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The improved $\left(\frac{G'}{G}\right)$-expansion method for solving the fifth-order KdV equation

T. A. Nofel, M. Sayed, Y. S. Hamed, S. K. Elagan

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Abstract. In this paper, the improved $\left(\frac{G'}{G}\right)-$ expansion method is used for construct explicit the traveling wave solution involving parameters of the fifth- order KdV equation, where $G = G(\xi)$ satisfies a second order linear differential equation. The travelling wave solution is expressed by the rational functions.

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Corresponding Author: Y. S. Hamed (eng_yaser_salah@yahoo.com)

1. Introduction

One can easily observe that searching for explicit solutions of nonlinear evolution equations by using various methods (see for example [1-49]) have been the object of extensive study from differential equations theorists. Many powerful methods have been presented by those authors such as the inverse scattering transform [3], the Backlund transform [14,15], the generalized Riccati equation [17,28], the Jacobi elliptic function expansion [7,13,26,28,30,34,37], the extended tanh- function method [1,8,35,36,45], the F-expansion method [2,19-21], the exp-function expansion method [6,9,31,43,44] the sub- ODE method [14,22], the extended sinh-cosh and sine-cosine methods [23], the complex hyperbolic function method [38], the truncated Painleve expansion [41], homotopy perturbation method [50, 51] and so on. The main purpose of this paper is to use a simple method which is called the improved $\left(\frac{G'}{G}\right)$- expansion method [5,25,40,41,48, 49]. This method is firstly proposed by Wang et al [25] for which the traveling wave solutions of nonlinear equations are obtained. The performance of this method is reliable, simple and gives many new solutions, its also standard and computerizable method which enable us to solve complicated nonlinear evolution equations in mathematical physics. The paper is organized as follows. In Section 2, we describe briefly the improved $\left(\frac{G'}{G}\right)$- expansion method, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$, where$\xi = lx - Vt$, where l and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the non-linear terms appearing in the given nonlinear equations. In Sections 3, we apply this method to the fifth-order Kdv equation. In section 4 some conclusions are given.

2. Description the improved $\frac{G'}{G}$− expansion method

In this section we will describe the improved $\left(\frac{G'}{G}\right)$-expansion method for finding out the traveling wave solutions of nonlinear evolution equations.

Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots) = 0 \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps [25] of the improved $\left(\frac{G'}{G}\right)$-expansion method:

Step 1. The traveling wave variable

$$u = u(\xi), \xi = lx - Vt \tag{2.2}$$

where l and V are constants, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P\left(u,\ -Vu', lu',\ V^2u'', -lVu'',\ l^2u'', ...\right) = 0 \tag{2.3}$$

Step 2. Suppose the solution of Eq. (2.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=\ -n}^{n} \alpha_i \left(\frac{G'}{G}\right)^i \tag{2.4}$$

where $G = G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{2.5}$$

where α_i, λ and μ are constants to be determined later. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

Step 3. By substituting (2.4) into Eq. (2.3) and using the second order linear ODE (2.5), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq. (2.3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$ Equating each coefficient of this polynomial to zero yields a set of algebraic equations for α_i, λ, μ and V.

Step 4. Since the general solution of Eq. (2.5) has been well known for us, then substituting α_i, V and the general solution of Eq. (2.5) into (2.4) we have more traveling wave solutions of the nonlinear partial differential equation (2.1).

3. The fifth-order Kdv equation

In this section, we apply the improved $\left(\frac{G'}{G}\right)$- expansion method to construct the traveling wave solutions for the fifth-order Kdv equation [10]as follows:

Let us first consider the fifth-order Kdv equation

$$u_t + uu_x + u_{xxxxx} = 0 \tag{3.1}$$

which was recently solved by Inan and ugurlu [10] using exp-function expansion method. In order to look for the traveling wave solution of Eq.(3.1) we suppose that

$$u(x,t) = u(\xi), \xi = lx - Vt \tag{3.2}$$

By using the traveling wave variable (3.2), Eq.(3.1) is converted into the following ODE for $u = u(\xi)$.

$$-Vu' + luu' + l^5 u^{(5)} = 0$$

By integrating we have

$$C - Vu + \frac{l}{2}u^2 + l^5 u^{(4)} = 0 \tag{3.3}$$

where C is the integration constant. Suppose that the solution of ODE (3.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows

$$u(\xi) = \sum_{i=-n}^{n} \alpha_i \left(\frac{G'}{G}\right)^i \tag{3.4}$$

where α_i are arbitrary constants and $G = G(\xi)$ satisfies the following second order linear ordinary differential equation (2.5).

