{\vee_{y=u+v}\left[\lambda^-(u)\wedge\mu^-(v),\lambda^+(u)\wedge\mu^+(v)\right]\right\}\\
&=\vee_{x=a+b,\ y=u+v}\left\{\left[\lambda^-(a)\wedge\mu^-(b),\lambda^+(a)\wedge\mu^+(b)\right]\wedge\right.\\
&\qquad\qquad\qquad\left.\left[\lambda^-(u)\wedge\mu^-(v),\lambda^+(u)\wedge\mu^+(v)\right]\right\}\\
&=\vee_{x=a+b,\ y=u+v}[\lambda^-(a)\wedge\mu^-(b)\wedge\lambda^-(u)\wedge\mu^-(v),\\
&\qquad\qquad\qquad\lambda^+(a)\wedge\mu^+(b)\wedge\lambda^+(u)\wedge\mu^+(v)]\\
&\leq\vee_{x=a+b,\ y=u+v}\left[\lambda^-(a+u)\wedge\mu^-(b+v),\lambda^+(a+u)\wedge\mu^+(b+v)\right]\\
&\leq\vee_{x+y=u'+v'}\left[\lambda^-(u')\wedge\mu^-(v'),\lambda^+(u')\wedge\mu^+(v')\right]\\
&=(\tilde{\lambda}+\tilde{\mu})(x+y)
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\lambda}+\tilde{\mu})(x) &= \vee_{x=u+v}[\lambda^-(u)\wedge\mu^-(v),\lambda^+(u)\wedge\mu^+(v)]\\
&\leq \vee_{x=u+v}\left[\lambda^-(yu)\wedge\mu^-(yv),\lambda^+(yu)\wedge\mu^+(yv)\right]\\
&\leq \vee_{yx=u'+v'}\left[\lambda^-(u')\wedge\mu^-(v'),\lambda^+(u')\wedge\mu^+(v')\right]\\
&= (\tilde{\lambda}+\tilde{\mu})(yx).
\end{aligned}$$

Thus $\left(\tilde{\lambda}+\tilde{\mu}\right)(x+y)\geq\left(\tilde{\lambda}+\tilde{\mu}\right)(x)\wedge\left(\tilde{\lambda}+\tilde{\mu}\right)(y)$ and $\left(\tilde{\lambda}+\tilde{\mu}\right)(yx)\geq\left(\tilde{\lambda}+\tilde{\mu}\right)(x)$ for all $x,y\in R$. Hence $\tilde{\lambda}+\tilde{\mu}$ is an interval valued fuzzy left ideal of R. □

Theorem 3.17. *If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of a hemiring R then $\tilde{\lambda}\tilde{\mu}$ is also an interval valued fuzzy left (respectively right) ideal of R.*

Proof. Let $x,y\in R$. Then

$$\left(\tilde{\lambda}\tilde{\mu}\right)(x)=\vee_{x=\Sigma_{i=1}^{n}y_iz_i}\{\wedge_i\left[\lambda^-\left(y_i\right)\wedge\mu^-\left(z_i\right),\lambda^+\left(y_i\right)\wedge\mu^+\left(z_i\right)\right]\}$$

and

$$\left(\tilde{\lambda}\tilde{\mu}\right)(y)=\vee_{y=\Sigma_{i=1}^{n}y'_iz'_i}\{\wedge_i\left[\lambda^-\left(y_i'\right)\wedge\mu^-\left(z_i'\right),\lambda^+\left(y_i'\right)\wedge\mu^+\left(z_i'\right)\right]\}.$$

