Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 3, No. 1, (January 2012), pp. 89- 101
ISSN 2093–9310
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

Fuzzy regions with holes and their topological relations in a special fuzzy topological space

Dibyajyoti Hazarika, Debajit Hazarika

Received 8 April 2011; Revised 25 May 2011; Accepted 28 May 2011

Abstract. In this paper a formal definition of a fuzzy region with holes in a crisp fuzzy topological space is proposed. Based on the same, we have obtained various topological relations between a fuzzy point and a fuzzy region with holes, a fuzzy line and a fuzzy region with holes as well as the relations between fuzzy regions with and without holes.

2010 AMS Classification: 54A40

Keywords: Fuzzy topological spaces, Fuzzy regions, Holes.

Corresponding Author: Dibyajyoti Hazarika (dibya@tezu.ernet.in)

1. Introduction

A fuzzy region is a region with imprecise boundary allowing flexibility of strict belongingness criteria of a point in space in relation to the region. Sometimes natural phenomena have discontinuity at the boundary and exterior in the form of cavities requiring representation of a fuzzy region with holes when they are modeled for application purpose such as GIS. Occurrence of oil underground, island in a river, puddle of water near coastline are some examples of regions with holes where the first object represents a hole on the second object. Holes are placed in host material that surround them and therefore cannot occur alone unless a surface for its occurrence is provided by the host. A fuzzy region was initially defined by Schneider[12] in terms of fuzzy open sets. A fuzzy region can be considered to be an extension of a crisp region that allows flexibility in belongingness. Since, crisp regions are usually considered to be closed sets, Schneider's definition of a fuzzy region appeared inconsistent with the definition of a crisp region. Tang [14] provided two definitions of a fuzzy region - one in a special type of fuzzy topological space viz., a crisp fuzzy topological space and the other in a general fuzzy topological space. His definition considered fuzzy regions to be closed sets making it consistent with its classical counterpart. However, incorporation of holes in a fuzzy region has not been considered in any of the approaches.

Egenhofer and Franzosa [4] provided the first framework for determining binary topological relations between two objects with connected boundary. They decomposed a spatial object into its boundary and interior and derived the intersection matrix between these parts of the objects by considering the emptiness and non-emptiness of the content. Egenhofer et. al. [5] also provided the concept of a region with holes in a classical topological sense. Visually a region with a single hole in Egenhofer's sense is similar to the region with indeterminate boundary of Cohn and Gotts [1], but their topologies are different. Egenhofer et. al. [6] also proposed a model of a region with holes in a classical topological space and determined the topological relations between two regions with finite number of holes. They considered two different cases: regions with holes that are completely surrounded by the interior of the region excluding spikes and holes that touch the boundary or another hole and region with hole that does not allow spikes but allows holes that touch the boundary of the region or another hole. Schneider [13] considered complex objects as a connection of large number of simple regions with holes and derived topological relations between them. Egenhofer and Vasardani [7] determined the topological relations between a region without hole and a region with hole. Further, they provided a model to derive topological relations between regions each with a single hole as well as between a region without hole and a region with multi-holes [8, 9]. All these models however, are not equipped to accommodate the uncertainty or imprecision which are typically present in a real object. Fuzzy topology therefore can be used to accommodate uncertainty of an object by assigning suitable membership grades to each point of the region. Du et. al. [2] fuzzified the famous 9-intersection model using fuzzy interior, boundary and exterior and partitioned the space to derive the topological relations. Zhan [16] considered a fuzzy region as a referential set and used it to represent topological relations using the α-cut operation.

In this paper we have proposed a definition of a fuzzy region with holes in crisp fuzzy topological space and derived topological relations in various cases.

2. Fuzzy topology

The following definitions and results can be found in [10].

Definition 2.1. Let X be a set. A fuzzy set in X is a function from X into the closed unit interval $[0, 1]$.

Definition 2.2. A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions

(i) $0_X, 1_X \epsilon T$,
(ii) If $A, B \epsilon T$ then $A \cap B \epsilon T$,
(iii) If $\{A_i : i \epsilon J\} \subset T$, where J is an index set, then $\cup_{i\epsilon J} A_i \epsilon T$.

