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Intuitionistic fuzzy mappings and intuitionistic fuzzy equivalence relations

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Abstract. We introduce the concepts of intuitionistic fuzzy equality and intuitionistic fuzzy mapping. And we obtain some fundamental properties of intuitionisic fuzzy mapping. Furthermore, we give the decomposition of an intuitionistic fuzzy mapping by using intuitionistic fuzzy equivalence relation.

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1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [14], Demirci[6] introduced the concept of fuzzy equality and fuzzy mapping. And he gave some their fundamental properties. In particular, Hur et al.l (Hur, Choi and Lim [10]) studied many properties of fuzzy mappings in the sense of Demirci. Moreover, they obtained the decomposition of a fuzzy mapping by using fuzzy equivalence relation.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues (Çoker [3], Çoker and Haydar Eş [4]) introduced the concept of intuitionistic fuzzy topology using intuitionistic fuzzy sets. Moreover, S. J. Lee and E. P. Lee [11] introduced the concepts of intuitionstic fuzzy point and intuitionistic fuzzy neighborhoods and investigated the properties of continuos, open and closed mappings in intuitionistic fuzzy topological sapces. In particular, Hur et al. (Hur, Kim and Ryou [8], Hur, Jang and Lim [9]) applied the concept of intuitionistic fuzzy sets to topology and semigroup theory, respectively. Bustince and P.Burillo.[2], and Hur et al. (Hur, Jang and Jun [6], Hur, jang and Ahn [7]) applied it to set theory, respectively.

In this paper, we introduce the concept of intuitionistic fuzzy equality and intuitionistic fuzzy mapping. And we obtain some fundamental properties of intuitionistic fuzzy mapping. Furthermore, we give the decomposition of an intuitionistic fuzzy mapping by using intuitionistic fuzzy equivalence relation.

2. Preliminaries

In this section, we will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval [resp. the set of all fuzzy sets in a set X] as I [resp. I^X] and $X, Y, Z, \cdots$, etc., will be nonempty crisp sets.

Definition 2.1([13]). Let $f : X \to Y$ be an(ordinary) mapping, let $A \in I^X$ and let $B \in I^X$. Then :

(a) The *image of A under f*, denoted by $f(A)$, is a fuzzy set in Y defined as follows : For each $y \in Y$,

$$[f(A)](y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1} \neq \phi, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The *preimage of B under f*, denoted by $f^{-1}(B)$, is a fuzzy set in X defined as follows : For each $x \in X$,

$$[f^{-1}(B)](x) = (B \circ f)(x) = B(f(x)).$$

Definition 2.2([1,3]). A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called a *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $\underset{\sim}{0}$ and $\underset{\sim}{1}$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $\underset{\sim}{0}(x) = (0, 1)$ and $\underset{\sim}{1}(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all the IFSs in X as IFS(X).

Definition 2.3([3]). Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A), <> A = (1 - \nu_A, \nu_A)$.

Definition 2.4([3]). Let $\{A_\alpha\}_{i \in \Gamma}$ be an arbitrary family of IFSs in X, where $A_\alpha = (\mu_{A_\alpha}, \nu_{A_\alpha})$ for each $\alpha \in \Gamma$. Then

(a) $\bigcap A_\alpha = (\bigwedge \mu_{A_\alpha}, \bigvee \nu_{A_\alpha})$.

(b) $\bigcup A_\alpha = (\bigvee \mu_{A_\alpha}, \bigwedge \nu_{A_\alpha})$.

Definition 2.5([3]). Let $f : X \to Y$ be an(ordinary) mapping, let $A \in$ IFS(X) and let $B \in$ IFS(Y). Then :

(a) The *image of* A *under* f, denoted by $f(A)$, is an intuitionistic fuzzy set in Y defined as follows : For each $y \in Y$,

$$[f(A)](y) = \begin{cases} (\bigvee_{x\in f^{-1}(y)} \mu_A(x), \bigwedge_{x\in f^{-1}(y)} \nu_A(x)) & \text{if } f^{-1} \neq \phi, \\ (0,1) & \text{otherwise.} \end{cases}$$

(b) The *preimage of* B *under f*, denoted by $f^{-1}(B)$, is an intuitionistic fuzzy set in X defined as follows : For each $x \in X$,

$$[f^{-1}(B)](x) = ([f^{-1}(\mu_B)](x), [f^{-1}(\nu_B)](x)) = (\mu_B(f(x)), \nu_B(f(x))).$$

Definition 2.6([2]). A complex mapping $R = (\mu_R, \nu_R) : X \times Y \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, IFR) from X to Y if $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for each $(x,y) \in X \times Y$, i.e., $R \in$IFS$(X \times Y)$.

In particular, if R is an intuitionistic fuzzy relation from X to itself, then f is called an *intuitionistic fuzzy relation on*(or *in*) X, and we will denote the set of all IFRs on s set X as IFR(X).

Definition 2.7([5]). Let E_X be a fuzzy relation on X. Then E_X is called a *fuzzy equality* on X if it satisfies the following conditions :

(e.1) $E_X(x,y) = 1 \Leftrightarrow x = y, \ \forall\ x, y \in X$,
(e.2) $E_X(x,y) = E_X(y,x), \ \forall\ x, y \in X$,
(e.3) $E_X(x,z) \geq E_X(x,y) \wedge E_X(y,z), \ \forall\ x, y, z \in X$.

Let E be a fuzzy equality on X and let $a, b \in X$. Then we interpret the value $E(a,b)$ as the grade of "a and b are nearly equal". We will denote the set of all fuzzy equalities on X as $E(X)$.

Definition 2.8([5]). Let f be a fuzzy relation from X to Y, i.e., $R \in I^{X\times Y}$. Let E_X and E_Y be fuzzy equalities on X and Y, respectively. Then R is called a *fuzzy mapping from* X *to* Y with respect to(in short, w.r.t.) E_X and E_Y denoted by $f : X \to Y$, if it satisfies the following conditions :

(f.1) $\forall\ x \in X, \ \exists\ y \in Y$ such that $f(x,y) > 0$,
(f.2) $\forall\ x, y \in X, \ \forall\ z, w \in Y, \ f(x,z) \wedge f(y,w) \wedge E_X(x,y) \leq E_Y(z,w)$.

Definition 2.9([5]). Let $f : X \to Y$ be a fuzzy mapping w.r.t. E_X and E_Y. Then f is said to be :

(a) *strong* if $\forall\ x \in X, \exists\ y \in Y$ such that $f(x,y) = 1$,
(b) *surjective* if $\forall\ y \in Y, \exists\ x \in X$ such that $f(x,y) > 0$,
(c) *strong surjective* if $\forall\ y \in Y, \exists\ x \in X$ such that $f(x,y) = 1$,
(d) *injective* if $f(x,z) \wedge f(y,w) \wedge E_Y(z,w) \leq E_X(x,y), \forall\ x, y \in X, \forall\ z, w \in Y$,
(e) *bijective* if it is surjective and injective,

(f) *strong bijective* if it is strong surjective and injective.

Definition 2.10([5]). Let I_X be the fuzzy relation on X defined by : For any $x, y \in X$,

$$I_X(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Then $I_X : X \to X$ is a strong bijective w.r.t. a fuzzy equality E_X on X. In this case, I_X is called the *unit fuzzy mapping* on X.

Definition 2.11([5]). Let $f : X \to Y$ be an(ordinary) mapping, let $A \in I^X$ and let $B \in I^X$. Then :

(a) The *image of A under f*, denoted by $f(A)$, is a fuzzy set in Y defined as follows : For each $y \in Y$,

$$[f(A)](y) = \bigvee_{x \in X}[A(x) \wedge f(x,y)].$$

(b) The *preimage of B under f*, denoted by $f^{-1}(B)$, is a fuzzy set in X defined as follows : For each $x \in X$,

$$[f^{-1}(B)](x) = \bigvee_{y \in Y}[B(y) \wedge f(x,y)].$$

3. Intuitionistic fuzzy equalities and intutionistic fuzzy mappings

Definition 3.1. Let X be a nonempty set and let $IE_X = (\mu_{IE_X}, \nu_{IE_X}) \in$IFR$(X)$. Then IE_X is called an *intuitionistic fuzzy equality* on X it satisfies the following conditions :

(ie.1) $IE_X(x,y) = (1,0) \Leftrightarrow x = y,\ \forall\ x, y \in X$,
(ie.2) $IE_X(x,y) = IE_X(y,x),\ \forall\ x, y \in X$,
(ie.3) $\mu_{IE_X}(x,y) \wedge \mu_{IE_X}(y,z) \leq \mu_{IE_X}(x,z)$
and

$$\nu_{IE_X}(x,y) \vee \nu_{IE_X}(y,z) \geq \nu_{IE_X}(x,z), \forall\ x, y, z \in X.$$

We will denote the set of all intuitionistic fuzzy equalities on X as IE(X).

Example 3.1. (1) Let $X = \{0, 1, 2, 3, 4, 5\}$, let $IE_X = (\mu_{IE_X}, \nu_{IE_X})$ be the intuitionistic fuzzy relation in X defined by : For any $x, y \in X$,

$$\mu_{IE_X}(x,y) = 1 - 0.2 \times |x - y| \text{ and } \nu_{IE_X}(x,y) = 0.2 \times |x - y|.$$

Then we can easily see that $IE_X = (\mu_{IE_X}, \nu_{IE_X}) \in$IE$(X)$.

(2) Equality of two points in the classical sense can be graded by the mapping $\mu_{E_X} : X \times X \longrightarrow 2 = \{0, 1\}$ defined by

$$\mu_{E_X}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \forall\ x, y \in X. \end{cases}$$

Then we can also easily see that $IE_X = (\mu_{E_X}, \mu_{E_X^c}) \in IE(X)$.

The following is the immediate results of Definition 2.7, Definitions 2.3 and 3.1.

Proposition 3.2. (1) If $E_X \in$E(X), then $(E_X, E_X^c) \in$ IE(X).

(2) If $IE_X = (\mu_{IE_X}, \nu_{IE_X}) \in$IE$(X)$, then $<> IE_X$, $[\,]IE_X \in$IE(X).
(3) If $IE_X = (\mu_{IE_X}, \nu_{IE_X}) \in$IE$(X)$, then μ_{IE_X}, ${\nu_{IE_X}}^c \in$E(X).

Definition 3.3. For any two nonempty sets X and Y, let IE_X and IE_Y be two intuitionistic fuzzy equalities on X and Y, respectively. Let $f \in$IFS$(X \times Y)$. Then f is called an *intuitionistic fuzzy mapping from X to Y w.r.t. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$*, denoted by $f : X \to Y$, if it satisfies the following conditions :
(if.1) $\forall\ x \in X, \exists\ y \in Y$ such that $\mu_f(x, y) > 0$ and $\nu_f(x, y) < 1$.
(if.2) $\forall\ x, y \in X, \forall\ z, w \in Y$,

$$\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_X}(x, y) \leq \mu_{IE_Y}(z, w)$$

and

$$\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_X}(x, y) \geq \nu_{IE_Y}(z, w).$$

Example 3.3. Let X and IE_X be same as Example 3.1 (1). Let $f = (\mu_f, \nu_f)$ be the intuitionistic fuzzy relation on X given by
$f(x, x) = (0.2 \times (5 - x), 1 - 0.2 \times (5 - x))\ \forall\ x \in X \backslash \{5\}$, $f(5, 5) = (0, 1)$,
$f(x, x - 1) = (0.2x, 1 - 0.2x)\ \forall\ x \in X \backslash \{0\}$,
$x < x^{'}$ or $x - 1 > x^{'}\ \Rightarrow\ f(x, x^{'}) = (0, 1)\ \forall\ x, x^{'} \in X$.
Then we can see that $f : X \to X$ is an intuitionistic fuzzy mapping w.r.t. IE_X.

The followings are the immediate results of Definitions 2.8, 3.1, and Proposition 3.2.

Proposition 3.4. (1) Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y). Then $(f, f^c) : X \to Y$ is an intuitionistic fuzzy mapping from X to Y w.r.t. (E_X, E_X^c) and (E_Y, E_Y^c).

(2) Let $f = (\mu_f, \nu_f) : X \to Y$ be an intuitionistic fuzzy mapping from X to Y w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Then $<> f$ and $<> f$ are intuitionistic fuzzy mapping from X to Y w.r.t. intuitionistic fuzzy equalities $<> IE_X$ and $<> IE_Y$, and $[\,]IE_X$ and $[\,]IE_Y$, respectively.

(3) Let $f = (\mu_f, \nu_f)$ be an intuitionistic fuzzy function from X to Y w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Then μ_f and ${\nu_f}^c$ are fuzzy mappings from X to Y w.r.t. fuzzy equalities μ_{IE_X} and μ_{IE_Y}, and ${\nu_{IE_X}}^c$ and ${\nu_{IE_Y}}^c$ on X and Y, respectively.