Now

$$
\begin{aligned}
(\tilde{\lambda}\tilde{\mu})(x) \wedge (\tilde{\lambda}\tilde{\mu})(y) = &\left\{\vee_{x=\Sigma_{i=1}^{n} y_i z_i}\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\}\right\} \\
&\wedge \left\{\vee_{y=\Sigma_{j=1}^{n} y'_j z'_j} \left\{\wedge_j \left[\lambda^-(y'_j) \wedge \mu^-(z'_j), \lambda^+(y'_j) \wedge \mu^+(z'_j)\right]\right\}\right\} \\
= \vee_{x=\Sigma_{i=1}^{n} y_i z_i}&\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\} \\
&\wedge \left\{\vee_{y=\Sigma_{j=1}^{n} y'_j z'_j} \left\{\wedge_j \left[\lambda^-(y'_j) \wedge \mu^-(z'_j), \lambda^+(y'_j) \wedge \mu^+(z'_j)\right]\right\}\right\} \\
= \vee_{x=\Sigma_{i=1}^{n} y_i z_i,\ y=\Sigma_{j=1}^{n} y'_j z'_j}& \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\
&\wedge \left\{\wedge_j \left[\lambda^-(y'_j) \wedge \mu^-(z'_j), \lambda^+(y'_j) \wedge \mu^+(z'_j)\right]\right\} \\
= \vee_{x+y=\Sigma_{i=1}^{n} y_i z_i+\Sigma_{j=1}^{n} y'_j z'_j}&[\{\wedge_i[\lambda^-(y_i) \wedge \mu^-(z_i)]\} \wedge \{\wedge_j[\lambda^-(y'_j) \wedge \mu^-(z'_j)]\}, \\
&\{\wedge_i[\lambda^+(y_i) \wedge \mu^+(z_i)]\} \wedge \{\wedge_j[\lambda^+(y'_j) \wedge \mu^+(z'_j)]\}] \\
\leq \vee_{x+y=\sum_{k=1}^{r} u_k v_k}&\{\wedge_k[\lambda^-(u_k) \wedge \mu^-(v_k), \lambda^+(u_k) \wedge \mu^+(v_k)]\} \\
= (\tilde{\lambda}\tilde{\mu})(x+y).&
\end{aligned}
$$

For $a, x \in R$

$$
\begin{aligned}
\left(\tilde{\lambda}\tilde{\mu}\right)(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i}\{\wedge_i \left[\lambda^-(ay_i) \wedge \mu^-(z_i), \lambda^+(ay_i) \wedge \mu^+(z_i)\right]\} \\
&= \vee_{ax=\Sigma_{i=1}^{n} ay_i z_i}\{\wedge_i \left[\lambda^-(ay_i) \wedge \mu^-(z_i), \lambda^+(ay_i) \wedge \mu^+(z_i)\right]\} \\
&\leq \vee_{ax=\Sigma_{i=1}^{n} y'_i z'_i}\{\wedge_i \left[\lambda^-(y'_i) \wedge \mu^-(z'_i), \lambda^+(y'_i) \wedge \mu^+(z'_i)\right]\} \\
&= (\tilde{\lambda}\tilde{\mu})(ax).
\end{aligned}
$$

Thus $(\tilde{\lambda}\tilde{\mu})(x+y) \geq (\tilde{\lambda}\tilde{\mu})(x) \wedge (\tilde{\lambda}\tilde{\mu})(y)$ and $\left(\tilde{\lambda}\tilde{\mu}\right)(ax) \geq \left(\tilde{\lambda}\tilde{\mu}\right)(x)$ for all $a, x, y \in R$. Hence $\tilde{\lambda}\tilde{\mu}$ is an interval valued fuzzy left ideal of R. □

Remark 3.18. If $\tilde{\lambda}$ and $\tilde{\mu}$ are interval valued fuzzy left (respectively right) ideals of R then $\tilde{\lambda} \cap \tilde{\mu}$ is an interval valued fuzzy left (respectively right) ideal of R.

In general, $\tilde{\lambda} \cap \tilde{\mu} \neq \tilde{\lambda}\tilde{\mu}$. Now we characterize regular and weakly regular hemirings by the properties of their interval valued fuzzy ideals.

Theorem 3.19. *The following assertions for a hemiring R are equivalent:*
(i) *R is von Neumann regular.*
(ii) *For any right ideal A and any left ideal B of R, $A \cap B = AB$.*
(iii) *For any interval valued fuzzy right ideal $\tilde{\lambda}$ and interval valued fuzzy left ideal μ, $\tilde{\lambda} \cap \tilde{\mu} = \tilde{\lambda}\tilde{\mu}$.*

Proof. (i) ⇔ (ii) follows from Theorem 2.4.

(i) $\Rightarrow$ (iii) Let $\tilde{\lambda}$ be an interval valued fuzzy right ideal and $\tilde{\mu}$ be an interval valued fuzzy left ideal of R. Then for any $x \in R$

$$\begin{aligned}
\left(\tilde{\lambda}\tilde{\mu}\right)(x) &= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \mu^-(y_i z_i), \lambda^+(y_i z_i) \wedge \mu^+(y_i z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left[\left\{\wedge_i \tilde{\lambda}(y_i z_i)\right\} \wedge \left\{\wedge_i \tilde{\mu}(y_i z_i)\right\}\right] \\
&\leq \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\tilde{\lambda}\left(\sum_{i=1}^{n} y_i z_i\right) \wedge \tilde{\mu}\left(\sum_{i=1}^{n} y_i z_i\right)\right\} \\
&= \vee_{x=\Sigma_{i=1}^{n} y_i z_i} \left\{\tilde{\lambda}(x) \wedge \tilde{\mu}(x)\right\} \\
&= \tilde{\lambda}(x) \wedge \tilde{\mu}(x) \\
&= \left(\tilde{\lambda} \cap \tilde{\mu}\right)(x).
\end{aligned}$$