Here 0_X and 1_X respectively denote the empty set and the whole set. The pair (X, T) is a fuzzy topological space (fts). The element in T are the open sets and their complements are the closed sets.

Definition 2.3. Let A be a fuzzy set in (X, T).

(i) The interior of A is the union of all open sets contained in A, denoted by A°.

(ii) The intersection of all closed sets containing A is the closure of A, denoted by $\overline{A}$.

Definition 2.4. The exterior of A is the complement of the closure of A.

Definition 2.5. A mapping from a fts to a fts is called a fuzzy homeomorphism if it is bijective, continuous and open. A fuzzy homeomorphism is union and crisp subset preserving. The properties of a fuzzy set that are invariant under fuzzy homeomorphism are said to be (fuzzy) topologically invariant.

Definition 2.6. Let X, Y be universal sets, then

$$R = \{((x, y), \mu_R(x, y))|(x, y) \in X \times Y\}$$

where $\mu_R : X \times Y \to [0, 1]$, is called a binary fuzzy relation on $X \times Y$.

Definition 2.7. Let R be a binary fuzzy relation from a fuzzy set $A \subset X$ to fuzzy set $B \subset X$ on fuzzy topological space X, then R is called a fuzzy topological relation from A to B on X if R is a topological invariant under a fuzzy homeomorphism.

Definition 2.8 ([10])**.** Two fuzzy sets A and B in (X, T) are said to be separated if there exist $U, V \in T$ such that $U \supset A$, $V \supset B$ and $U \cap B = V \cap A = \phi$.

Definition 2.9 ([10])**.** A fuzzy topological space X is called connected if there are no separated fuzzy sets C and D such that $X = C \cup D$.

Definition 2.10 ([10])**.** A connected component in a fts is a maximally connected subset.

Definition 2.11. The fuzzy topological space (X, C) is called a crisp fuzzy topological space if all open sets in (X, C) are crisp.

Remark: A crisp fuzzy topological space allows the existence of fuzzy set.

Definition 2.12 (Pu and Liu)**.** The fuzzy boundary of A is defined as $\overline{A} \cap \overline{A^c}$.

There are other forms of fuzzy boundaries in literature. However, since crisp fuzzy topological spaces behave similar to crisp topological spaces we have considered used of fuzzy boundary in the sense of Pu-Liu.

Definition 2.13. A fuzzy point is a fuzzy subset of a set X with support x which is defined by

$$x_a(y) = \begin{cases} a, & \text{if } y = x; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.14 ([12])**.** A fuzzy line is a continuous curve with smooth transition of membership grade over the neighboring point i.e., a fuzzy subset l in a fuzzy set X is called a fuzzy line if support of l is a line in X.

Definition 2.15 ([10])**.** A fuzzy set is said to be regular closed if $\overline{A} = \overline{(\overline{A})^\circ}$.

A fuzzy set is said to be regular open if its complement is regular closed.

Definition 2.16 ([14])**.** A fuzzy set A is called a simple fuzzy region in a connected fuzzy topological space if it meets the following conditions:

(SR1) The closure of A is a proper non-empty connected regular closed subset.
(SR2) The support of A is equal to the closure.
(SR3) The interior A is a non empty connected regular open set.
(SR4) The boundary, the interior of the boundary and the exterior of A are connected.

3. Fuzzy region with holes

We shall require the following definitions:

Definition 3.1. Inner exterior of a fuzzy region is a fuzzy region which fills the cavities (i.e. discontinuity in the boundary and exterior which are connected in piece) of the fuzzy region.

Definition 3.2. A hole is the closure of the inner exterior.

Definition 3.3. Outer exterior of the fuzzy region corresponds to the exterior of the region.

Definition 3.4. A component of boundary is a boundary of the hole which separates the interior and inner exterior of the fuzzy region.

Definition 3.5. A generalized fuzzy region is the union of interior and the components of boundary.

For simplicity, the term point, line and region in place of fuzzy point, fuzzy line and fuzzy region respectively wherever there is no confusion. We proceed to provide a formal definition of a fuzzy region with holes. For simplicity, holes are considered to be contained completely inside the region and disjoint from each other. Further, the region should contain at most a finite number of holes which are not along the boundary of the region.