Definition 3.5. For sets X and Y, let $f : X \to Y$ be an intuitionistic fuzzy mapping from X to Y w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Then f is said to be :
(a) *strong* if $\forall\ x \in X, \exists\ y \in Y$ such that $f(x, y) = (1, 0)$,
(b) *surjective* if $\forall\ y \in Y, \exists\ x \in X$ such that $\mu_f(x, y) > 0$ and $\nu_f(x, y) < 1$,
(c) *strong surjective* if $\forall\ y \in Y, \exists\ x \in X$ such that $f(x, y) = (1, 0)$,
(d) *injective* if

$$\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_Y}(z, w) \leq \mu_{IE_X}(x, y)$$

and

$$\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_Y}(z, w) \geq \nu_{IE_X}(x, y), \quad \forall\ x, y \in X, \forall\ z, w \in Y,$$

(e) *bijective* if it is surjective and injective,
(f) *strong bijective* if it is strong surjective and injective.

The followings are the immediate results of Definitions 2.3, 2.8, 2.9, 3.5, Propositions 3.2 and 3.4.

Proposition 3.6. (1) Let $f : X \to Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] fuzzy mapping w.r.t. fuzzy equalities E_X and E_Y on X and Y, respectively, then $(f, f^c) : X \to Y$ is a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy w.r.t. $(E_X, E_X^c) \in$ IE(X) and $(E_Y, E_Y^c) \in$IE(Y).

(2) Let $f = (\mu_f, \nu_f) : X \to Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$ IE(Y). Then $<> f$ and $[\,]f$ are a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. intuitionistic fuzzy equalities $<> IE_X$ and $<> IE_Y$, and $[\,]IE_X$ and $[\,]IE_Y$ on X and Y, respectively.

(3) Let $f = (\mu_f, \nu_f) : X \to Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$IE(Y). Then μ_f and ${\nu_f}^c$ are strong [surjective, strong surjective, injective, bijective, strong bijective] fuzzy mappings w.r.t. fuzzy equalities μ_{IE_X} and μ_{IE_Y}, and ${\nu_{IE_X}}^c$ and ${\nu_{IE_Y}}^c$ on X and Y, respectively.

The following is the immediate result of Definition 3.3.

Proposition 3.7. Let Δ_X be the intuitionistic fuzzy relation on a set X defined by : For each $(x, y) \in X \times X$,

$$\Delta_X(x, y) = \begin{cases} (1, 0) & \text{if } x = y, \\ (0, 1) & \text{if } x \neq y. \end{cases}$$

Then Δ_X is a strong and strong bijective intuitionistic fuzzy mapping on X w.r.t. an intuitionistic fuzzy equality IE_X on X. In fact, Δ_X is an intuitionistic fuzzy equality on X. In this case, Δ_X is called an *i*dentity intuitionistic fuzzy mapping on X.

Definition 3.8([2]). Let R be an intuitionistic fuzzy relation from X to Y and let S be an intuitionistic fuzzy relation from Y to Z.

(a) R^{-1} is called the *inverse* of R if $R^{-1}(y, x) = R(x, y), \forall\, x \in X, \forall\, y \in Y$.

(b) The *sup-min composition* of R and S, denoted by $S \circ R$, is an intuitionistic fuzzy relation from X to Z defined by : For each $(x, z) \in X \times Y$,

$$\mu_{S \circ R}(x, z) = \bigvee_{y \in Y} [\mu_R(x, y) \wedge \mu_S(y, z)]$$

and

$$\nu_{S \circ R}(x, z) = \bigwedge_{y \in Y} [\nu_R(x, y) \vee \nu_S(y, z)].$$

Proposition 3.9. Let $f : X \to Y$ and $g : Y \to Z$ be intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE$(X), IE_Y \in$IE(Y) and $IE_Z \in$IE(Z). Then the sup-min composition $g \circ f$ is an intuitionistic fuzzy mapping $g \circ f : X \to Z$ w.r.t. $IE_X \in$ IE(X) and

IE_Z $\in$IE(Z).

Proof (i) Let $x \in X$. Since f and g are intuitionistic fuzzy mapping, $\exists y_0 \in Y$ and $z_0 \in Z$ such that

$$\mu_f(x, y_0) > 0, \mu_g(y_0, z_0) > 0 \text{ and } \nu_f(x, y_0) < 1, \nu_f(y_0, z_0) < 1.$$

Then $\mu_f(x, y_0) \wedge \mu_g(y_0, z_0) > 0$ and $\nu_f(x, y_0) \vee \nu_f(y_0, z_0) < 1$. Thus

$$\mu_{g\circ f}(x, z_0) = \bigvee_{y\in Y}[\mu_f(x, y) \wedge \mu_g(y, z_0)] > 0$$

and

$$\nu_{g\circ f}(x, z_0) = \bigwedge_{y\in Y}[\nu_f(x, y) \vee \nu_g(y, z_0)] < 1.$$

So $g \circ f$ satisfies the condition (if.1).

(ii) Let $x_1, x_2 \in X$ and let $z_1, z_2 \in Z$. Then by the proof of Proposition 2.1 in [5], it is clear that

$$\mu_{g\circ f}(x_1, z_1) \wedge \mu_{g\circ f}(x_2, z_2) \wedge \mu_{IE_X}(x_1, x_2) \le \mu_{IE_I}(z_1, z_2)\ . \tag{3.1}$$

Let $\mu = \nu_{g\circ f}(x_1, z_1)$ and let $\lambda = \nu(x_2, x_2)$. We show that

$$\mu \vee \lambda \vee \nu_{IE_X}(x_1, x_2) \ge \nu_{IE_I}(z_1, z_2). \tag{3.2}$$

If $\mu = 1$ or $\lambda = 1$, then clearly, the inequality (3.2) holds.
Suppose $\mu < 1$ and $\lambda < 1$. Then, by the definition of $\nu_{g\circ g}$ and μ, for $\mu \vee \lambda < \epsilon < 1, \exists y_1(\epsilon), y_2(\epsilon) \in Y$ such that

$$\nu_f(x_1, y_1(\epsilon)) \vee \nu_g(y_1(\epsilon), z_1) < \mu + \epsilon$$

and

$$\nu_f(x_2, y_2(\epsilon)) \vee \nu_g(y_2(\epsilon), z_2) < \lambda + \epsilon.$$

Thus

$$\begin{aligned}&\nu_f(x_1, y_1(\epsilon) \vee \nu_f(x_2, y_2(\epsilon)) \vee \nu_g(y_1(\epsilon), z_1) \vee \nu_g(y_2(\epsilon), z_2) \vee \nu_{IE_X}(x_1, x_2)\\ &< (mu + \epsilon) \vee (\lambda + \epsilon).\end{aligned} \tag{3.3}$$

Since f is an intuitionistic fuzzy mapping,

$$\nu_f(x_1, y_1(\epsilon)) \vee \nu_f(x_2, y_2(\epsilon)) \vee \nu_{IE_X}(x_1, x_2) \ge \nu_{IE_Y}(y_1(\epsilon), y_2(\epsilon)). \tag{3.4}$$

So

$$\begin{aligned}\nu_{IE_Z}(z_1, z_2) &\le \nu_g(y_1(\epsilon), z_1) \vee \nu_g(y_2(\epsilon), z_2)\\ &\quad\text{[Since } g \text{ is an intuitionistic fuzzy mapping]}\\ &< (\mu + \epsilon) \vee (\lambda + \epsilon) \vee \nu_{IE_X}(x_1, x_2). \text{ [By (3.3) and (3.4)]}\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the inequality (3.2) holds. Hence

$$\nu_{g\circ f}(x_1, z_1) \vee \nu_{g\circ f}(x_2, z_2) \vee \nu_{IE_X}(x_1, x_2) \ge \nu_{IE_Z}(z_1, z_2). \tag{3.5}$$

Hence, by (3.1) and (3.5), $g \circ f$ satisfies the condition (if.2). Therefore $g \circ f$ is an intuitionistic fuzzy mapping. □

The following are the immediate results of Proposition 3.9 and Definitions 3.5 and 3.8 (b).

Corollary 3.9-1. Let $f : X \to Y$ and $g : Y \to Z$ be intuitionistic fuzzy mappings w.r.t. IE_X $\in$IE(X), IE_Y $\in$IE(Y) and IE_Z $\in$IE(Z), respectively. If f and g are strong [resp. injective, surjective, strong surjective, bijective and strong bijective], then so is $g \circ f$.

Corollary 3.9-2. Let $f : X \to Y$ and $g : Y \to Z$ be intuitionistic fuzzy mappings w.r.t. $IE_X \in$ IE(X), $IE_Y \in$ IE(Y) and IE_Z $\in$IE(Z), respectively.

(a) If $g \circ f$ is strong [resp. injective], then so is f.
(b) If $g \circ f$ is surjective [resp. strong surjective], then so is g.
(c) If $g \circ f$ is bijective [resp. strong bijective], then f is injective and g is surjective [resp. strong surjective].

Definition 3.10. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$ IE(Y). Then f is said to be *invertible* if the intuitionistic fuzzy relation f^{-1} on $Y \times X$ is an intuitionistic fuzzy mapping $f^{-1} : Y \to X$ w.r.t. IE_Y and IE_X.

Lemma 3.11. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). If f is invertible, then f is bijective.

Proof. Suppose f is invertible and let $y \in Y$. Since f^{-1}: $Y \to X$ is an intuitionistic fuzzy mapping w.r.t. IE_Y and IE_X, $\exists x_0 \in X$ such that $\mu_{f^{-1}}(y, x_0) > 0$ and $\nu_{f^{-1}}(y, x_0) < 1$. Then $\mu_f(x_0, y) > 0$ and $\nu_f(x_0, y) < 1$. Thus f is surjective. Let $x_1, x_2 \in X$ and let $y_1, y_2 \in Y$. Since $f^{-1} : Y \to X$ is an intuitionistic fuzzy mapping,

$$\mu_{f^{-1}}(y_1, x_1) \wedge \mu_{f^{-1}}(y_2, x_2) \wedge \mu_{IE_Y}(y_1, y_2) \leq \mu_{IE_X}(x_1, x_2)$$

and

$$\nu_{f^{-1}}(y_1, x_1) \vee \nu_{f^{-1}}(y_2, x_2) \vee \nu_{IE_Y}(y_1, y_2) \geq \nu_{IE_X}(x_1, x_2).$$

Thus

$$\mu_f(x_1, y_1) \wedge \mu_f(x_2, y_2) \wedge \mu_{IE_Y}(y_1, y_2) \leq \mu_{IE_X}(x_1, x_2)$$

and

$$\nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_Y}(y_1, y_2) \geq \nu_{IE_X}(x_1, x_2).$$

So f is injective. Hence f is bijective. □

Lemma 3.12. Let $f : X \to Y$ be a bijective intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Then the intuitionistic fuzzy relation f^{-1} on $Y \times X$ is an intuitionistic fuzzy mapping $f^{-1} : Y \to X$ w.r.t. IE_Y and IE_X.

Proof. (i) Let $y \in Y$. Since f is surjective, $\exists x_0 \in X$ such that $\mu_f(x_0, y) > 0$ and $\nu_f(x_0, y) < 1$. Then $\mu_{f^{-1}}(y, x_0) > 0$ and $\nu_{f^{-1}}(y, x_0) < 1$. Thus f^{-1} satisfies the condition (if.1). Let $y_1, y_2 \in Y$ and let $x_1, x_2 \in X$. Since f is injective,

$$\mu_f(x_1, y_1) \wedge \mu_f(x_2, y_2) \wedge \mu_{IE_Y}(y_1, y_2) \leq \mu_{IE_X}(x_1, x_2)$$

and

$$\nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_Y}(y_1, y_2) \leq \nu_{IE_X}(x_1, x_2)$$

Then

$$\mu_{f^{-1}}(y_1, x_1) \wedge \mu_{f^{-1}}(y_2, x_2) \wedge \mu_{IE_Y}(y_1, y_2) \leq \mu_{IE_X}(x_1, x_2).$$

$$\nu_{f^{-1}}(y_1, x_1) \vee \nu_{f^{-1}}(y_2, x_2) \vee \nu_{IE_Y}(y_1, y_2) \leq \nu_{IE_X}(x_1, x_2).$$

Thus f^{-1} satisfies the condition (if.2). So $f^{-1} : Y \to X$ is an intuitioninstic fuzzy mapping w.r.t. IE_Y and IE_X.