considering the homogeneous balance between $u^{(4)}$ and u'^2 in Eq. (3.2) we required that

$n + 4 = 2n$, then $n = 4$,so we can write (3.4) as

$$u(\xi) = \sum_{i=-4}^{4} \alpha_i \left(\frac{G'}{G}\right)^i \tag{3.5}$$

Substituting (3.5) into (3.3) along with (2.5), collecting all terms with the same powers of $\left(\frac{G'}{G}\right)$ and setting them to zero. Consequently, we have the following system of algebraic equations:

$$840l^5\mu^4\alpha_{-4} + \frac{1}{2}l\alpha_{-4}^2 = 0$$

$$l\alpha_{-4}\alpha_{-3} + 2640l^5\mu^3\lambda\alpha_{-4} + 360l^5\mu^4\alpha_{-3} = 0$$

$$l\alpha_{-4}\alpha_{-2} + 120l^5\mu^4\alpha_{-2} + 2080l^5\mu^3\alpha_{-4} + 1080l^5\mu^3\lambda\alpha_{-3} + 3020l^5\mu^2\lambda^2\alpha_{-4} + \frac{1}{2}l\alpha_{-3}^2 = 0$$

$$1476l^5\mu\lambda^3\alpha_{-4}+1164l^5\mu^2\lambda^2\alpha_{-3}+l\alpha_{-1}\alpha_{-4}+816l^5\mu^3\alpha_{-3}$$
$$+4608l^5\mu^2\lambda\alpha_{-4}+336l^5\mu^3\lambda\alpha_{-2}+l\alpha_{-3}\alpha_{-2}+24l^5\mu^4\alpha_{-1}=0$$

$$3232l^5\mu\lambda^2\alpha_{-4}+1696l^5\mu^2\alpha_{-4}+256l^5\lambda^4\alpha_{-4}+l\alpha_{-4}\alpha_0+330l^5\lambda^2\mu^2\alpha_{-2}$$
$$+\frac{1}{2}l\alpha_{-2}^2+60l^5\mu^3\lambda\alpha_{-1}+525l^5\mu\lambda^3\alpha_{-3}-V\alpha_{-4}+l\alpha_{-3}\alpha_{-1}+1680l^5\lambda\mu^2\alpha_{-3}$$
$$+240l^5\mu^3\alpha_{-2}=0$$

$$130l^5\mu\lambda^3\alpha_{-2}+l\alpha_{-3}\alpha_{-1}+576l^5\mu^2\alpha_{-3}+50l^5\lambda^2\mu^2\alpha_{-1}+1062l^5\lambda^2\mu\alpha_{-3}$$
$$+2240l^5\lambda\mu\alpha_{-4}+700l^5\lambda^3\alpha_{-4}+l\alpha_{-4}\alpha_1+81l^5\lambda^4\alpha_{-3}+40l^5\mu^3\alpha_{-1}$$
$$+440l^5\lambda\mu^2\alpha_{-2}+l\alpha_{-3}\alpha_0-V\alpha_{-3}=0$$

$$l\alpha_{-3}\alpha_1+l\alpha_{-2}\alpha_0+660l^5\lambda^2\alpha_{-4}+480l^5\mu\alpha_{-4}+15l^5\lambda^3\mu\alpha_{-1}+60l^5\lambda\mu^2\alpha_{-1}$$
$$+\frac{1}{2}l\alpha_{-1}^2+195l^5\lambda^3\alpha_{-3}-V\alpha_{-2}+660l^5\lambda\mu\alpha_{-3}+l\alpha_{-4}\alpha_2+16l^5\lambda^4\alpha_{-2}$$
$$+136l^5\mu^2\alpha_{-2}+232l^5\lambda^2\alpha_{-2}\mu=0$$

$$30l^5\lambda^3\alpha_{-2}+l\alpha_{-4}\alpha_3+l\alpha_{-1}\alpha_0-V\alpha_{-1}+22l^5\lambda^2\mu\alpha_{-1}+l^5\lambda^4\alpha_{-1}+120l^5\lambda\mu\alpha_{-2}$$
$$+l\alpha_{-3}\alpha_2+240l^5\lambda\alpha_{-4}+150l^5\lambda^2\alpha_{-3}+l\alpha_{-2}\alpha_1+120l^5\mu\alpha_{-3}+16l^5\mu^2\alpha_{-1}=0$$