Definition 3.6. A fuzzy set A is called a fuzzy region with hole in a connected crisp fuzzy topological space (X, C), if it satisfies (SR1), (SR2), (SR3) and the following conditions:

(SR5) Boundary is the union of disjoint connected components.
(SR6) Exterior is the union of disjoint inner exterior and outer exterior.
(SR7) Outer and inner exterior are closed.

Conditions (SR6) and (SR7) signify that holes are disjoint from each other and are not along the boundary of the region. Closedness ensures that the region does not contain spikes so that if we eliminate the holes it will be a fuzzy region without holes [14] and when we draw a plane through 1 of the interval $[0, 1]$ then the shadow of fuzzy region in these plane will be the crisp region with hole [6] in a classical topological space.

This definition is illustrated by considering a fuzzy region with hole in reality in a crisp fuzzy topological space (R^2, C).

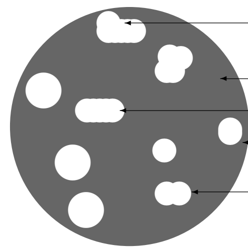


Figure 1: A fuzzy region with holes

3.1. **Topological relations between a fuzzy point and a fuzzy region with 'n' holes.** Since we have considered generalized region, holes and region without hole as topological invariant to determine the relational matrix, we shall consider 8 relations of regional variations (disjoint, meet, overlap, covered by, inside, equal, covers, contains) depending on the nature of intersection. In practical situations region with holes contains at most finite number of holes. Suppose a fuzzy region consists of n holes $H_1, H_2, ..., H_n$. Let A^* be the generalized region and P_A be a point, then the intersection matrix will be of the form

	A^*	H_1	H_2	.	.	H_n	P_A
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, P_A)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, P_A)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, P_A)$
.	.	.	.	.	.	.	.
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$	.	.	$t(H_n, H_n)$	$t(H_n, P_A)$
P_A	$t(P_A, A^*)$	$t(P_A, H_1)$	$t(P_A, H_2)$	.	.	$t(P_A, H_n)$	$t(P_A, P_A)$

where $t(A^*, H_1)$ represents the topological relation between the object parts.
This matrix follows a kind of symmetric matrix relation i.e. relation $t(A^*, H_1)$ can be obtained from $t(H_1, A^*)$ and vice-versa. So the topological relation in the above matrix reduces to an equivalent upper or lower triangular matrix which is given by

	A^*	H_1	H_2	.	.	H_n	P_A
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, P_A)$
H_1	.	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, P_A)$
H_2	.	.	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, P_A)$
.	.	.	.	.	.	.	.
H_n	.	.		.	.	$t(H_n, H_n)$	$t(H_n, P_A)$
P_A	.	.	.	.	.	.	$t(P_A, P_A)$

To determine the number of necessary relation we add up the entries with distinct relation in each row of the upper triangular matrix.

Number of elements in the 1^{st} row $=n+1$
Number of elements in the 2^{nd} row $=n$
....
Number of elements in the $(n+1)^{th}$ row $=1$
Number of elements in the $(n+2)^{th}$ row $=0$

Therefore, total number of elements in this matrix$=(n+1)+n+...+2+1=\frac{(n+1)(n+2)}{2}$, where 'n' is the number of holes.
Since, an equal relation between region with hole and holes are considered to be a single relation and an equal relation due to the points is considered to be another relation, so adding two to total number of distinct entries give us the required number of consistent relation as each relation in the diagonal has the equal relation. Therefore, the total number of necessary relations is $\frac{(n+1)(n+2)}{2}+2$
As per our assumption (a) the relation between each of the hole and the embedding region is that of containment and (b) each pair of holes are disjoint from each other. Therefore, it is possible to further reduce the number of implied relations. So that, in the first row of the above intersection matrix the relation $t(A^*, H_1)$, $t(A^*, H_2)$,..., $t(A^*, H_n)$ are considered to be a single relation, in the second row $t(H_2, H_3)$, ...,$t(H_2, H_n)$ are considered as a single relation and proceeding similarly