(ii) Let $x \in X$. Since f is an intuitionistic fuzzy mapping, $\exists y_0 \in Y$ such that $\mu_f(x, y_0) > 0$ and $\nu_f(x, y_0) < 1$. Then $\mu_{f^{-1}}(y_0, x) > 0$ and $\nu_f(x, y_0) > 0$. Thus f^{-1} is surjective. Now let $y_1, y_2 \in Y$ and let $x_1, x_2 \in X$. Since f is an intuitionistic fuzzy mapping,

$$\mu_f(x_1, y_1) \wedge \mu_f(x_2, y_2) \wedge \mu_{IE_X}(x_1, x_2) \leq \mu_{IE_Y}(y_1, y_2).$$

and

$$\nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_X}(x_1, x_2) \leq \nu_{IE_Y}(y_1, y_2).$$

Then

$$\mu_{f^{-1}}(y_1, x_1) \wedge \mu_{f^{-1}}(y_2, x_2) \wedge \mu_{IE_X}(x_1, x_2) \leq \mu_{IE_Y}(y_1, y_2).$$

and

$$\nu_{f^{-1}}(y_1, x_1) \vee \nu_{f^{-1}}(y_2, x_2) \vee \nu_{IE_X}(x_1, x_2) \leq \nu_{IE_Y}(y_1, y_2).$$

Thus f^{-1} is injective. So f^{-1} is bijective. This competes the proof. □

The following is the immediate result of Lemma 3.12 and Definition 3.5.

Corollary 3.12. If $f : X \to Y$ is strong bijective, then $f^{-1} : Y \to X$ is strong bijective.

The following is the immediate result of Lemmas 3.11 and 3.12.

Theorem 3.13 Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Then f is invertible if and only if f is bijective.

Lemma 3.14. Let $f : X \to Y$ is strong and injective w.r.t. $IE_X = \Delta_X \in$ IE(X) and $IE_Y \in$IE(Y), then $f^{-1} \circ f = \Delta_X$.

Proof. We show that $\mu_{f^{-1} \circ f} = \mu_{\Delta_X}$ and $\nu_{f^{-1} \circ f} = \nu_{\Delta_X}$. Since $\mu_{f^{-1} \circ f} = \mu_{\Delta_X}$, by the proof of Proposition 2.3 in [5], it is sufficient to show that $\nu_{f^{-1} \circ f} = \nu_{\Delta_X}$.

Let $x, x' \in X$. Then

$$\begin{aligned}\nu_{f^{-1} \circ f}(x, x') &= \bigwedge_{y \in Y}[\nu_f(x, y) \vee \nu_{f^{-1}}(y, x')]\\ &= \bigwedge_{y \in Y}[\nu_f(x, y) \vee \nu_f(x', y) \vee \nu_{IE_Y}(y, y)]\\ &\quad \text{[By the definition of } IE_Y, \nu_{IE_Y}(y, y) = 0.\text{]}\\ &\leq \nu_{IE_X}(x, x') \text{ [since } f \text{ is injective]}\\ &= \nu_{\Delta_X}(x, x'). \text{ [since } IE_X = \Delta_X]\end{aligned}$$

Thus $\nu_{\Delta_X} \leq \nu_{f^{-1} \circ f}$. On the other hand, $\nu_{\Delta_X}(x, x') = 1$ or $\nu_{\Delta_X}(x, x') = 0$.

If $\nu_{\Delta_X}(x,x')=1$, then clearly $\nu_{f^{-1}\circ f}(x,x')\le\nu_{\Delta_X}(x,x')$. Suppose $\nu_{\Delta_X}(x,x')=0$, i.e., $x=x'$. Since f is strong, for $x\in X,\exists y_0\in Y$ such that $\nu_f(x,y_0)=0$.
Thus

$$\begin{aligned}\nu_{f^{-1}\circ f}(x,x)&=\bigwedge_{y\in Y}[\nu_f(x,y)\vee\nu_{f^{-1}}(y,x)]\\&=\bigwedge_{y\in Y}\nu_f(x,y)\\&=0.\end{aligned}$$

So, in either cases, $\nu_{\Delta_X}\le\nu_{f^{-1}\circ f}$. Hence $\nu_{f^{-1}\circ f}=\nu_{\Delta_X}$. This complete the proof. □

Lemma 3.15. If $f:X\to Y$ is strong surjective w.r.t. $IE_X\in$IE(X) and $IE_Y=\Delta_Y\in$IE(Y), then $f\circ f^{-1}=\Delta_Y$.

Proof. Since $\mu_{f\circ f^{-1}}=\mu_{\Delta_Y}$, by the proof of Lemma 3.11 in [10], it is sufficient to show that $\nu_{f\circ f^{-1}}=\nu_{\Delta_Y}$.
Let $y,y'\in Y$. Then

$$\begin{aligned}\nu_{f\circ f^{-1}}(y,y')&=\bigwedge_{x\in X}[\nu_{f^{-1}}(y,x)\vee\nu_f(x,y')]\\&=\bigwedge_{x\in X}[\nu_f(x,y)\vee\nu_f(x,y')\vee\nu_{IE_Y}(x,x)]\\&\quad\text{[By the definition of } IE_X,\nu_{IE_X}(x,x)=0.\text{]}\\&\ge\nu_{IE_Y}(y,y')\ \text{[since } f \text{ is intuitionistic fuzzy mapping]}\\&=\nu_{\Delta_Y}(y,y').\ \text{[since } IE_Y=\Delta_Y\text{]}\end{aligned}$$

Thus $\nu_{\Delta_{f\circ f^{-1}}}$. On the other hand, $\nu_{\Delta_Y}(y,y')=1$ or $\nu_{\Delta_Y}(y,y')=0$. If $\nu_{\Delta_Y}(y,y')=1$, then clearly $\nu_{f\circ f^{-1}}(y,y')\le\nu_{\Delta_Y}(y,y')$. Suppose $\nu_{\Delta_Y}(y,y')=0$, i.e., $y=y'$. Since f is strong, for $y\in Y,\exists x_0\in Y$ such that $\nu_f(x_0,y)=0$.
Thus

$$\begin{aligned}\nu_{f\circ f^{-1}}(y,y)&=\bigwedge_{x\in X}[\nu_{f^{-1}}(y,x)\vee\nu_f(x,y)]\\&=\bigwedge_{x\in X}\nu_f(x,y)\\&=0.\end{aligned}$$

So, in either cases, $\nu_{f\circ f^{-1}}\le\nu_{\Delta_Y}$. Hence $\nu_{f\circ f^{-1}}=\nu_{\Delta_Y}$. This complete the proof. □

The following is the immediate result of Lemmas 3.14 and 3.15.

Theorem 3.16. Let $f:X\to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X\in IE(X)$ and $IE_Y\in$ IE(Y). If f is strong and strong bijective, $IE_X=\Delta_X$ and $IE_Y=\Delta_Y$, then $f^{-1}\circ f=\Delta_X$ and $f\circ f^{-1}=\Delta_Y$.

Proposition 3.17. Let $f:X\to Y$ and $g:Y\to Z$ be bijective w.r.t. $IE_X\in$IE(X), $IE_Y\in$ IE(Y) and $IE_Z\in$IE(Z). Then $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ and the intuitionistic fuzzy relation $(g\circ f)^{-1}$ is an intuitionistic fuzzy mapping $(g\circ f)^{-1}:Z\to X$ w.r.t. IE_Z and IE_X.

Proof. From Definition 3.8, it can be easily seen that $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$. Moreover, by this equality and Proposition 3.9 and Theorem 3.13, it is directly obtained that the intuitionistic fuzzy relation $(g\circ f)^{-1}$ is an intuitionistic fuzzy mapping. □

Definition 3.18. Let $f : X \to Y$ be an intuitionistic fuzzy mapping, and let $A \in$ IFS(X), $B \in$ IFS(Y). Then :

(a) The *image of* A *under* f, denoted by $f(A)$, is an intuitionistic fuzzy set in Y given by : For each $y \in Y$,

$$\mu_{f(A)}(y) = \bigvee_{x \in X} [\mu_A(x) \wedge \mu_f(x, y)]$$

and

$$\nu_{f(A)}(y) = \bigwedge_{x \in X} [\nu_A(x) \vee \nu_f(x, y)].$$

(b) The *preimage of* B *under* f, denoted by $f^{-1}(B)$, is an intuitionistic fuzzy set in X given by : For each $x \in X$,

$$\mu_{f^{-1}(B)}(x) = \bigvee_{y \in Y} [\mu_B(y) \wedge \mu_f(x, y)]$$

and

$$\nu_{f^{-1}(B)}(x) = \bigwedge_{y \in Y} [\nu_B(y) \vee \nu_f(x, y)].$$

Remark 3.18. (a) If $f : X \to Y$ is an (ordinary) mapping, then it is clear that Definition 3.18 is identical with Definition 2.5.

(b) If $f : X \to Y$ is strong surjective, then

$$f(A)(y) = (\bigvee_{f(x,y)=1} \mu_A(x), \bigwedge_{f(x,y)=1} \nu_A(y)),\ \forall y \in Y.$$

(ii) If $f : X \to Y$ is strong, then

$$f^{-1}(B)(x) = (\bigvee_{f(x,y)=1} \mu_B(x), \bigwedge_{f(x,y)=1} \nu_B(y)),\ \forall x \in X.$$

Proposition 3.19. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$ IE(Y). Let $A \in$IFS(X) and let $B \in$IFS(Y). Then :

(a) If f is strong, then $A \subset f^{-1}(f(A))$.

(b) If $IE_X = \Delta_X$ and f is injective, then $f^{-1}(f(A)) \subset A$.

(c) If f is strong surjective, then $B \subset f(f^{-1}(B))$.

(d) If $IE_Y = \Delta_Y$, then $f(f^{-1}(B)) \subset B$.

Proof. (a) Suppose f is strong and let $A \in$IFS(X). Since $\mu_A \leq \mu_{f^{-1}(f(A))}$, by the proof of Proposition 2.5(a) in [5], it is sufficient to show that $\nu_{f^{-1}(f(A))} \leq \nu_A$. For each $x \in X$, let $\nu_{f^{-1}(f(A))}(x) = \lambda$.
Then

$$\begin{aligned}\nu_{f^{-1}(f(A))}(x) &= \textstyle\bigwedge_{y \in Y}\{\bigwedge_{x' \in X}[\nu_A(x') \vee \nu_f(x', y)] \vee \nu_f(x, y)\}\\ &= \textstyle\bigwedge_{y \in Y}\{\bigwedge_{x' \in X}[\nu_A(x') \vee \nu_f(x', y) \vee \nu_f(x, y)]\}\\ &= \lambda. \end{aligned} \tag{3.6}$$

Thus $\nu_A(x') \vee \nu_f(x', y) \vee \nu_f(x, y) \geq \lambda,\ \forall y \in Y.$ (3.7)

Since f is strong, for $x \in X, \exists y_0 \in Y$ such that $\nu_f(x, y_0) = 0$. In (3.7), let $x' = x$ and let $y = y_0$. Then $\nu_{f^{-1}(f(A))} \leq \nu_A$.
Hence $A \subset f^{-1}(f(A))$.

(b) Suppose $f : X \to Y$ is injective w.r.t. $IE_X = \Delta_X$ and IE_Y. Since $\mu_{f^{-1}(f(A))} \leq \mu_A$, by the proof of Proposition 2.5(b) in [5], it is sufficient to show that $\nu_A \leq \nu_{f^{-1}(f(A))}$. For each $x \in X$, let $\lambda = \nu_{f^{-1}(f(A))}(x)$. If $\lambda = 1$, then clearly

$\nu_A(x) \leq \nu_{f^{-1}(f(A))}(x)$. Suppose $\lambda < 1$. Let $\lambda < \epsilon < 1$. Then, by (3.6), $\exists u(\epsilon) \in X$ and $y(\epsilon) \in Y$ such that

$$\nu_A(u(\epsilon)) \vee \nu_f(u(\epsilon), y(\epsilon)) \vee \nu_f(x, y(\epsilon)) < \lambda + \epsilon < 1. \quad (3.8)$$

Thus $\nu_f(u(\epsilon), y(\epsilon)) \vee \nu_f(x, y(\epsilon)) < 1$ Since f is injective,

$$\begin{aligned} 1 &> \nu_f(u(\epsilon), y(\epsilon)) \vee \nu_f(x, y(\epsilon)) \\ &= \nu_f(u(\epsilon), y(\epsilon)) \vee \nu_f(x, y(\epsilon)) \vee \nu_{IE_Y}(y(\epsilon), y(\epsilon)) \\ &\geq \nu_{IE_X}(u(\epsilon), x). \end{aligned} \quad (3.9)$$

Since $IE_X = \Delta_X$, $\nu_{IE_X}(u(\epsilon), x) = \nu_{\Delta_X}(u(\epsilon), x) = 0$. So $u(\epsilon) = x$.
By (3.8) and (3.9),

$$\begin{aligned} \lambda + \epsilon &> \nu_A(u(\epsilon)) \vee \nu_f(u(\epsilon), y(\epsilon)) \vee \nu_f(x, y(\epsilon)) \\ &\geq \nu_A(u(\epsilon)) \vee \nu_{IE_X}(u(\epsilon), x) \\ &= \nu_A(x). \end{aligned}$$

Since ϵ is arbitrary, $\nu_A(x) \leq \lambda$. In either cases, $\nu_A \leq \nu_{f^{-1}(f(A))}$. Hence $f^{-1}(f(A)) \subset A$.

By using a similar way as that in (a) and (b), it can be easily see that the properties (c) and (d) hold. □

The following is the immediate result of the definition of a mapping and Definition 3.18.

Proposition 3.20. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y).