$$-V\alpha_0+36l^5\lambda\alpha_{-3}+14l^5\lambda^2\alpha_{-2}+l\alpha_{-4}\alpha_4+4l^5\mu^4\alpha_{-4}+l\alpha_{-1}\alpha_1+l^5\lambda^3\alpha_{-1}$$
$$+16l^5\mu^3\alpha_2+l\alpha_{-2}\alpha_2+l^5\lambda^3\alpha_1\mu+l\alpha_{-3}\alpha_3+16l^5\mu\alpha_{-2}+C+\frac{1}{2}l\alpha_0^2$$
$$+14l^5\lambda^2\alpha_2\mu^2+36l^5\lambda\alpha_3\mu^3+8l^5\lambda\alpha_1\mu^2+8l^5\lambda\alpha_{-1}\mu+24l^5\alpha_{-4}=0$$

$$l^5\lambda^4\alpha_1+l\alpha_{-2}\alpha_3+16l^5\mu^2\alpha_1+l\alpha_{-1}\alpha_2+120l^5\lambda\alpha_2\mu^2-V\alpha_1+150l^5\lambda^2\alpha_3\mu^2$$
$$+22l^5\lambda^2\mu\alpha_1+l\alpha_{-3}\alpha_4+120l^5\mu^3\alpha_3+240l^5\lambda\mu^3\alpha_4+30l^5\lambda^3\mu\alpha_2+l\alpha_0\alpha_1=0$$

$$-V\alpha_2+l\alpha_{-2}\alpha_4+660l^5\lambda\alpha_3\mu^2+480l^5\mu^3\alpha_4+l\alpha_{-1}\alpha_3+136l^5\mu^2\alpha_2+195l^5\lambda^3\alpha_3\mu$$
$$+660l^5\lambda^2\alpha_4\mu^2+l\alpha_0\alpha_2+15l^5\lambda^3\alpha_1+232l^5\lambda^2\alpha_2\mu+60l^5\lambda\alpha_1\mu+\frac{1}{2}l\alpha_1^2+16l^5\lambda^4\alpha_2=0$$

$$50l^5\lambda^2\alpha_1+l\alpha_0\alpha_3+700l^5\lambda^3\mu\alpha_4+130l^5\lambda^3\alpha_2+576l^5\mu^2\alpha_3+440l^5\lambda\alpha_2\mu+l\alpha_1\alpha_2$$
$$-V\alpha_3+2240l^5\lambda\alpha_4\mu^2+1062l^5\lambda^2\mu\alpha_3+40l^5\mu\alpha_1+81l^5\lambda^4\alpha_3+l\alpha_{-1}\alpha_4=0$$

$$240l^5\mu\alpha_2+l\alpha_0\alpha_4-V\alpha_4+525l^5\lambda^3\alpha_3+l\alpha_1\alpha_3+330l^5\lambda^2\alpha_2+1696l^5\mu^2\alpha_4+60l^5\lambda\alpha_1$$
$$+\frac{1}{2}l\alpha_2^2+3232l^5\lambda^2\mu\alpha_4+256l^5\lambda^4\alpha_4+1680l^5\lambda\mu\alpha_3=0$$

$$l\alpha_2\alpha_3+336l^5\lambda\alpha_2+816l^5\mu\alpha_3+1476l^5\lambda^3\alpha_4+l\alpha_1\alpha_4+1164l^5\lambda^2\alpha_3+4608l^5\lambda\alpha_4\mu$$
$$+24l^5\alpha_1=0$$

$$2080l^5\mu\alpha_4+120l^5\alpha_2+l\alpha_2\alpha_4+\frac{1}{2}l\alpha_3^2+3020l^5\lambda^2\alpha_4+1080l^5\lambda\alpha_3=0$$

$$l\alpha_3\alpha_3+2640l^5\lambda\alpha_4+360l^5\alpha_3=0$$

$$840l^5\alpha_4+\frac{1}{2}l\alpha_4^2=0$$

On solving the above algebraic equations using the Maple, we obtain two cases

The first case:

$$\alpha_{-4} = -\frac{105}{16}l^4\lambda^8,\ \alpha_{-3} = -\frac{105}{2}l^4\lambda^7, \alpha_{-2} = -\frac{315}{2}l^4\lambda^6,\ \alpha_{-1} = -210l^4\lambda^5,$$
$$\alpha_0 = \alpha_0,\ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0,$$
$$C = 105\alpha_0 l^5\lambda^4 + \frac{1}{2}l\alpha_0^2 + \frac{11025}{2}l^9\lambda^8, V = 105l^5\lambda^4 + l\alpha_0,\ \lambda = \lambda,\ \mu = \frac{1}{4}\lambda^2$$