Number of elements in the 1^{st} row $=2$
Number of elements in the 2^{nd} row $=2$
....
Number of elements in the $(n-1)^{th}$ row $=2$
Number of elements in the n^{th} row $=2$
Number of elements in the $(n+1)^{th}$ row $=1$
Number of elements in the $(n+2)^{th}$ row $=0$

Therefore, total number of elements $=2n+1$
be equal to itself for the region with hole and other equal relation between point which are two distinct relations. So, to obtain total number of relation we add two to the total number of elements so that the number of necessary relations in the matrix will be $2n+3$.
Further, to obtain the total number of relations, we add two (one due to the equal relation between region with hole and other equal relation due to the point which are topologically considered to be two distinct relations) to the total number of relations so that number of necessary relations becomes $2n+3$.

Example 3.1. *If there is only one hole then only 5 topological relations are realizable between a fuzzy point and a fuzzy region with the hole in a crisp fts.*

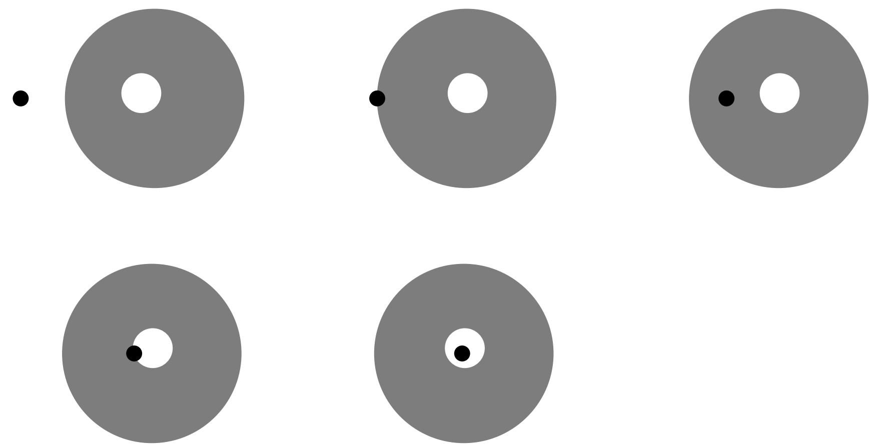


Figure 2: Topological relations between a fuzzy region with a hole and a fuzzy point

3.2. **Topological relations between a fuzzy region with 'n' holes and a fuzzy line.** Suppose a region A^* consists of n holes $H_1, H_2, ..., H_n$. Let L_A be a line. Then the relation between the line and the region with n holes is determined by a relational matrix similar to the relation between point and region with hole.

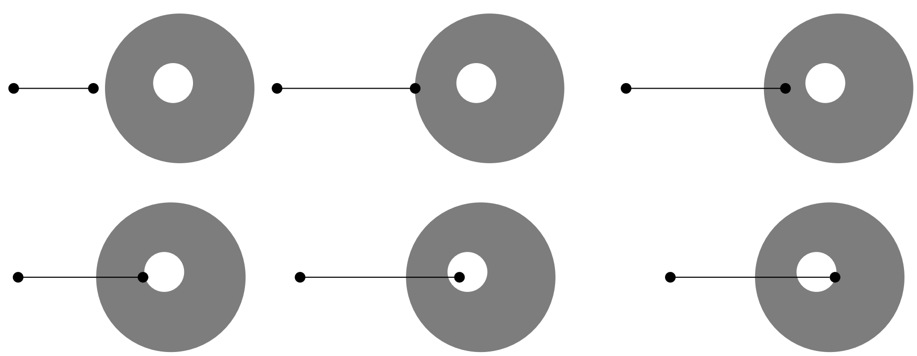
	A^*	H_1	H_2	.	.	H_n	L_A
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, L_A)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, L_A)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, L_A)$
.	.	.	.	.	.	.	.
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$	.	.	$t(H_n, H_n)$	$t(H_n, L_A)$
L_A	$t(L_A, A^*)$	$t(L_A, H_1)$	$t(L_A, H_2)$	.	.	$t(L_A, H_n)$	$t(L_A, L_A)$

From regional connection calculus [3], we know that only 8 recognizable relations exist between two spatial objects with connected boundary. So each of the entries in the intersection matrix can be filled in 8 ways. Therefore the total number of relations between a line and region with hole is 8^{n+2} where 'n' is the number of holes.
In particular, For $n = 1$ i.e. region with a single hole there are total 512 relations between line and the region. It is, however, observed that under certain conditions only a few of them will actually be realized in R^2.