(a) Define the (ordinary) relation $\bar{f}$ from IFS(X) to IFS(Y) as follows : $\bar{f}(A) = f(A)$, $\forall A \in$ IFS(X). Then $\bar{f}$:IFS(X) $\to$IFS(Y) is an (ordinary) mapping,

(b) Define the (ordinary) relation $\overline{\overline{f}}$ from IFS(Y) to IFS(X) as follows : $\overline{\overline{f}}(B) = f^{-1}(B)$, $\forall B \in$IFS(Y). Then $\overline{\overline{f}}$:IFS(Y) $\to$IFS(X) is an (ordinary) mapping.

Corollary 3.20-1. Let $f : X \to Y$ be strong surjective w.r.t. $IE_X = \Delta_Y \in$IE(X). Then $\overline{f} \circ \overline{\overline{f}} \circ \overline{f} = \overline{f}$.

Proof. Let $A \in IFS(X)$. Since$f$ is stron surjective, by Propositions 3.20 and 3.19 (c), $\overline{f}(A) = f(A) \subseteq f(f^{-1}(f(A))) = (\overline{f} \circ \overline{\overline{f}} \circ \overline{f})(A)$.
Then $\overline{f}(A) \subset (\overline{f} \circ \overline{\overline{f}} \circ \overline{f})(A)$. Since $IE_Y = \Delta_Y$, by Proposition 3.20 and 3.19(d),
$(\overline{f} \circ \overline{\overline{f}} \circ \overline{f})(A) = f(f^{-1}(f(A))) \subset f(A) = \overline{f}(A)$. Thus $(\overline{f} \circ \overline{\overline{f}} \circ \overline{f})(A) \subset \overline{f}(A)$. So $(\overline{f} \circ \overline{\overline{f}} \circ \overline{f})(A) = \overline{f}(A)$. Hence $\overline{f} \circ \overline{\overline{f}} \circ \overline{f} = \overline{f}$. □

Corollary 3.20-2. Let $f : X \to Y$ be an intuionistic fuzzy mapping w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$IE(Y).

(a) If f is strong, injective and $IE_X = \Delta_X$, then, $\forall A_1, A_2 \in$IFS(X), $\overline{f}(A_1) = \overline{f}(A_2) \Rightarrow A_1 = A_2$.

(b) If f is strong surjective and $IE_Y = \Delta_y$, then, $\forall B_1, B_2 \in$IFS(Y), $\overline{\overline{f}}(B_1) = \overline{\overline{f}}(B_2) \Rightarrow B_1 = B_2$.

Proposition 3.21. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$ IE(Y).

(a) If f is strong, injective and $IE_X = \Delta_X$, then $\overline{\overline{f}} \circ \overline{f}$ is bijective.

(b) If f is strong surjective and $IE_Y = \Delta_Y$, then $\overline{f} \circ \overline{\overline{f}}$ is bijective.

Proof.(a) Clearly $\overline{\overline{f}} \circ \bar{f}$:IFS$(X) \to$IFS(X) is an (ordinary) mapping. Suppose $(\overline{\overline{f}} \circ \bar{f})(A_1) = (\overline{\overline{f}} \circ \bar{f})(A_2)$, $\forall A_1, A_2 \in$IFS(X). Then, by Proposition 3.20, $f^{-1}(f(A_1)) = f^{-1}(f(A_2))$. Thus, by Proposition 3.19, $A_1 = A_2$. So $\overline{\overline{f}} \circ \bar{f}$ is injective. Let $A \in$IFS(X). Then clearly $\overline{f}(A) = f(A) \in$IFS$(Y)$. Moreover, by Proposition 3.19, $(\overline{\overline{f}} \circ \bar{f})(A) = f^{-1}(f(A)) = A$. Thus $\overline{\overline{f}} \circ \bar{f}$ is surjective. Hence $\overline{\overline{f}} \circ \bar{f}$ is bijective.

(b) Clearly $\bar{f} \circ \overline{\overline{f}}$:IFS(Y) $\to$IFS(Y) is an (ordinary) mapping. Suppose $(\bar{f} \circ \overline{\overline{f}})(B_1) = (\bar{f} \circ \overline{\overline{f}})(B_2)$, $\forall B_1, B_2 \in$IFS(Y). Then, by Proposition 3.20, $f(f^{-1}(B_1)) = f(f^{-1}(B_2))$. Thus, by Proposition 3.19, $B_1 = B_2$. So $\bar{f} \circ \overline{\overline{f}}$ is injective. Let $B \in$IFS(Y). Then clearly $f^{-1}(B) \in$ IFS(X) and $f(f^{-1}(B)) = B$. Thus $(\bar{f} \circ \overline{\overline{f}})(B) = B$. So $\bar{f} \circ \overline{\overline{f}}$ is surjective. Hence $\bar{f} \circ \overline{\overline{f}}$ is bijective. □

Proposition 3.22. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y).

(a) If f is strong, injective and $IE_X = \Delta_X$, then $\overline{f}$ is injective and $\overline{\overline{f}}$ is surjective.

(b) If f is strong surjective and $IE_Y = \Delta_Y$, then $\bar{f}$ is surjective and $\overline{\overline{f}}$ is injective.

(c) If f is strong, strong bijective, $IE_X = \Delta_X$ and $IE_Y = \Delta_Y$, then $\bar{f}$ and $\overline{\overline{f}}$ are bijective.

Proof. (a) Suppose $\overline{f}(A_1) = \overline{f}(A_2)$, $\forall A_1, A_2 \in$IFS(X). Then, by Corollary 3.20-2, $A_1 = A_2$. Thus $\overline{f}$ is injective. Let $A \in$ IFS(X). Then clearly $\overline{f}(A) = f(A) \in$ IFS(Y). Thus, by Proposition 3.19, $f^{-1}(f(A)) = A$, i.e., $\overline{\overline{f}}(f(A)) = A$. So $\overline{\overline{f}}$ is surjective.

(b) Let $B \in$IFS(Y). Then clearly $\overline{\overline{f}}(f(B)) = f^{-1}(B) \in$ IFS(X). Thus, by Proposition 3.19, $f(f^{-1}(B)) = B$, i.e., $\overline{f}(f^{-1}(B)) = B$. So $\overline{f}$ is surjective. Now suppose $\overline{\overline{f}}(f(B_1)) = \overline{\overline{f}}(f(B_2))$, $\forall B_1, B_2 \in IFS(Y)$. Then, by Corollary 3.20-2, $B_1 = B_2$. Thus $\overline{\overline{f}}$ is injective.

(c) It is clear from (a) and (b). □

Proposition 3.23. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$IE(Y). Let $A \in$IFS(X) and let $B \in$IFS(Y). Then :

(a) $f(A^c) \subset [f(A)]^c$. In particular, if f is strong surjective, then $[f(A)]^c = f(A^c)$.

(b) $f^{-1}(B^c) \subset [f^{-1}(B)]^c$. In particular, if f is strong, then $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Proof. (a) From Definition 2.3 and 3.18, it is clear that : For each $y \in Y$,

$$[f(A^c)](y) = (\bigvee_{x \in X}[\nu_A(x) \wedge \mu_f(x,y)], \bigwedge_{x \in X}[\mu_A(x) \vee \nu_f(x,y)])$$

and

$$[f(A)]^c(y) = (\bigwedge_{x\in X}[\nu_A(x) \vee \nu_f(x,y)], \bigvee_{x\in X}[\mu_A(x) \wedge \mu_f(x,y)]).$$

Moreover, for each $x \in X$,

$$\nu_A(x) \wedge \mu_f(x,y) \le \nu_A(x) \vee \nu_f(x,y)$$

and

$$\mu_A(x) \vee \nu_f(x,y) \ge \mu_A(x) \wedge \mu_f(x,y).$$

Then

$$\bigvee_{x\in X}[\nu_A(x) \wedge \mu_f(x,y)] \le \bigwedge_{x\in X}[\nu_A(x) \vee \nu_f(x,y)]$$

and

$$\bigwedge_{x\in X}[\mu_A(x) \vee \nu_f(x,y)] \ge \bigvee_{x\in X}[\mu_A(x) \wedge \mu_f(x,y)].$$

Thus $f(A^c) \subset [f(A)]^c$.

Suppose f is strong surjective and let $y \in Y$. Then $\exists x_0 \in X$ such that $f(x_0, y) = (1, 0)$. Thus

$$[f(A^c)](y) = (\nu_A(x_0), \mu_A(x_0)) = [f(A)]^c(y).$$

So $f(A^c) = [f(A)]^c$.

(b) It is clear that : For each $x \in X$,

$$[f(B)]^c(x) = (\bigwedge_{y\in Y}[\nu_B(y) \vee \nu_f(x,y)], \bigvee_{y\in Y}[\mu_B(y) \wedge \mu_f(x,y)])$$

and

$$[f(B^c)](x) = (\bigvee_{y\in Y}[\nu_B(y) \wedge \mu_f(x,y)], \bigwedge_{y\in Y}[\nu_B(y) \vee \nu_f(x,y)]).$$

By the similar way of the proof of (a), it can be easily seen that $f^{-1}(B^c) \subset [f^{-1}(B)]^c$.

Suppose f is strong and let $x \in X$. Then $\exists y_0 \in Y$ such that $f(x, y_0) = (1, 0)$. Thus

$$[f^{-1}(B^c)](x) = (\nu_B(y_0), \mu_B(y_0)) = [f^{-1}(B)]^c(x).$$

So $f^{-1}(B^c) = [f^{-1}(B)]^c$. This completes the proof. □

Proposition 3.24. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$IE(Y). Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$IFS($X$) and let $\{B_\alpha\}_{\alpha\in\Gamma} \subset$IFS($Y$). Then :

(a) $f(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f(A_\alpha)$.

(b) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.

(c) $f(\bigcap_{\alpha\in\Gamma} A_\alpha) \subset \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.

(d) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) \subset \bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.

(e) If $A_\alpha \subset A_\beta$ for $\alpha, \beta \in \Gamma$, then $f(A_\alpha) \subset f(A_\beta)$.

(f) If $B_\alpha \subset B_\beta$ for $\alpha, \beta \in \Gamma$, then $f^{-1}(B_\alpha) \subset f^{-1}(B_\beta)$.

(g) If $IE_X = \Delta_X$ and f is injective, then $\bigcap_{\alpha\in\Gamma} f(A_\alpha) \subset f(\bigcap_{\alpha\in\Gamma} A_\alpha)$.

(h) If $IE_Y = \Delta_Y$, then $\bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha) \subset f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha)$.

Proof. By Definition 3.18, it can be easily seen that the properties (a) - (f) hold.

(g) Suppose f is injective and $IE_X = \Delta_X$. For each $y \in Y$, let

$$\begin{aligned}&[\textstyle\bigcap_{\alpha\in\Gamma} f(A_\alpha](y)\\ &= (\textstyle\bigwedge_{\alpha\in\Gamma}\mu_{f(A_\alpha)}(y), \bigvee_{\alpha\in\Gamma}\nu_{f(A_\alpha)}\\ &=(\textstyle\bigwedge_{\alpha\in\Gamma}\{\bigvee_{x\in X}[\mu_{A_\alpha}(x)\wedge\mu_f(x,y)]\}, \bigvee_{\alpha\in\Gamma}\{\bigwedge_{x\in X}[\nu_{A_\alpha}(x)\vee\nu_f(x,y)]\})\\ &=(\lambda,\mu) \qquad (3.10)\end{aligned}$$

and let

$$\begin{aligned}&[f(\textstyle\bigcap_{\alpha\in\Gamma} A_\alpha)](y)\\ &= (\mu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y), \nu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y))\\ &= (\textstyle\bigvee_{x\in X}[\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(x)\wedge\mu_f(x,y)], \bigwedge_{x\in X}[\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(x)\vee\nu_f(x,y)])\\ &= (\textstyle\bigvee_{x\in X}\{\bigwedge_{\alpha\in\Gamma}[\mu_{A_\alpha}(x)\wedge\mu_f(x,y)]\}, \bigwedge_{x\in X}\{\bigvee_{\alpha\in\Gamma}[\nu_{A_\alpha}(x)\vee\nu_f(x,y)]\})\\ &= (\lambda',\mu') \qquad (3.11)\end{aligned}$$

By the proof of Proposition 2.9 (g) in [5],

$$\mu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y) \le \mu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y), \text{ i.e., } \lambda \le \lambda'. \qquad (3.12)$$

Thus it is sufficient to show that

$$\nu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y) \ge \nu_{f(\cap_{\alpha\in\Gamma}A_\alpha)}(y), \quad \text{i.e., } \mu \ge \mu'. \qquad (3.13)$$

By (3.10), $\bigwedge_{x\in X}[\nu_{\alpha\in\Gamma}(x)\vee\nu_f(x,y)] \le \mu, \quad \forall\alpha\in\Gamma$ (3.14)