Thus $\tilde{\lambda}\tilde{\mu} \subseteq \tilde{\lambda} \cap \tilde{\mu}$. Since R is regular so for $x \in R$ there exists $a \in R$ such that $x = xax$. Now $\left(\tilde{\lambda} \cap \tilde{\mu}\right)(x) = \tilde{\lambda}(x) \wedge \tilde{\mu}(x) \leq \tilde{\lambda}(x) \wedge \tilde{\mu}(ax)$. Thus

$$\left(\tilde{\lambda} \cap \tilde{\mu}\right)(x) \leq \vee_{x=\Sigma_{j=1}^{m} y_j z_j} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} = \left(\tilde{\lambda}\tilde{\mu}\right)(x).$$

This implies $\tilde{\lambda} \cap \tilde{\mu} \subseteq \tilde{\lambda}\tilde{\mu}$, and hence $\tilde{\lambda}\tilde{\mu} = \tilde{\lambda} \cap \tilde{\mu}$.

(iii) $\Rightarrow$ (i) Let I be a right ideal and J be a left ideal of R. Then the interval valued characteristic functions $\tilde{C}_I$ and $\tilde{C}_J$ are interval valued fuzzy right and interval valued fuzzy left ideals of R and by hypothesis

$$\tilde{C}_{IJ} = \tilde{C}_I.\tilde{C}_J = \tilde{C}_I \cap \tilde{C}_J$$

which implies that $\tilde{C}_{IJ} = \tilde{C}_{I \cap J}$. Thus $IJ = I \cap J$. Hence R is regular. □

Theorem 3.20. *The following assertions for a hemiring R with 1 are equivalent:*
(i) *R is right weakly regular.*
(ii) *All right ideals of R are idempotent.*
(iii) *$IJ = I \cap J$ for all right ideals I and two-sided ideas J of R.*
(iv) *All interval valued fuzzy right ideals of R are fully idempotent.*
(v) *$\tilde{\lambda}\tilde{\mu} = \tilde{\lambda} \cap \tilde{\mu}$ for all interval valued fuzzy right ideals $\tilde{\lambda}$ and interval valued fuzzy two-sided ideals $\tilde{\mu}$ of R.*
If R is commutative, then the above assertions are equivalent to
(vi) *R is von-Neumann regular.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (vi) follows from Theorem 2.7.

(i)⇒ (iv) Let $\tilde{\lambda}$ be an interval valued fuzzy right ideal of R and $x \in R$. Then

$$\begin{aligned}
\tilde{\lambda}^2(x) &= \left(\tilde{\lambda}.\tilde{\lambda}\right)(x) \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \lambda^-(z_i), \lambda^+(y_i) \wedge \lambda^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \lambda^-(z_i), \lambda^+(y_i z_i) \wedge \lambda^+(z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\left\{\wedge_i \lambda^-(y_i z_i)\right\} \wedge \lambda^-(z_i), \left\{\wedge_i \lambda^+(y_i z_i)\right\} \wedge \lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-\left(\sum_{i=1}^n y_i z_i\right) \wedge \lambda^-(z_i), \lambda^+\left(\sum_{i=1}^n y_i z_i\right) \wedge \lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-(x) \wedge \lambda^-(z_i), \lambda^+(x) \wedge \lambda^+(z_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\lambda^-(x), \lambda^+(x)\right] \\
&= \tilde{\lambda}(x)
\end{aligned}$$

and so $\tilde{\lambda}^2 \subseteq \tilde{\lambda}$. Now since R is right weakly regular so $x \in xRxR$. Hence we can write $x = \sum_{i=1}^n xa_i xb_i$ where $a_i, b_i \in R$ and $n \in \mathbb{N}$. Now $\tilde{\lambda}(x) \wedge \tilde{\lambda}(x) \leq \tilde{\lambda}(xa_i) \wedge \tilde{\lambda}(xb_i)$ for all i, which implies that

$$\begin{aligned}
\lambda(x) &\leq \wedge_i \{\lambda(xa_i) \wedge \lambda(xb_i)\} \\
&= \wedge_i \left[\lambda^-(xa_i) \wedge \lambda^-(xb_i), \lambda^+(xa_i) \wedge \lambda^+(xb_i)\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n xa_i xb_i} \left\{\wedge_i \left[\lambda^-(xa_i) \wedge \lambda^-(xb_i), \lambda^+(xa_i) \wedge \lambda^+(xb_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{j=1}^m y_j z_j} \left\{\wedge_j \left[\lambda^-(y_j) \wedge \lambda^-(z_j), \lambda^+(y_j) \wedge \lambda^+(z_j)\right]\right\} \\
&= \tilde{\lambda}.\tilde{\lambda}(x) \\
&= \tilde{\lambda}^2(x).
\end{aligned}$$

Hence $\tilde{\lambda} \subseteq \tilde{\lambda}^2$, and thus $\tilde{\lambda} = \tilde{\lambda}^2$. Therefore λ is fully idempotent.