3.2.1. *Conditions that reduce redundant relations between fuzzy region with hole and fuzzy line.* The number of necessary relations that can be realized between a line and region with hole depends on their codimension (i.e. the difference between the dimension of the embedding space and the object). Egenhofer and Herring [5] listed 8 geometric conditions between a region and line in R^2. These conditions can be extended to determine the relations between a fuzzy line and a region with hole in a crisp topological space (R^2, C) since crisp fuzzy topology is equal to crisp topology on R^2. These conditions are

(i) If the line does not intersect the region with hole (i.e. generalized region), then it also does not intersect or meet the hole.
(ii) If the line intersects the region, then the relation between the line and the hole will be either 'meet' or 'intersect' or 'disjoint'.
(iii) If the line is inside the hole, then the line and region must be 'disjoint'.
(iv) If the line is inside the region, then the line and the hole will be 'disjoint'.
(v) If both ends of the line meet the hole then relation between the line and the region must also be 'meet'.
(vi) If the line intersects the hole, then the relation between the line and the region must be non empty.
(vii) If the line is along the connected component of the boundary and intersects the region with hole interior, then the relation between the line and the hole as well as the line and the region is non empty.
(viii) If the line meets the region from outside, then the intersection of the hole and the line is empty.
(ix) If the line is inside the hole and meets the boundary of the hole, then intersection of region and the hole is non empty.

The relational matrix of the existing relations between a line and a region with hole can be determined by successively applying the conditions and cancelling the corresponding non existing relation from the set of 512 relations. Out of 512 relations, only 19 relations in the relational matrix satisfy these conditions. The geometric representation for relation between a line and a region is shown in Figure 3.