If $\mu = 1$, then it is obvious that (3.13) holds. Suppose $\mu < 1$. Let $\mu < \epsilon < 1$ and let $\alpha \in \Gamma$. Then, by (3,14),

$$\exists x_\alpha(\epsilon) \in X \text{ such that } \nu_{\alpha\in\Gamma}(x_\alpha(\epsilon), y) < \mu + \epsilon < 1. \qquad (3.15)$$

Since f is injective, for any $\alpha, \beta \in \Gamma$,

$$\begin{aligned}1 &> \nu_f(x_\alpha(\epsilon,y)\vee\nu_f(x_\beta(\epsilon),y)\\ &=\nu_f(x_\alpha(\epsilon),y)\vee\nu_f(x_\beta(\epsilon),y)\vee\nu_{IE_Y}(y,y)\\ &\qquad [\text{Since } \nu_{IE_Y}(y,y)=0]\\ &\le \nu_{IE_X}(x_\alpha(\epsilon),x_\beta(\epsilon))\\ &=\nu_{\Delta_X}(x_\alpha(\epsilon),x_\beta(\epsilon)). \quad [\text{Since } IE_X=\Delta_X]\end{aligned}$$

Thus $x_\alpha(\epsilon) = x_\beta(\epsilon)$. So, for a fixed $\gamma \in \Gamma$, (3.15) implies that

$$\nu_{A_\alpha}(x_\gamma(\epsilon))\vee\nu_f(x_\gamma(\epsilon),y) < \mu+\epsilon, \quad \forall\alpha\in\Gamma,$$

i.e.,

$$\begin{aligned}\mu' &= \textstyle\bigwedge_{x\in X}\{\bigvee_{\alpha\in\Gamma}[\nu_{A_\alpha}(x)\vee\nu_f(x,y)]\}\\ &\le \textstyle\bigvee_{\alpha\in\Gamma}[\nu_{A_\alpha}(x_\gamma(\epsilon))\vee\nu_f(x_\gamma(\epsilon),y)]\\ &\le \mu+\epsilon.\end{aligned}$$

Since ϵ is arbitray, $\mu' \le \mu$, So (3.13) holds. Hence, by (3.12) and (3.13), $\bigcap_{\alpha\in\Gamma} f(A_\alpha) \subset f(\bigcap_{\alpha\in\Gamma} A_\alpha)$.

(h) Suppose $IE_Y = \Delta_Y$. For each $x \in X$, let

$$[\bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha)](x) = (\mu_{\cap_{\alpha\in\Gamma} f^{-1}(B_\alpha)}(x), \nu_{\cap_{\alpha\in\Gamma} f^{-1}(B_\alpha)}(x)) = (\lambda, \mu)$$

and let

$$[f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha)](x) = (\mu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x), \nu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x)) = (\lambda', \mu').$$

By the proof of Proposition 2.7 (h),

$$\mu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x) \leq \mu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x)\ , \quad \text{i.e., } \lambda \leq \lambda'. \tag{3.16}$$

Thus, it is sufficient to show that

$$\nu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x) \geq \nu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x)\ , \quad \text{i.e., } \mu \geq \mu'. \tag{3.17}$$

If $\mu = 1$, then it is clear that (3.17) holds. Suppose $\mu < 1$.
Let $\mu < \epsilon < 1$ and let $\alpha \in \Gamma$. Then, by the definition of μ,

$$\exists y_\alpha(\epsilon) \in Y \text{ such that } \nu_{B_\alpha}(y_\gamma(\epsilon)) \vee \nu_f(x, y_\gamma(\epsilon)) < \mu + \epsilon < 1. \tag{3.18}$$

Let $\alpha, \beta \in \Gamma$. Since f is an intuitionistic fuzzy mapping,

$$\begin{aligned} 1 &> \nu_f(x, y_\alpha)(\epsilon) \vee \nu_f(x, y_\beta(\epsilon)) \vee \nu_{IE_X}(x, x) \\ &\qquad [\text{Since } \nu_{IE_X}(x,x) = 0] \\ &\geq \nu_{IE_Y}(y_\alpha(\epsilon), y_\beta(\epsilon)) \\ &= \nu_{\Delta_Y}(y_\alpha(\epsilon), y_\beta(\epsilon)). \quad [\text{Since } IE_Y = \Delta_Y] \end{aligned}$$

Then $y_\alpha(\epsilon) = y_\beta(\epsilon)$. Thus, for a fixed $\gamma \in \Gamma$, (3.18) implies that

$$\nu_{B_\alpha}(y_\gamma(\epsilon)) \vee \nu_f(x, y_\gamma(\epsilon)) < \mu + \epsilon, \quad \forall \alpha \in \Gamma,$$

i.e.,

$$\begin{aligned} \mu' &= \nu_{f^{-1}(\cap_{\alpha\in\Gamma} B_\alpha)}(x) \\ &= \bigwedge_{y\in Y} \{\bigvee_{\alpha\in\Gamma} [\nu_{B_\alpha}(y) \vee \nu_f(x,y)]\} \\ &\leq \bigvee_{\alpha\in\Gamma} [\nu_{B_\alpha}(y_\gamma(\epsilon)) \vee \nu_f(x, y_\gamma(\epsilon))] \\ &\leq \mu + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $\mu' \leq \mu$. Hence, by (3.16), $\bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha) \subset f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha)$. This completes the proof. □

The following is the immediate result of Definition 3.1.

Proposition 3.25. Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a family of sets and let $X = \prod_{\alpha\in\Gamma} X_\alpha$ be the product of $\{X_\alpha\}_{\alpha\in\Gamma}$. If IE_{X_α} be an intuitionistic fuzzy equality on X_α for each $\alpha \in \Gamma$, then $IE_X = \prod_{\alpha\in\Gamma} IE_{X_\alpha}$ is an intuitionistic fuzzy equality on X, where $IE_X = (\mu_{IE_X}, \nu_{IE_X}) : X \times X \to I \times I$ is the complex mapping defined as follows :

$$IE_X((x_\alpha), (y_\alpha)) = (\bigwedge_{\alpha\in\Gamma} \mu_{IE_{X_\alpha}}(x_\alpha, y_\alpha), \bigvee_{\alpha\in\Gamma} \nu_{IE_{X_\alpha}}(x_\alpha, y_\alpha)),$$

for any $(x_\alpha), (y_\alpha) \in X$.

The following is the immediate result of Definition 3.3 and Proposition 3.25.

Proposition 3.26. Let $X = \prod_{\alpha\in\Gamma} X_\alpha$ be the product of a family $\{X_\alpha\}_{\alpha\in\Gamma}$ of sets. For each $\alpha \in \Gamma$, we define the intuitionistic fuzzy relation $\pi_\alpha = (\mu_{\pi_\alpha}, \nu_{\pi_\alpha})$ on $X \times X_\alpha$ as follows :

$$\pi_\alpha((x_\alpha), x) = (1, 0), \quad \text{if } x = x_\alpha$$

and

$$\mu_{\pi_\alpha}((x_\alpha), x) \geq 0 \text{ and } \nu_{\pi_\alpha}(x_\alpha), x) \leq 1, \quad \text{if } x \neq x_\alpha,$$

for each $(x_\alpha) \in X$ and each $x \in X_\alpha$.
Then $\pi_\alpha : X \to X_\alpha$ is an intuitionistic fuzzy mapping w.r.t. $IE_X = \prod_{\alpha\in\Gamma} IE_{X_\alpha} \in$ IE(X) and $IE_{X_\alpha} \in$ IE(X), $\forall \alpha \in \Gamma$. In this case, π_α is called the *intuitionistic fuzzy projection of X to X_α*. In fact, π_α is strong and strong surjective.

Proposition 3.27. Let $\pi_\alpha : X = \prod_{\alpha\in\Gamma} X_\alpha \to X_\alpha$ be the intuitionistic fuzzy projection of X to X_α and let $B_\alpha \in$IFS(X_α), $\forall \alpha \in \Gamma$. Then $\bigcap_{\alpha\in\Gamma} \pi_\alpha^{-1}(B_\alpha) = \prod_{\alpha\in\Gamma} B_\alpha$, where $\prod_{\alpha\in\Gamma} B_\alpha$ is the intuitionistic fuzzy set in X defined as follows : For each $(x_\alpha) \in X$,

$$(\prod_{\alpha\in\Gamma} B_\alpha)((x_\alpha)) = (\bigwedge_{\alpha\in\Gamma} \mu_{B_\alpha}(x_\alpha), \bigvee_{\alpha\in\Gamma} \nu_{B_\alpha}(x_\alpha)).$$

Proof. Let $(x_\alpha) \in X$. Then

$$\begin{aligned}
[\bigcap_{\alpha\in\Gamma} \pi_\alpha^{-1}(B_\alpha)]((x_\alpha)) &= (\bigwedge_{\alpha\in\Gamma} \mu_{\pi_\alpha^{-1}(B_\alpha)}((x_\alpha)), \bigvee_{\alpha\in\Gamma} \nu_{\pi_\alpha^{-1}(B_\alpha)}((x_\alpha))) \\
&= (\bigwedge_{\alpha\in\Gamma}\{\bigvee_{y\in X_\alpha}[\mu_{B_\alpha}(y) \wedge \pi_\alpha((x_\alpha), y)]\}, \bigvee_{\alpha\in\Gamma}\{\bigwedge_{y\in X_\alpha}[\nu_{B_\alpha}(y) \vee \pi_\alpha((x_\alpha), y)]\}) \\
&= (\bigwedge_{\alpha\in\Gamma} \mu_{B_\alpha}(x_\alpha), \bigvee_{\alpha\in\Gamma} \nu_{B_\alpha}(x_\alpha)) \quad [\text{Since } \pi_\alpha \text{ is strong}] \\
&= (\prod_{\alpha\in\Gamma} B_\alpha)((x_\alpha)).
\end{aligned}$$

Hence $\bigcap_{\alpha\in\Gamma} \pi_\alpha^{-1}(B_\alpha) = \prod_{\alpha\in\Gamma} B_\alpha$. □

The following is the immediate result of Definition 3.3 and Proposition 3.25.

Proposition 3.28. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. IE_X ∈IE(X) and IE_Y ∈IE(Y). We defined the intuitionistic fuzzy relation g on $(X \times X) \times (Y \times Y)$ as follws :

$$g((x, x'), (y, y')) = (\mu_f(x, y) \wedge \mu_f(x', y'), \nu_f(x, y) \vee \nu_f(x', y')),$$

for each $(x, x') \in X \times X$ and each $(y, y') \in Y \times Y$. Then $g : X \times X \to Y \times Y$ is an intuitionistic fuzzy mapping w.r.t. $IE_{X\times X} = IE_X \times IE_X$ ∈IE($X \times X$) and $IE_{Y\times Y} = IE_Y \times IE_Y$ ∈IE($Y \times Y$). In this case, g is called the *intuitionistic fuzzy product mapping of f* and is denoted by $g = f \times f = f^2$.

4. Preimage and quotient of intuitionistic fuzzy equivalence relations

Definition 4.1[2]. An intuitionistic fuzzy relation on X is called an *intuitionistic fuzzy equivalence relation* (in short, $IFER$) on X if it satisfies the following conditions :

(i) it is intuitionistic fuzzy reflexive, i.e., $R(x, x) = (1, 0) \quad \forall x \in X$,
(ii) it is intuitionistic fuzzy symmetric, i.e., $R^{-1} = R$,
(iii) it is intuitionistic fuzzy transitive, i.e., $R \circ R = R$.

We will denote the set of all IFERs on X as IFE(X).

Proposition 4.2. Let $f : X \to Y$ be an intuitionistic fuzzy mapping w.r.t. $IE_X \in$IE(X) and $IE_Y \in$ IE(Y), and let R be an intuitionistic fuzzy equivalence relation on Y. Then $f^{-2}(R)$ is an intuitionistic fuzzy equivalence relation on X. In this case, $f^{-1}(R)$ is called the *preimage of* G *under* f, where $f^{-2} = (f^2)^{-1}$.

proof. It is clear that $f^{-1}(R)$ is an intuitionistic fuzzy relation on X.

(i) Let $x \in X$. Then, by the proof of proposition 4.1 in [10], $\mu_{f^{-2}(R)}(x,x) = 1$. Thus, it is sufficient to show that $\nu_{f^{-2}(R)}(x,x) = 0$.

$$\begin{aligned}
\nu_{f^{-2}(R)}(x,x) &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_R(y,y') \vee (\nu_f \times \nu_f)((x,x),(y,y'))] \\
&\quad \text{[By Definition 3.18 and Notation } f^2 = f \times f] \\
&= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_R(y,y') \vee (\nu_f(x,y) \wedge \nu_f(x',y'))] \\
&\quad \text{[By Proposition 3.28.]} \\
&\le \nu_R(y_0,y_0) \\
&\quad \text{[Since } f \text{ is strong, } \exists y_0 \in Y \text{ such that } \nu_f(x,y_0) = 1.] \\
&=0.
\end{aligned}$$

So $\nu_{f^{-2}(R)}(x,x) = 0$. Hence $f^{-1}(R)$ is reflexive.