The second case:

$$\alpha_{-4} = \alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0,\ \alpha_0 = \alpha_0, \alpha_1 = -840l^4\lambda^3,\ \alpha_2 = -2520l^4\lambda^2,$$
$$\alpha_3 = -3360l^4\lambda,\ \alpha_4 = -1680l^4,$$
$$C = 105\alpha_0 l^5\lambda^4 + \frac{1}{2}l\alpha_0^2 + \frac{21525}{4}l^9\lambda^8, V = 105l^5\lambda^4 + l\alpha_0,\ \lambda = \lambda,\ \mu = \frac{1}{4}\lambda^2$$

For the first case: Expression (3.5) can be written as

$$u(\xi) = -\frac{105}{16}l^4\lambda^8 \left(\frac{G'}{G}\right)^{-4} - \frac{105}{2}l^4\lambda^7 \left(\frac{G'}{G}\right)^{-3} - \frac{315}{2}l^4\lambda^6 \left(\frac{G'}{G}\right)^{-2}$$

$$-210l^4\lambda^5\left(\frac{G'}{G}\right)^{-1} + \alpha_0 \tag{3.6}$$

where $\xi = lx - \left(105l^5\lambda^4 + l\alpha_0\right)t$.

According to this case , the solution of Eq.(2.5) is given by,

$$G = e^{-\frac{\lambda}{2}\xi}\ (C_1 + C_2\xi)$$

$$\frac{G'}{G} = \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2} \tag{3.7}$$

where C_1 and C_2 are arbitrary constants. Substituting Eq. (3.7) into (3.6) , we obtain the travelling wave solution of the fifth-order Kdv equation (3.1) as follows:

$$\begin{aligned} u_1(\xi) = \quad & -\frac{105}{16}l^4\lambda^8 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^{-4} - \frac{105}{2}l^4\lambda^7 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^{-3} \\ & -\frac{315}{2}l^4\lambda^6 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^{-2} - 210l^4\lambda^5\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^{-1} + \alpha_0. \end{aligned}$$

This solution can be simplified to

$$u_1(\xi) = \frac{V}{l} - 1680l^4\lambda^4\left(\frac{C_2}{-2C_2 + \lambda C_1 + \lambda C_2\xi}\right)^4$$

This is error 5 in [52], where $\xi = lx - \left(105l^5\lambda^4 + l\alpha_0\right)t$, C_1 and C_2 are arbitrary constants. We see that this solution has too many constants in it. We better write

$$u_1(\xi) = \alpha_0 + 105l^4\lambda^4 - 1680l^4\lambda^2\left(\frac{1}{\xi - \xi_0}\right)^4.$$

This is error 7 in [52].

For the second case : The second solution is given

$$\begin{aligned} u_2(\xi) &= \alpha_0 - 840 l^4 \lambda^3 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) - 2520 l^4 \lambda^2 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^2 \\ & \quad -3360 l^4 \lambda \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^3 - 1680^4 \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^4 \end{aligned}$$

This solution can be simplified to

$$u_2(\xi) = \alpha_0 + 105 l^4 \lambda^4 - 1680 l^4 \left(\frac{C_2}{C_1 + C_2\xi}\right)^4.$$

We see that the solution has too many constants in it. It should be written as

$$u_2(\xi) = \alpha_0 + 105 l^4 \lambda^4 - 1680 l^4 \lambda^2 \left(\frac{1}{\xi - \xi_0}\right)^4.$$

so u_1 and u_2 are the same solution.

4. Conclusions

In this paper, we have seen the main idea of the improved $\left(\frac{G'}{G}\right)$- expansion method, which is that the traveling wave solutions of nonlinear partial differential equations can be expressed as polynomials in $\frac{G'}{G}$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation
$G'' + \lambda G' + \mu G = 0$, where $\xi = lx - Vt$, where l, and V are constants. By using this method, we have obtained an explicit exact solution for complicated nonlinear evolution equation in the mathematical physics. Also in this article we obtained the traveling wave solution in term of rational function for the fifth-order Kdv equation.

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T. A. Nofel (ta_nofal@hotmail.com) – Department of Mathematics, Faculty of Science, El-Minia University, El-Minia, Egypt.

M. Sayed (moh_6_11@yahoo.com) – Department of Engineering Mathematics, Faculty of Electronic Engineering, Menouf 32952, Menofia University, Egypt.

Y. S. Hamed (eng_yaser_salah@yahoo.com) – Department of Engineering Mathematics, Faculty of Electronic Engineering, Menouf 32952, Menofia University, Egypt.

S. K. Elagan (sayed_khalil2000@yahoo.com) – Department of Mathematics, Faculty of Science, Shebin El-Kom, Menofia University, Egypt.

The Present adress for all authors – Department of Mathematics,Faculty of Science,Taif university, Taif , El-Haweiah, P.O.Box 888, Zip Code 21974, Kingdom of Saudi Arabia (KSA).