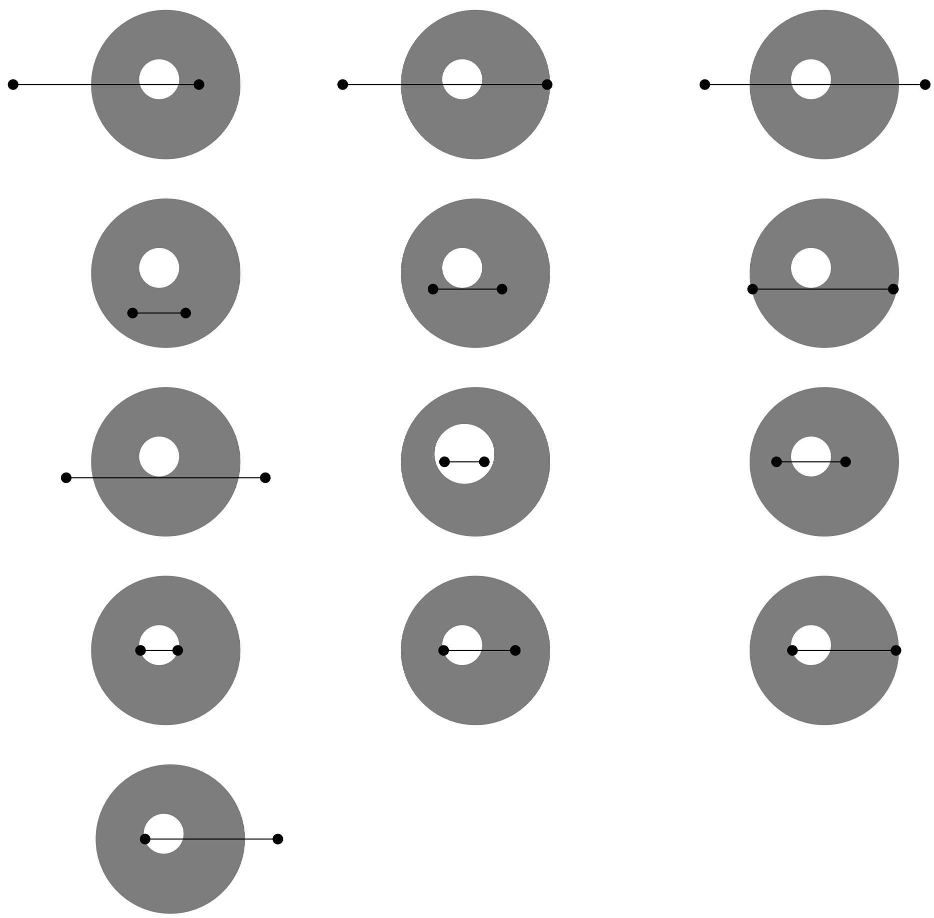


Figure 3: Topological relation between region with a hole and a line

3.3. **Topological relations between a simple fuzzy region and a fuzzy region with 'n' holes.** As in previous cases, if A^* be a fuzzy region consisting of n holes $H_1, H_2, ..., H_n$ and B be a fuzzy region without holes then the topological relational matrix between the region with hole and region without hole is given by

	A^*	H_1	H_2	.	.	H_n	B
A^*	$t(A^*, A^*)$	$t(A^*, H_1)$	$t(A^*, H_2)$	.	.	$t(A^*, H_n)$	$t(A^*, B)$
H_1	$t(H_1, A^*)$	$t(H_1, H_1)$	$t(H_1, H_2)$	.	.	$t(H_1, H_n)$	$t(H_1, B)$
H_2	$t(H_2, A^*)$	$t(H_2, H_1)$	$t(H_2, H_2)$	.	.	$t(H_2, H_n)$	$t(H_2, B)$
.	.	.	.	.	.	.	.
H_n	$t(H_n, A^*)$	$t(H_n, H_1)$	$t(H_n, H_2)$	.	.	$t(H_n, H_n)$	$t(H_n, B)$
B	$t(B, A^*)$	$t(B, H_1)$	$t(B, H_2)$	.	.	$t(B, H_n)$	$t(B, B)$

Here we used a spatial scene (i.e. considering region without hole, generalized region and hole together with eight binary topological relations among these regions) to know exactly which spatial relation exists between a fuzzy region and a fuzzy region with 'n' holes. Since these relations are subject to the variations between the region B in relation to the generalized region A^* and variation of B in relation to the holes, therefore the total number of relations is 8^{n+1} where 'n' is the number of holes.
In particular, n=1 i.e. a region with one hole there are a total of $8^2 = 64$ relations. However, under certain conditions, the number of these relation reduces to a great extent in R^2.

3.3.1. *Conditions to reduce redundant relations between region without hole and region with a hole.* We now identify the topological relations between a fuzzy region without hole and a fuzzy region with a hole. Similar to the case of line and region with hole in (R^2, C), the relations between region without and with hole can be reduced to a small number under following conditions:

(i) If both the regions are disjoint then, region without hole does not intersect the hole.
(ii) If both the region meet each other then intersection of hole and hole free region is non empty.
(iii) If the region without hole is inside the hole of the other region with hole then the generalized fuzzy region does not intersect region with hole.
(iv) If the region without hole meets the hole then intersection of the region without hole and hole as well as region with and without hole is non empty.
(v) If the region without hole covers the hole of the opposite region then the intersection between hole and region without hole as well as region without hole and with hole is non empty.
(vi) If the region without hole is inside the interior of the region with hole then the region without hole does not intersect the hole.
(vii) If the region without hole intersect both generalized regions and its hole, then region and generalized fuzzy region, region and holes shall have non empty intersection.
(viii) If the region without hole intersects the interior of the generalized region, then the region without hole and the hole does not intersect each other.
(ix) If the hole free region covers the holed region, then hole free region intersects the hole and the generalized region.

Under these condition only 23 relations are consistent between a region without hole and a region with hole in R^2 which are shown in Figure 4.