(ii) By the definition of $f^{-2}(R)$, it is clear that $f^{-2}(R)$ is symmetirc.

(iii) Let $x, x'' \in X$, Then, by the proof of Proposition 4.1 in [10], $\mu_{f^{-2}(R)\circ f^{-2}(R)}(x,x'') \le \mu_{f^{-2}(R)}(x,x'')$. Thus, it is sufficient to show that $\nu_{f^{-2}(R)\circ f^{-2}(R)}(x,x'') \ge \nu_{f^{-2}(R)}(x,x'')$.

$$\begin{aligned}
&\nu_{f^{-2}(R)\circ f^{-2}(R)}(x,x'') \\
&= \textstyle\bigwedge_{x'\in X}[\nu_{f^{-2}(R)}(x,x') \vee \nu_{f^{-2}(R)}(x',x'')] \\
&= \textstyle\bigwedge_{x'\in X}\{(\bigwedge_{(y,y')\in Y\times Y}[\nu_R(y,y') \vee (\nu_f \times \nu_f)((x,x'),(y,y'))]) \\
&\qquad \vee(\textstyle\bigwedge_{(y',y'')\in Y\times Y}[\nu_R(y',y'') \vee (\nu_f \times \nu_f)((x',x''),(y',y''))]\} \\
&= \textstyle\bigwedge_{x'\in X}\{(\bigwedge_{(y,y')\in Y\times Y}[\nu_R(y,y') \vee \nu_f(x,y) \vee \nu_f(x',y')]) \\
&\qquad \vee(\textstyle\bigwedge_{(y',y'')\in Y\times Y}[\nu_R(y',y'') \vee \nu_f(x',y') \vee \nu_f(x'',y'')])\} \\
&= \textstyle\bigwedge_{(y,y'')\in Y\times Y}[\nu_R(y,y_0) \vee \nu_R(y_0,y'') \vee f(x,y) \vee \nu_f(x'',y'')] \\
&\qquad \text{[Since } f \text{ is strong, } \exists y_0 \in Y \text{ such that } \nu_f(x',y_0) = 0.] \\
&\ge \textstyle\bigwedge_{(y,y'')\in Y\times Y}[\nu_R(y,y'') \vee (\nu_f \times \nu_f)((x,x''),(y,y''))] \\
&\qquad \text{[Since } f \text{ is transitive.]} \\
&= \nu_{f^{-2}(R)}(x,x'').
\end{aligned}$$

So $f^{-2}(R) \circ f^{-2}(R) \subset f^{-2}(R)$, i.e., $f^{-2}(R)$ is transitive. Hence $f^{-2}(R)$ is an intuitionistic fuzzy equivalence relation on X. □

Corollary 4.2. Let f and R be same as in Proposition 4.2. Then $f^{-2}(R) = f^{-1} \circ R \circ f$.

Proof. Let $a, b \in X$. Then

$$\begin{aligned}
&[f^{-2}(R)](a,b) \\
&= (\mu_{f^{-2}(R)}(a,b), \nu_{f^{-2}(R)}(a,b)) \\
&= (\textstyle\bigvee_{(c,d)\in Y\times Y}[\mu_R(c,d) \wedge (\mu_f \times \mu_f)((a,b),(c,d))], \\
&\qquad \textstyle\bigwedge_{(c,d)\in Y\times Y}[\nu_R(c,d) \vee (\nu_f \times \nu_f)((a,b),(c,d))]) \\
&= (\textstyle\bigvee_{(c,d)\in Y\times Y}[\mu_R(c,d) \wedge \mu_f(a,c) \wedge \mu_f(b,d)],
\end{aligned}$$

$$\begin{aligned}
&\quad \textstyle\bigwedge_{(c,d)\in Y\times Y}[\nu_R(c,d) \vee \nu_f(a,c) \vee \nu_f(b,d)]) \\
&= (\textstyle\bigvee_{d\in Y}\{(\bigvee_{c\in Y}[\mu_f(a,c) \wedge \mu_R(c,d)]) \wedge \mu_f(b,d)\}, \\
&\quad \textstyle\bigwedge_{d\in Y}\{(\bigwedge_{c\in Y}[\nu_f(a,c) \vee \nu_R(c,d)]) \vee \nu_f(b,d)\}) \\
&= (\textstyle\bigvee_{d\in Y}[\mu_{R\circ f}(a,d) \wedge \mu_f(b,d)], \bigwedge_{d\in Y}[\nu_{R\circ f}(a,d) \vee \nu_f(b,d)]) \\
&= (\textstyle\bigvee_{d\in Y}[\mu_{R\circ f}(a,d) \wedge \mu_{f^{-1}}(d,b)], \bigwedge_{d\in Y}[\nu_{R\circ f}(a,d) \vee \nu_{f^{-1}}(d,b)] \\
&= (f^{-1} \circ (R \circ f))(a,b).
\end{aligned}$$

Hence $f^{-2}(R) = f^{-1} \circ R \circ f$. □

Let R be an intuitionistic fuzzy equivalence relation on X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : For each $x \in X$,

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in IFS(X)$. The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class of R containing $a \in X$*. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of X by R* and denoted by X/R.

Result 4.A[7, Theorem 2.5]. Let $R \in$IFE(X). Then :

(a) $Ra = Rb$ if and only if $R(a, b) = (1, 0)$ for any $a, b \in X$.
(b) $Ra = Rb$ if and only if $Ra \cap Rb = 0_\sim$ for any $a, b \in X$
(c) $\bigcup_{a\in X} Ra = 1_\sim$.
(d) $\exists$ the surjection $\pi : X \to X/R$(called *the natural mapping*) defined by
$\pi(x) = Rx$ for each $x \in X$.

We obtain the generalization of Result 4.A(d).

Proposition 4.3. If R is an intuitionistic fuzzy equivalence relation on X, then $\exists$ the strong and strong surjective intuitionistic fuzzy mapping $\pi : X \to X/R$ w.r.t. $\Delta_X \in$IE(X) and $IE_{X/R} \in$IE(X/R), where $IE_{X/R} : X/R \times X/R \to I \times I$ is the intuitionistic fuzzy equality on X/R defined as follows : For any $a, b \in X$,

$$IE_{X/R}(Ra, Rb) = R(a, b).$$

In this case, π is called the *natural* (or *canonical*) *fuzzy mapping*.

Proof. We define the intuitionistic fuzzy relation $\pi : X \times X/R \to I \times I$ as follows : For any $a, b \in X$,

$$\pi(a, Rb) = Rb(a) = R(b, a).$$

Then, by Result 4.A(a), π satisfies the condition (if.1). Let $a_1, a_2, b_1, b_2 \in X$. If $a_1 \neq a_2$, then clearly $\Delta_X(a_1, a_2) = (0, 1)$. Thus

$$\mu_\pi(a_1, Rb_1) \wedge \mu_\pi(a_2, Rb_2) \wedge \mu_{\Delta_X}(a_1, a_2) = 0 \leq \mu_{IE_{X/R}}(Rb_1, Rb_2)$$

and

$$\nu_\pi(a_1, Rb_1) \vee \nu_\pi(a_2, Rb_2) \vee \nu_{\Delta_X}(a_1, a_2) = 1 \geq \nu_{IE_{X/R}}(Rb_1, Rb_2).$$

Suppose $a_1 = a_2$. Then, by the proof of Proposition 4.2 in [10],

$$\mu_\pi(a_1, Rb_1) \wedge \mu_\pi(a_2, Rb_2) \wedge \mu_{\Delta_X}(a_1, a_2) \leq \mu_{IE_{X/R}}(Rb_1, Rb_2).$$

Thus it is sufficient to show that

$$\nu_\pi(a_1, Rb_1) \vee \nu_\pi(a_2, Rb_2) \vee \nu_{\Delta_X}(a_1, a_2) \geq \nu_{IE_{X/R}}(Rb_1, Rb_2).$$

$$\begin{aligned}
&\nu_\pi(a_1, Rb_1) \vee \nu_\pi(a_2, Rb_2) \vee \nu_{\Delta_X}(a_1, a_2)\\
&= \nu_R(a_1, b_1) \vee \nu_R(a_1, b_2) \vee \nu_{\Delta_X}(a_1, a_1) \quad \text{[By the hypothesis.]}\\
&= \nu_R(b_1, a_1) \vee \nu_R(a_1, b_2) \text{ [Since } R \text{ is symmetric and } \nu_{\Delta_X}(a_1, a_1) = 0.]\\
&\geq \nu_R(b_1, b_2) \text{ [Since } R \text{ is transitive.]}\\
&= \nu_{IE_{X/R}}(Rb_1, Rb_2).
\end{aligned}$$

So π satisfies the condition (if.2). Hence $\pi : X \to X/R$ is an intuitionistic fuzzy mapping w.r.t. Δ_X and $IE_{X/R}$. Furthermore, it is clear that π is strong and strong surjective from the definition of π. □

Proposition 4.4. Let R and G be intuitionistic fuzzy equivalent relations on X such that $R \subset G$. We define the complex mapping $G/R : X/R \times X/R \to I \times I$ as follows :

$$[G/R](Ra, Rb) = G(a, b), \quad \forall a, b \in X.$$

Then G/R is an intuitionistic fuzzy equivalence relation on X/R. In this case, G/R is called the *intuitionistic fuzzy quotient of* G *by* R.

Proof. It is clear that G/R is intuitionistic fuzzy reflexive and symmetric. Let $a, c \in X$. Then, by the proof of Proposition 4.3 in [10], $\mu_{G/R\circ G/R}(Ra, Rc) \leq \mu_{G/R}(Ra, Rc)$.

On the other hand,

$$\begin{aligned}
\nu_{G/R\circ G/R}(Ra, Rc) &= \textstyle\bigwedge_{b\in X}[\nu_{G/R}(Ra, Rb) \vee \nu_{G/R}(Rb, Rc)]\\
&= \textstyle\bigwedge_{b\in X}[\nu_G(a, b) \vee \nu_G(b, c)]\\
&= \nu_{G\circ G}(a, c)\\
&\geq \nu_G(a, c) \text{ [Since } G \text{ is transitive.]}\\
&= \nu_{G/R}(Ra, Rc).
\end{aligned}$$

Thus G/R is intuitionistic fuzzy transitive. This completes the proof. □

The following is the immediate result of Proposition 4.4.

Corollary 4.4. Let $R, G, H \in$IFE(X) such that $R \subset G \subset H$. Then $G/R \subset H/R$.

Proposition 4.5. Let R, G and H be same as in Corollary 4.4.

(a) $R \subset G \circ H$.

(b) If $G \circ H \in$IFE(X), then $(G \circ H)/R$ is an intuitionistic fuzzy equivalence relation on X/R and $G/R \circ H/R = (G \circ H)/R$.

(c) $G/R \circ H/R \in$IFE(X/R).

Proof. (a) Let $a, c \in X$. Then, by the proof of Proposition 4.4(a) in [10], $\mu_{G\circ H}(a, c) \geq \mu_R(a, c)$.

On the other hand,

$$\begin{aligned}
\nu_{(G\circ H)}(a, c) &= \textstyle\bigwedge_{b\in X}[\nu_H(a, b) \vee \nu_G(b, c)]\\
&\leq \textstyle\bigwedge_{b\in X}[\nu_R(a, b) \vee \nu_R(b, c)] \text{ [Since } R \subset G \subset H]\\
&\leq \nu_R(a, c) \vee \nu_R(c, c)\\
&= \nu_R(a, c). \text{ [Since } \nu_R(c, c) = 0]
\end{aligned}$$

Thus $R \subset G \circ H$.

(b) By the hypothesis, (a) and Proposition 4.4, it is clear that $(G \circ H)/R$ is an intuitionistic fuzzy equivalence relation on X/R. Let $a, c \in X$. Then

$$\begin{aligned}
&(G/R \circ H/R)(Ra, Rc)\\
&\quad = (\textstyle\bigvee_{b\in X}[\nu_{H/R}(Ra, Rb)\wedge\nu_{G/R}(Rb, Rc)], \bigwedge_{b\in X}[\nu_{H/R}(Ra, Rb)\vee\nu_{G/R}(Rb, Rc)])\\
&\quad = (\textstyle\bigvee_{b\in X}[\nu_H(a, b) \wedge \nu_G(b, c)], \bigwedge_{b\in X}[\nu_H(a, b) \vee \nu_G(b, c)]\\
&\quad = (\nu_{G\circ H}(a, c), \nu_{G\circ H}(a, c))\\
&\quad = [(G \circ H)/R](Ra, Rc).
\end{aligned}$$

Thus $G/R \circ H/R = (G \circ H)/R$.

(c) It is obvious from (b). □

Proposition 4.6. Let R ∈IFE(X) and G ∈IFE(Y) and let the intuitiionistic fuzzy product of R and G, denoted by $R \cdot G$, be an intuitionistic fuzzy relation on $(X \times Y) \times (X \times Y)$ defined as follows : $\forall x_1, x_2 \in X, \forall y_1, y_2 \in Y$,

$$\begin{aligned}
&(R \cdot G)((x_1, y_1), (x_2, y_2))\\
&\qquad = (\mu_R(x_1, x_2) \wedge \mu_G(y_1, y_2), \nu_R(x_1, x_2) \vee \nu_G(y_1, y_2).
\end{aligned}$$

Then $R \cdot G \in IER(X \times Y)$.

