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On soft ternary semigroups

Muhammad Shabir, Ali Ahmad

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Abstract. In this paper we introduce the notions of soft ternary semigroups, soft ideals, soft quasi-ideals, soft bi-ideals and characterize some classes of the ternary semigroups by the properties of these soft ideals.

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Corresponding Author: Ali Ahmad (ali.ahmad200873@yahoo.com)

1. Introduction

The introduction of the mathematical literature of ternary algebraic system dated back to 1932. In 1932, Lehmer [8], investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroup was introduced by Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. In [7], Good and Hughes introduced the notion of bi-ideals and in [14], Steinfeld introduced the notion of quasi-ideals in semigroups. In [13], Sioson studied some properties of quasi-ideals of ternary semigroups. In [4], Dixit and Dewan studied about the quasi-ideals and bi-ideals in ternary semigroups. In [12], Shabir and Bano initiated the concept of prime, semiprime and strongly prime bi-ideals in ternary semigroups.

In dealing with uncertainties, many theories have been recently developed, including the theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets and theory of rough sets and so on. Although many new techniques have been developed as a result of these theories, yet difficulties are still there. The major difficulties posed by these theories are probably due to the inadequacy of parameters.

In 1999, Molodtsov [10], initiated the novel concept of soft set theory, which was a completely new approach for modeling uncertainty and had a rich potential for applications in several directions. This so-called soft set theory is free from the difficulties affecting existing methods. Later on, Maji et al. have defined some binary operations on soft sets in [9]. These binary operations were corrected by Ali et al.

in [3]. In [5], Feng et al. established a connection between soft sets and rough sets, they initiated the concept of soft rough approximations, soft rough sets and some related notions. Applications of soft set theory in algebraic structures were initiated by Aktas and Cagman [2]. Feng et al. introduced the notion of soft semirings [6]. In [1], Acar et al. introduced the initial concepts of soft rings. In [11], Shabir and Ali, introduced the notion of soft semigroups and soft ideals over a semigroup and discussed some of their basic properties.

By a soft ternary semigroup, we mean a collection of ternary subsemigroups of a ternary semigroup, whereas a soft left (right, lateral, quasi, bi) ideal is a collection of left (right, lateral, quasi, bi) ideals of a given ternary semigroup. In this paper, we have introduced the concept of soft ternary semigroups, soft left (right, lateral, quasi, bi) ideal of ternary semigroups and also proved some different results of soft left (right, lateral, quasi, bi) ideal of ternary semigroups.

2. Preliminaries

A ternary semigroup is an algebraic structure (S, f) such that S is a non-empty set and $f : S^3 \to S$ is a ternary operation satisfying the following associative law:

$$f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e))$$

for all $a, b, c, d, e \in S$. For simplicity we write $f(a, b, c)$ as abc and consider the ternary operation f as "$\cdot$". A ternary semigroup $(S, \cdot)$ is called commutative (laterally commutative) if $abc = bac = bca$ $(abc = cba)$ for all $a, b, c \in S$. A non-empty subset T of a ternary semigroup S is said to be a ternary subsemigroup of S if $TTT = T^3 \subseteq T$. By a left (right, lateral) ideal of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ $(ASS \subseteq A,\ SAS \subseteq A)$. By a two sided ideal, we mean a subset of S which is both a left and a right ideal of S. If a non-empty subset of S is a left, right and a lateral ideal of S, then it is called an ideal of S. A left (right, lateral, two sided) ideal I of a ternary semigroup S is idempotent if $I^3 = I$. A non-empty subset Q of a ternary semigroup S is called a quasi-ideal of S if $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q$, $(QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$. Every left, right and lateral ideal in a ternary semigroup is a quasi-ideal but the converse is not true in general. A non-empty subset A of a ternary semigroup S is called a bi-ideal of S if $AAA \subseteq A$ and $ASASA \subseteq A$. Every quasi-ideal of a ternary semigroup is a bi-ideal. An element a in a ternary semigroup S is called regular if there exists an element $x \in S$ such that $a = axa$, that is $a \in aSa$. A ternary semigroup S is called regular if all its elements are regular. Throughout this paper, S will denote a ternary semigroup unless stated otherwise.

3. Soft Sets

Definition 3.1 ([10])**.** Let U be an initial universe and E a set of parameters. Let $P(U)$ denotes the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$ approximate elements of the soft set (F, A).

Clearly, a soft set is not a set.