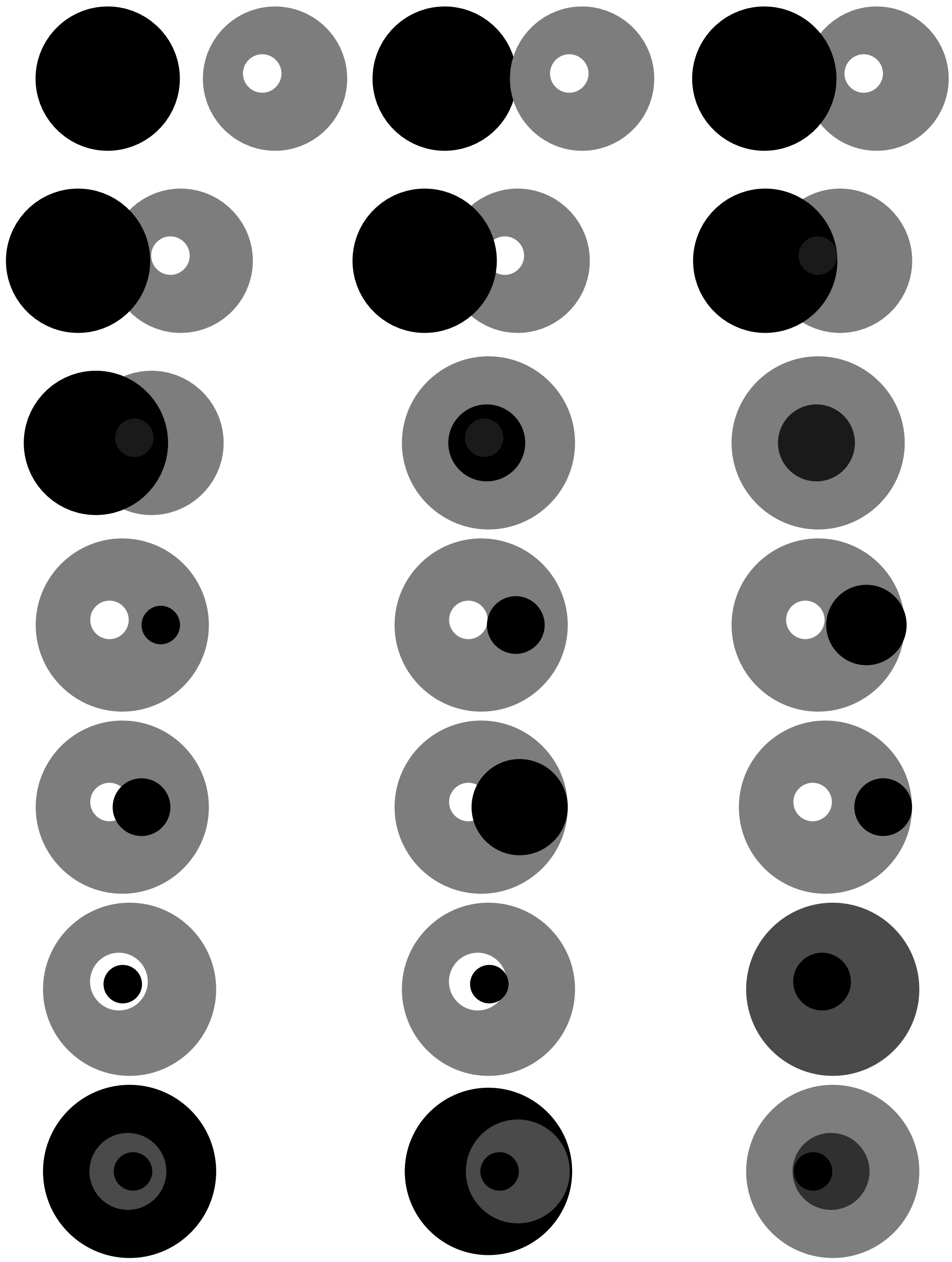


Figure 4: Topological relations between fuzzy regions with and without hole

4. Conclusion

In this paper we have proposed a definition of a fuzzy region with holes in a special fuzzy topological space i.e. crisp fuzzy topological space. We have derived the topological relations between a fuzzy point and a fuzzy region with finite number holes, a fuzzy line and a fuzzy region with finite number holes, a simple fuzzy region and a fuzzy region with finite number holes to infer their differences with the existing model for region without holes. In case of relation between a point and a fuzzy region with hole there are only 5 realizable relations where as Liu and Shi's model shows that there are only three consistent relations between a point and a region without hole. There are only 19 recognizable relation between a line and a region with hole which is 3 more than the number of relations recognizable between a line and a region without hole as shown by Tang and Kainz as well as Liu and Shi's model. It is, however, same as the number of relations determined by Egenhofer and Herring [5] in the crisp case. For fuzzy regions without hole and with a hole, our model gives only 23 necessary relations while Tang and Kainz's model [15] shows that there are 44 recognizable relations between two regions without hole which are same as in case of Liu and Shi model [11]. The challenge, however remains to fully accommodate the membership grades of points in the region with holes and for the same it will be required to formulate a suitable fuzzy region with holes in the general framework of fuzzy topological spaces.

References

[1] A. G. Cohn and N. M. Gotts, The egg-yolk representation of regions with indeterminate boundaries, In: P. A. Burrough, A. U. Frank (Eds), Geographic objects with indeterminate boundaries. Taylor and Francis, London, (1996) 171–187.

[2] S. Du, Q. Qin and Q. Wang, Fuzzy description of topological relations I: a unified fuzzy 9-intersection model, In: Advances in Natural Computation, LNCS, Springer-Verlag 3612 (2005) 1260–1273.

[3] M. J. Egenhofer and J. Herring, A mathematical framework for the definition of topological relationships, In: Proceedings of Fourth International Symposium on Spatial Data Handling (SDH), Switzerland, (1989) 803–813.

[4] M. J. Egenhofer and R. Franzosa, Point-set topological spatial relations, Int. J. Geographical Information Systems 5(2) (1990) 161–174.