Proof. By the definition of $R \cdot G$, it can be easily seen that $R \cdot G$ is intuitionistic fuzzy reflexive and symmetric. Let $(a_1, b_1), (a_2, b_2) \in X \times Y$. Then, by the proof of Proposition 4.5 in [10],

$$\mu_{(R\cdot G)\circ(R\cdot G)}((a_1, b_1), (a_2, b_2)) \le \mu_{R\cdot G}((a_1, b_1), (a_2, b_2)).$$

On the other hand,

$$\begin{aligned}
&\nu_{(R\cdot G)\circ(R\cdot G)}((a_1.b_1), (a_2, b_2))\\
&= \textstyle\bigwedge_{(a,b)\in X\times Y}[\nu_{R\cdot G}((a_1, b_1), (a, b)) \vee \nu_{R\cdot G}((a, b), (a_2, b_2))]\\
&= \textstyle\bigwedge_{(a,b)\in X\times Y}[\nu_R(a_1, a) \vee \nu_G(b_1, b) \vee \nu_R(a, a_2) \vee \nu_G(b, b_2)]\\
&= (\textstyle\bigwedge_{a\in X}[\nu_R(a_1, a) \vee \nu_R(a, a_2)]) \vee (\bigwedge_{b\in Y}[\nu_G(b_1, b) \vee \nu_G(b, b_2)])\\
&= \nu_{(R\circ R)}(a_1, a_2) \vee \nu_{(G\circ G)}(b_1, b_2)\\
&\ge \nu_R(a_1, a_2) \vee \nu_G(b_1, b_2) \text{ [Since } R \text{ and } G \text{ are intuitionistic fuzzy transitive]}\\
&= \nu_{R\cdot G}((a_1, b_1), (a_2, b_2)).
\end{aligned}$$

Thus $(R \cdot G) \circ (R \cdot G) \subset R \cdot G$. So $R \cdot G$ is intuitionistic fuzzy transitive. Hence $R \cdot G \in (X \times Y)$. □

5. Intuitionistic fuzzy equivalence relations and intuitionistic fuzzy mappings

Proposition 5.1 Let $f : X \to Y$ be a strong intuitionistic fuzzy mapping w.r.t. IE_X ∈IE(X) and IE_Y ∈ IE(Y). We define the mapping $R : X \times X \to I \times I$ as follows : For each $(x, x') \in X \times X$,

$$\begin{aligned}
R(x, x') = (&\textstyle\bigvee_{(y,y')\in Y\times Y}[\mu_f(x, y) \wedge \mu_f(x', y') \wedge \mu_{IE_Y}(y, y')],\\
&\textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_f(x, y) \vee \nu_f(x', y') \vee \nu_{IE_Y}(y, y')]).
\end{aligned}$$

Then R ∈IFE(X). In this case, R is called *the intuitionistic fuzzy equivalence relation on X determined by f* and will be denoted by R_f.

Proof. By the definition of R, it can be easily seen that R is intuitionistic fuzzy reflexive and symmetric. Let $a, c \in X$. Then, by the proof of Proposition 5.1 in [10], $\mu_{R\circ R}(a, c) \le \mu_R(a, c)$.

On the other hand,

$$\begin{aligned}\nu_{R\circ R}(a,c) &= \bigwedge_{x\in X}[\nu_R(a,x)\vee\nu_R(x,c)]\\ &= \bigwedge_{x\in X}\{(\bigwedge_{(b,b')\in Y\times Y}[\nu_f(a,b)\vee\nu_f(x,b')\vee\nu_{IE_Y}(b,b')])\\ &\qquad \vee(\bigwedge_{(b',b'')\in Y\times Y}[\nu_f(x,b')\vee\nu_f(c,b'')\vee\nu_{IE_Y}(b','')])\}\\ &= (\bigwedge_{(b,b_0)\in Y\times Y}[\nu_f(a,b)\vee\nu_f(x,b_0)\vee\nu_{IE_Y}(b,b_0)])\\ &\qquad \vee(\bigwedge_{(b_0,b'')\in Y\times Y}[\nu_f(x,b_0)\vee\nu_f(c,b'')\vee\nu_{IE_Y}(b_0,b'')])\\ &\qquad [\text{Since } f \text{ is strong, } \exists b_0\in Y \text{ such that } \nu_f(x,b_0)=0.]\\ &\geq \bigwedge_{(b,b'')\in Y\times Y}[\nu_f(a,b_0)\vee\nu_f(c,b'')\vee\nu_{IE_Y}(b,b'')]\\ &= \nu_R(a,c).\end{aligned}$$

Thus $R\circ R\subset R$. So R is intuitionistic fuzzy transitive. Hence $R\in$IFE(X). □

Corollary 5.1. Let $R\in$IFE(X). If $\pi : X\to X/R$ is the natural intuitionistic fuzzy mapping w.r.t. $\Delta_X\in$IE(X) and $IE_{X/R}\in$IE(X/R), then $R=R_\pi$.

Proof. By Proposition 4.3, it is obvious that π is strong. Let $a,b\in X$, Then, by the proof of Corollary 5.1 in [10], $\mu_R=\mu_{R_\pi}$. On the other hand,

$$\begin{aligned}\nu_{R_\pi}(a,b) &= \bigwedge_{(c,d)\in X\times X}[\nu_\pi(a,Rc)\vee\nu_\pi(b,Rb)\vee\nu_{IE_{X/R}}(Rc,Rd)]\\ &= \bigwedge_{(c,d)\in X\times X}[\nu_R(c,a)\vee\nu_R(d,b)\vee\nu_R(c,d)]\\ &\qquad [\text{By the definitions of } \pi \text{ and } IE_{X/R}.]\\ &= \bigwedge_{d\in X}\{(\bigwedge_{c\in X}(\nu_R(a,c)\vee\nu_R(c,d)])\wedge\nu_R(d,b)\}\\ &\qquad [\text{Since } R \text{ is symmetric}]\\ &= \bigwedge_{d\in X}[\nu_{R\circ R}(a,d)\vee\nu_R(d,b)]\\ &\geq \bigwedge_{d\in X}[\nu_R(a,d)\vee\nu_R(d,b)]\ [\text{Since } R \text{ is transitive}]\\ &= \nu_{R\circ R}(a,b)\\ &\geq \nu_R(a,b).\ [\text{Since } R \text{ is transitive}]\end{aligned}$$

Thus $\nu_R(a,b)\leq\nu_{R_\pi}(a,b)$.

Also,

$$\begin{aligned}\nu_R(a,b) &= \nu_R(a,a)\vee\nu_R(b,b)\vee\nu_R(a,b)\\ &=\nu_\pi(a,Ra)\vee\nu_\pi(b,Rb)\vee\nu_{IE_{X/R}}(Ra,Rb)\\ &\qquad [\text{By the definitions of } \pi \text{ and } IE_{X/R}.]\\ &\geq \bigwedge_{(c,d)\in X\times X}[\nu_\pi(a,Rc)\vee\nu_\pi(b,Rd)\vee\nu_{IE_{X/R}}(Rc,Rd)]\\ &=\nu_{R_\pi}(a,b).\ [\text{By the definitions of } R_\pi]\end{aligned}$$

Thus $\nu_{R_\pi}\leq\nu_R$. So $\nu_R=\nu_{R_\pi}$. Hence $R=R_\pi$. □

Remark 5.1. Corollary 5.1 is the generalization of Theorem 3.22 in [12] in intuitionistic fuzzy setting.

Proposition 5.2. Let $f : X\to Y$ be a strong intuitionistic fuzzy mapping w.r.t. $\Delta_X\in$IE(X) and $IE_Y\in$IE(Y) and let

$$ranf=\{y\in Y : \exists x\in X \text{ such that } \mu_f(x,y)>0 \text{ and } \nu_f(x,y)<1\}\subset Y.$$

Let R be the intuitionistic fuzzy equivalence relation determined by *f*. We define two intuitiionistic fuzzy relations s and t on $X/R\times ranf$ and $ranf\times Y$, respectively as follows:

$$s(Ra,y)=f(a,y),\forall a\in X,\forall y\in ran\ f$$

and

$$t(y, y') = \begin{cases} (1, 0) & \text{if} \quad y = y', \\ (0, 1) & \text{if} \quad y \neq y', \forall y \in ran\ f, \forall y' \in Y. \end{cases}$$

Then s is strong and bijective, t is strong and injective and $f = t \circ s \circ \pi$.

Proof. (i) By Proposition 4.3, it is obvious that $\pi : X \to X/R$ is a strong and strong surjective intuitionistic fuzzy mapping w.r.t. Δ_X and $IE_{X/R} \in E(X/R)$.

(ii) It can be easily seen that $s : X/R \to ran\ f$ is an intuitionistic fuzzy mapping w.r.t. $IE_{X/R}$ and IE_Y. Moreover, by the defninition of s, s is surjective. Let $x_1, x_2 \in X$ and let $y_1, y_2 \in ran\ f$. Then, by the proof of Proposition 5.2 in [10],

$$\mu_f(Rx_1, y_1) \wedge \mu_s(Rx_2, y_2) \wedge \mu_{IE_Y}(y_1, y_2) \leq \mu_{IE_{X/R}}(Rx_1, Rx_2).$$

On the other hand,

$$\begin{aligned}
&\nu_{IE_{X/R}}(Rx_1, Rx_2) = \nu_R(x_1, x_2) \\
&\quad = \textstyle\bigwedge_{(c,d) \in Y \times Y}[\nu_f(x_1, c) \vee \nu_f(x_2, d) \vee \nu_{IE_Y}(c, d)] \\
&\qquad \text{[Since } R \text{ is an intuitionistic fuzzy equivalence relation determined by } f] \\
&\quad \leq \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_Y}(y_1, y_2) \\
&\quad = \nu_s(Rx_1, y_1) \vee \nu_s(Rx_2, y_2) \vee \nu_{IE_Y}(y_1, y_2). \\
&\qquad \text{[By the definition of } s.]
\end{aligned}$$

Thus s is injective. Since f is strong, it is clear that s is strong. Hence s is strong and bijective.

(iii) From the definition of t, it can be easily seen that t: $ran\ f \to Y$ is strong and injective intuitionistic fuzzy mapping w.r.t. IE_Y and IE_Y.

(iv) Let $x \in X$ and let $y \in Y$. Then

$$\begin{aligned}
&(t \circ s \circ \pi)(x, y) = [(t \circ s) \circ \pi](x, y) \\
&\quad = (\textstyle\bigvee_{Ra \in X/R}[\mu_\pi(x, Ra) \wedge \mu_{t \circ s}(Ra, y)], \bigwedge_{Ra \in X/R}[\nu_\pi(x, Ra) \vee \nu_{t \circ s}(Ra, y)]) \\
&\quad = (\textstyle\bigvee_{Ra \in X/R}[\mu_R(a, x) \wedge (\bigvee_{z \in ranf}[\mu_s(Ra, z) \wedge \mu_t(z, y)])], \\
&\qquad \textstyle\bigwedge_{Ra \in X/R}[\nu_R(a, x) \vee (\bigwedge_{z \in ranf}[\nu_s(Ra, z) \vee \nu_t(z, y)])]) \\
&\qquad\quad \text{[By the definitions of } \pi \text{ and } t \circ s.] \\
&\quad = \textstyle\bigvee_{a \in X}[\mu_R(a, x) \wedge (\bigvee_{z \in ranf}[\mu_f(a, z) \wedge \mu_t(z, y)])], \\
&\qquad \textstyle\bigwedge_{a \in X}[\nu_R(a, x) \vee (\bigwedge_{z \in ranf}[\nu_f(a, z) \vee \nu_t(z, y)])]) \\
&\qquad\quad \text{[By the definition of } s.] \\
&\quad = (\textstyle\bigvee_{z \in ranf}[\mu_f(x, z) \wedge \mu_t(z, y)], \\
&\qquad \textstyle\bigwedge_{z \in ranf}[\nu_f(x, z) \vee \nu_t(z, y)] \\
&\qquad\quad \text{[Since } R \text{ is reflexive.]} \\
&\quad = (\mu_f(x, y), \nu_f(x, y)) \qquad \text{[By the definition of } t] \\
&\quad = f(x, y).
\end{aligned}$$

Thus $t \circ s \circ \pi = f$. This completes the proof. □

The following is the immediate result of Propositions 5.2

Corollary 5.2. Let f,s,t and R be same as in Proposition 5.2. If f is surjective [resp. strong surjective], then t : $ran\ f \to Y$ is strong and bijective [resp. strong bijective] and hence $s : X/R \to Y$ is strong and bijective [resp. strong bijective].