(iv)⇒ (i) Let $x \in R$ and let $A = xR$ be a right ideal of R generated by x. Then $x \in A$ and the characteristic function $\tilde{C}_A$ of A is interval valued fuzzy right ideal of R and by hypothesis $\tilde{C}_A = \tilde{C}_A \cdot \tilde{C}_A = \tilde{C}_{A^2}$. Hence $A = A^2$ and so $x \in A^2 = (xR)^2$ because $x \in A$. Hence $x \in xRxR$. Thus R is right weakly regular.

(i) ⇒ (v) Let λ be an interval valued fuzzy right ideal and $\tilde{\mu}$ be an interval valued fuzzy two-sided ideal of R. Then for any $x \in R$

$$\begin{aligned}
\left(\tilde{\lambda}\tilde{\mu}\right)(x) &= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\wedge_i \left[\lambda^-(y_i z_i) \wedge \mu^-(y_i z_i), \lambda^+(y_i z_i) \wedge \mu^+(y_i z_i)\right]\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left[\{\wedge_i \lambda(y_i z_i)\} \wedge \{\wedge_i \tilde{\mu}(y_i z_i)\}\right] \\
&\leq \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\tilde{\lambda}\left(\sum_{i=1}^n y_i z_i\right) \wedge \tilde{\mu}\left(\sum_{i=1}^n y_i z_i\right)\right\} \\
&= \vee_{x=\Sigma_{i=1}^n y_i z_i} \left\{\tilde{\lambda}(x) \wedge \tilde{\mu}(x)\right\} \\
&= \tilde{\lambda}(x) \wedge \tilde{\mu}(x) \\
&= \left(\tilde{\lambda} \cap \tilde{\mu}\right)(x)
\end{aligned}$$

and so $\tilde{\lambda}\tilde{\mu} \subseteq \tilde{\lambda} \cap \tilde{\mu}$. Since R is right weakly regular so each $x \in R$ can be written as $x = \sum_{i=1}^{n} xa_i x b_i$, where $a_i, b_i \in R$, $n \in \mathbb{N}$. Now

$$\left(\tilde{\lambda} \cap \tilde{\mu}\right)(x) = \tilde{\lambda}(x) \wedge \tilde{\mu}(x) \leq \tilde{\lambda}(xa_i) \wedge \tilde{\mu}(xb_i)$$

for all i and thus

$$\begin{aligned} (\lambda \cap \tilde{\mu})(x) &\leq \wedge_i \{(\lambda(xa_i) \wedge \mu(xb_i))\} \\ &= \wedge_i \left[\lambda^-(xa_i) \wedge \mu^-(xb_i), \lambda^+(xa_i) \wedge \mu^+(xb_i)\right] \\ &\leq \vee_{x=\Sigma_{i=1}^{n} xa_i xb_i} \left\{\wedge_i \left[\lambda^-(xa_i) \wedge \mu^-(xb_i), \lambda^+(xa_i) \wedge \mu^+(xb_i)\right]\right\} \\ &\leq \vee_{x=\Sigma_{j=1}^{m} y_j z_j} \left\{\wedge_i \left[\lambda^-(y_i) \wedge \mu^-(z_i), \lambda^+(y_i) \wedge \mu^+(z_i)\right]\right\} \\ &= (\lambda\tilde{\mu})(x) \end{aligned}$$

which implies $\lambda \cap \tilde{\mu} \subseteq \lambda\tilde{\mu}$. Hence $\lambda\tilde{\mu} = \lambda \cap \tilde{\mu}$.

(v) $\Rightarrow$ (iii) Let I be a right ideal and J be a two-sided ideal of R. Then the characteristic functions $\tilde{C}_I$ and $\tilde{C}_J$ are interval valued fuzzy right and interval valued fuzzy two-sided ideals of R and by hypothesis

$$\tilde{C}_{IJ} = \tilde{C}_I . \tilde{C}_J = \tilde{C}_I \cap \tilde{C}_J$$

which implies $\tilde{C}_{IJ} = \tilde{C}_{I \cap J}$. Hence $IJ = I \cap J$. □

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