[5] M. J. Egenhofer and J. Herring, Categorizing binary topological relationships between regions, lines and points in geographic databases, Technical report, Department of Survey Engineering, University of Maine, Orono (1991).

[6] M. J. Egenhofer, E. Clementini and P. Di Felice, Topological relation between regions with holes, Int. J. Geographical Information Systems 8(2)(1994) 129–142.
[7] M. J. Egenhofer and M. Vasardani, Spatial reasoning with a hole, Spatial Information Theory-8th Int. Conference-COSIT07, LNCS, Springer, New York, 4736 (2007) 303–320.
[8] M. J. Egenhofer and M. Vasardani, Single holed regions: their relations and inferences, Fifth Int. Conference on Geographic Science-GIS2008, LNCS, Springer-Verlag Berlin Heidelberg, 5266 (2008) 337–353.
[9] M. J. Egenhofer and M. Vasardani, Comparing relations with a multi-holed region, Conference on Spatial Information Theory (COSIT09), LNCS, Springer-Verlag Berlin Heidelberg, 5756(2009) 159–176.
[10] Y. M. Liu and M. K. Luo, Fuzzy Topology, Advances in Fuzzy Systems - Applications and Theory, World Scientific, 9(1997).
[11] K. Liu and W. Shi, Quantitative fuzzy topological relations of spatial objects by induced fuzzy topology, Int. J. of Applied Earth Observation and Geoinformation, 11(2009) 38–45.
[12] M. Schneider, Uncertainty management for spatial database: fuzzy spatial data types, The 6th International Symposium on Advances in Spatial Databases (SSD), LNCS, Springer Verlag 1651 (1999) 330–351.
[13] M. Schneider and T. Behr, Topological relationship between complex spatial objects, ACM Trans. Database Systems, 31(1) (2006) 39–81.
[14] X. M. Tang, Spatial object modelling in fuzzy topological spaces: with applications to land cover change, Ph. D. Thesis, University of Twente, Netherlands (2004).
[15] X. Tang, W. Kainz and H. Wang, Topological relations between fuzzy regions in a fuzzy topological space, Int. J. of Applied Earth Observation and Geoinformation, 12(2) (2010) 151–165.
[16] F. B. Zhan, Topological relations between fuzzy regions, In: Proceedings of the ACM Symposium on Applied Computing (1997) 192–196.

Dibyajyoti Hazarika (dibya@tezu.ernet.in) – Department of Mathematical Sciences, Tezpur University, Napam - 784028, Assam (INDIA)

Debajit Hazarika (debajit@tezu.ernet.in) – Department of Mathematical Sciences, Tezpur University, Napam - 784028, Assam, INDIA