Remark 5.2. Proposition 5.2 and Corollary 5.2 are the generalizations of Theorems 3.23 and 3.24 in [12] in intuitionistic fuzzy setting.

Proposition 5.3. Let $f : X \to Y$ be a strong intuitionistic fuzzy mapping w.r.t. $\Delta_X \in$IE(X) and $IE_Y \in$ IE(Y). Let R be the intuitionistic fuzzy equivalence relation on X determined by f and let $G \in$IFE(X) such that $G \subset R$. We define the intuitionistic fuzzy relation f/G on $X/G \times Y$ as follows:

$$[f/G](Gx, y) = f(x, y), \forall x \in X, \forall y \in Y.$$

Then $f/G : X/G \to Y$ is a strong intuitionistic fuzzy mapping w.r.t. $IE_{X/G} \in$IE(X/G) and E_Y. In this case, f/G is called the *fuzzy quatient of f by G*.

Proof. From the definition of f/G, it is clear that f/G satisfies the condition (if.1). Let $Gx_1, Gx_2 \in X/G$ and let $y_1, y_2 \in Y$. Then, by the proof of Proposition 5.3 in [10],

$$\mu_{f/G}(Gx_1, y_1) \wedge \mu_{f/G}(Gx_2, y_2) \wedge \mu_{IE_{X/R}}(Gx_1, Gx_2) \leq \mu_{IE_Y}(y_1, y_2). \tag{5.1}$$

On the other hand,

$$\begin{aligned}
&\nu_{f/G}(Gx_1, y_1) \vee \nu_{f/G}(Gx_1, y_2) \vee \nu_{IE_{X/G}}(Gx_1, Gx_2)\\
&= \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_G(x_1, x_2)\\
&\geq \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_R(x_1, x_2) \quad \text{[Since } G \subset R.]\\
&= \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee (\textstyle\bigwedge (c, d) \in Y \times Y[\nu_f(x_1, c) \vee \nu_f(x_2, d) \vee \nu_{IE_Y}(c, d)])\\
&\quad \text{[Since } R \text{ is the intuitionistic fuzzy equivalence relation determined by } f]\\
&= \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_Y}(y_1, y_2)\\
&\quad \text{[Since } f \text{ is strong, } \exists c_0, d_0 \in Y \text{ such that } f(x_1, c_0) = f(x_2, d_0) = (1, 0).]\\
&= \nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{IE_Y}(c_0, c_0).
\end{aligned} \tag{5.2}$$

Since $f : X \to Y$ is an intuitionistic fuzzy mapping w.r.t. Δ_X and IE_Y,

$$\nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{\Delta_X}(x_1, x_2) \geq \nu_{IE_Y}(y_1, y_2). \tag{5.3}$$

By (5.1) and (5.2),

$$\begin{aligned}
&\nu_f(x_1, y_1) \vee \nu_f(x_2, y_2) \vee \nu_{\Delta_X}(x_1, x_2) \vee \nu_{IE_Y}(c_0, d_0)\\
&\geq \nu_{IE_Y}(c_0, d_0) \vee \nu_{IE_Y}(y_1, y_2) \geq \nu_{IE_Y}(y_1, y_2).
\end{aligned}$$

Thus

$$\nu_{f/G}(Gx_1, y_1) \vee \nu_{f/G}(Gx_2, y_2) \vee \nu_{IE_{X/R}}(Gx_1, Gx_2) \geq \nu_{IE_Y}(y_1, y_2). \tag{5.4}$$

So, by (5.1) and (5.4), f/G satisfies the condition (if.2). Since f is strong, it is clear that f/G is strong. Hence $f/G : X/G \to Y$ is strong w.r.t. $IE_{X/G}$ and IE_Y. □

Proposition 5.4. Let f, R, G and f/G be same as in Proposition 5.3. Then R/G is the intuitionistic fuzzy equivalence relation on X/G determined by f/G.

Proof. Let $R_{f/G}$ be the intuitionistic fuzzy equivalence relation on X/G determined by f/G and let $Ga, Gb \in X/G$. Then, by the proof of Proposition 5.4 in [10],

$$\mu_{R_{f/G}}(Ga, Gb) = \mu_{R/G}(Ra, Rb).$$

On the other hand,

$$\begin{aligned}
\nu_{f/G}(Ga, Gb) &= \textstyle\bigwedge_{(c,d) \in Y \times Y}[\nu_{f/G}(Ga, c) \vee \nu_{f/G}(Gb, d) \vee \nu_{IE_Y}(c, d)]\\
&= \textstyle\bigwedge_{(c,d) \in Y \times Y}[\nu_f(a, c) \vee \nu_f(b, d) \vee \nu_{IE_Y}(c, d)]\\
&= \nu_R(a, b) \text{ [By Proposition 5.1]}
\end{aligned}$$

$=\nu_{R/G}(Ga, Gb)$. [By Proposition 4.4]

Thus $R_{f/G} = R/G$. So R/G is the intuitionistic fuzzy equivalence relation on X/G determined by f/G. $\square$

Remark 5.4. Proposition 5.4. is the generalization of Theorem 3.26 in [12] in intuitionistic fuzzy setting.

Proposition 5.5. Let $R, G \in$IFE(X) such that $G \subset R$. Then $\exists$ a strong and strong bijective intuitionistic fuzzy mapping $h : (X/G)/(R/G) \to X/R$.

Proof. By Proposition 4.3, $\exists$ a strong and strong surjective intuitionistic fuzzy mapping $\pi : X \to X/R$ w.r.t. $\Delta_X \in$IE(X) and $IE_{X/R} \in$IE(X/R). By Corollary 5.1, it is clear that $R = R_\pi$. Then, by Proposition 5.3, $\pi/G : X/G \to X/R$ is strong w.r.t. $IE_{X/G} \in$IE(X/G) and $IE_{X/R}$. Thus, by Proposition 5.4, $R/G = R_{\pi/G}$. Since π is strong surjective, π/G is strong surjective. So, $\pi/G : X/G \to X/R$ is strong and strong surjective. Hence, by Corollary 5.2, $\exists$ a strong and strong bijective intuitionistic fuzzy mapping $h : (X/G)/(R/G) \to X/R$. $\square$

The following is the immediate result of Proposition 5.5.

Corollary 5.5. Let $R, G \in$IFE(X). Then :

(a) $\exists$ a bijective intuitionistic fuzzy mapping $g : X/(R \circ G) \to (X/R)/(R \circ G/R)$.

(b) $\exists$ a bijective intuitionistic fuzzy mapping $h : X/R \to (X/R \cap G)/(R/R \cap G)$.

Proposition 5.6. Let $f : X \to Y$ be a strong and strong surjective intuitionistic fuzzy mapping w.r.t. $\Delta_X \in$ IE(X) and $IE_Y \in$IE(Y), and let $R \in$IFE(X). Then $f^2(R) \in$IFE(Y). In this case, $f^2(R)$ is called the *image of* G *under* f.

Proof. By the definition of $f^2(R)$, it can be easily seen that $f^2(R)$ is intuitionistic fuzzy reflexive and symmetric. Let $y, y'' \in Y$. Then, by the proof of Proposition 5.6 in [10],

$$\mu_{f^2(R)\circ f^2(R)}(y, y'') \leq \mu_{f^2(R)}(y, y'').$$

On the other hand,

$$\begin{aligned}
&\nu_{f^2(R)\circ f^2(R)}(y,'')\\
=&\textstyle\bigwedge_{y' \in Y}[\nu_{f^2(R)}(y, y') \vee \nu_{f^2(R)}(y', y'')]\\
=&\textstyle\bigwedge_{y' \in Y}\{(\bigwedge_{(x,x')\in X\times X}[\nu_R(x, x') \vee \nu_f(x, y) \vee \nu_f(x', y')])\\
&\quad \vee(\textstyle\bigwedge_{(x',x'')\in X\times X}[\nu_R(x', x'') \vee \nu_f(x', y') \vee \nu_f(x'', y'')])\}\\
=&\textstyle\bigwedge_{(x,x'')\in X\times X}[\nu_R(x, x_0) \vee \nu_R(x_0, x'') \vee \nu_f(x, y) \vee f(x'', y'')]\\
&\quad \text{[Since } f \text{ is strong surjective, } \exists x_0 \in X \text{ such that } f(x_0, y') = (1, 0).]\\
\geq& \textstyle\bigvee_{(x,x'')\in X\times X}[\nu_R(x, x'') \vee \nu_f(x, y) \vee \nu_f(x'', y'')]\\
&\quad \text{[Since } R \text{ is transitive.]}\\
=&\nu_{f^2(R)}(y, y'').
\end{aligned}$$

Thus $f^2(R) \circ f^2(R) \subset f^2(R)$. So $f^2(R)$ is intuitionistic fuzzy transitive. Hence $f^2(R) \in$IFE(X). $\square$

Theorem 5.7. Let $f : X \to Y$ be strong and strong surjective w.r.t. $\Delta_X \in$IE(X) and $IE_Y \in$IE(Y), let $R = R_f$ and let $G \in$IFE(Y). Then :

(a) $R \subset f^{-2}(G)$.

(b) $H = f^{-2}(G)$ if and only it $G = f^2(H)$.

Hence $\exists$ a bijection h :IFE$(Y) \to$IFE$_R(X)$, where IFE$_R(X)$ denotes the set of all intuitionistic fuzzy equivalence relations on X containing R.

Proof. (a) Let $x, x' \in X$. Then, by the proof of Theorem 5.7(a),

$$\mu_R(x,x') \le \mu_{f^{-2}(G)}(x,x').$$

On the other hand,

$$\begin{aligned}\nu_R(x,x') &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_f(x,y) \vee \nu_f(x',y') \vee \nu_{IE_Y}(y,y')] \quad \text{[By Proposition 5.1]}\\ &\ge \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_f(x,y) \vee \nu_f(x',y')]\\ &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_G(y_0,y_0) \vee \nu_f(x,y) \vee \nu_f(x',y')] \quad \text{[Since } G(y_0,y_0) = 1]\\ &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_G(y,y') \vee \nu_f(x,y) \vee \nu_f(x',y')]\\ &= \nu_{f^{-2}(G)}(x,x').\end{aligned}$$

Thus $R \subset f^{-2}(G)$.

(b) ($\Rightarrow$): Suppose $H = f^{-2}(G)$ and let $y, y' \in Y$. Then, by the proof of Theorem 5.7(b) in [10],

$$\mu_{f^2(H)}(y,y') = \mu_G(y,y').$$

On the other hand,

$$\begin{aligned}&\nu_{f^2(H)}(y,y')\\ &= \textstyle\bigwedge_{(x,x')\in X\times X}[\nu_H(x,x') \vee \nu_{f^2}((x,x'),(y,y'))]\\ &= \textstyle\bigwedge_{(x,x')\in X\times X}[\nu_{f^{-2}(G)}(x,x') \vee \nu_f(x,y) \vee \nu_f(x',y')]\\ &= \nu_{f^{-2}(G)}(x_0,x'_0)\\ &\text{[Since } f \text{ is strong surjective, } \exists x_0, x'_0 \in X \text{ such that } f(x_0,y) = f(x'_0,y') = (1,0).]\\ &= \textstyle\bigwedge_{(z,z')\in Y\times Y}[\nu_G(z,z') \vee \nu_f(x_0,z) \vee \nu_f(x'_0,z')]\\ &= \nu_G(y,y'). \quad \text{[Since } f(x,y_0) = f(x',y'_0) = (1,0).]\end{aligned}$$

Thus $f^2(H) = G$.

($\Leftarrow$): Suppose $f^2(H) = G$ and let $x, x' \in X$. Then, by the proof of Theorem 5.7(b) in [10],

$$\mu_{f^{-2}(G)}(x,x') = \mu_H(x,x').$$

$$\begin{aligned}&\nu_{f^{-2}(G)}(x,x')\\ &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_G(y,y') \vee \nu_{f^2}((x,x'),(y,y'))]\\ &= \textstyle\bigwedge_{(y,y')\in Y\times Y}[\nu_{f^2(H)}(y,y') \vee \nu_f(x,y) \vee \nu_f(x',y')]\\ &= \nu_{f^2(H)}(y_0,y'_0)\\ &\text{[Since } f \text{ is strong, } \exists y_0, y'_0 \in Y \text{ such that } f(x,y_0) = f(x',y'_0) = (1,0).]\\ &= \textstyle\bigwedge_{(a,b)\in X\times X}[\nu_H(a,b) \vee \nu_f(a,y_0) \vee \nu_f(b,y'_0)]\\ &= \nu_H(x,x'). \quad \text{[Since } f(x,y_0) = f(x',y'_0) = (1,0).]\end{aligned}$$

Thus $f^{-2}(G) = H$.

Now we define h :IFE$(Y) \to$IFE$_R(X)$ as follows: $\forall G \in$IFE(Y), $h(G) = f^{-2}(G)$. Then, by Proposition 5.6 and (a), clearly $h(G) \in$IFE$_R(X)$. It is easy to see that h is bijective. This completes the proof. $\square$

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