Definition 3.2 ([9])**.** Let (F, A) and (G, B) be two soft sets over a common universe U. Then (F, A) is called a soft subset of (G, B) if,
(1) $A \subseteq B$ and
(2) $F(a) \subseteq G(a)$ for all $a \in A$.
We write $(F, A) \subseteq (G, B)$.

(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A) \supseteq (G, B)$.

Definition 3.3 ([9])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 3.4 ([9])**.** Let (F, A) and (G, B) be two soft sets over a common universe U. Then "(F, A) OR (G, B)" denoted by $(F, A) \vee (G, B)$ is a soft set defined by $(K, A \times B) = (F, A) \vee (G, B)$, where $K(\alpha, \beta) = F(\alpha) \cup G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 3.5 ([9])**.** Let (F, A) and (G, B) be two soft sets over a common universe U. Then "(F, A) AND (G, B)" denoted by $(F, A) \wedge (G, B)$ is a soft set defined by $(H, A \times B) = (F, A) \wedge (G, B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 3.6 ([3])**.** Let (F, A) and (G, B) be two soft sets over a universe U.
(1) The extended union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cup_E (G, B) = (H, C)$.
(2) The restricted union of two soft sets (F, A) and (G, B) is denoted by $(F, A) \cup_R (G, B)$, and is defined as the soft set $(F, A) \cup_R (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.
(3) The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cap G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A) \cap_E (G, B) = (H, C)$.
(4) The restricted intersection of (F, A) and (G, B) is denoted by $(F, A) \cap_R (G, B)$, and is defined as $(F, A) \cap_R (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

4. Major Section

Definition 4.1. A soft set (S, E) over S is said to be an absolute soft set over S, if for all $e \in E$, $S(e) = S$.

Definition 4.2. The restricted product (K, D) of three soft sets (F, A), (G, B) and (H, C) over a ternary semigroup S is defined as the soft set

$$(K, D) = (F, A)\hat{o}(G, B)\hat{o}(H, C),$$

where $D = A \cap B \cap C$ is non-empty and K is a set valued function from D to $P(S)$ defined as $K(d) = F(d)G(d)H(d)$ for all $d \in D$.

Definition 4.3. A soft set (F, A) over a ternary semigroup S is called a soft ternary semigroup over S if,

$$(F, A)\hat{o}(F, A)\hat{o}(F, A) \subseteq (F, A).$$

Proposition 4.4. *A soft set (F, A) over a ternary semigroup S is a soft ternary semigroup over S if and only if for all $a \in A$, $F(a) \neq \varnothing$ is a ternary subsemigroup of S.*

Proof. Suppose that (F, A) is a soft ternary semigroup over S. We prove that $F(a) \neq \varnothing$ is a ternary subsemigroup of S. By definition

$$(F, A)\hat{o}(F, A)\hat{o}(F, A) = (H, A \cap A \cap A) = (H, A)$$

where H is defined by

$$H(a) = F(a)F(a)F(a), \text{ for all } a \in A.$$

As $(F, A)\hat{o}(F, A)\hat{o}(F, A) \subseteq (F, A)$; so $(H, A) \subseteq (F, A)$. That is, $H(a) \subseteq F(a)$ for all $a \in A$, and so $F(a)F(a)F(a) \subseteq F(a)$. This implies that $F(a)$ is a ternary subsemigroup of S.

Conversely, suppose that $F(a) \neq \varnothing$ is a ternary subsemigroup of S. We show that (F, A) is a soft ternary semigroup over S. By definition

$$(F, A)\hat{o}(F, A)\hat{o}(F, A) = (H, A \cap A \cap A) = (H, A)$$

where $H(a) = F(a)F(a)F(a)$, for all $a \in A$. Since $F(a)$ is a ternary subsemigroup of S, we have $F(a)F(a)F(a) \subseteq F(a)$, that is, $H(a) \subseteq F(a)$. Thus, $(H, A) \subseteq (F, A)$. This implies that

$$(F, A)\hat{o}(F, A)\hat{o}(F, A) \subseteq (F, A).$$

Hence, (F, A) is a soft ternary semigroup over S. □

Example 4.5. Let $S = \{a, b, c, d\}$ be a semigroup under the operation ( ), given below :

( )	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define the ternary operation [ ], as $[xyz] = x\,(yz) = (xy)\,z$. Then $(S, [\ \ ])$ is a ternary semigroup. We define $F : S \to P(S)$ by $F(a) = \{a\}$, $F(b) = \{a, b\}$, $F(c) = \{a, b, c\}$, $F(d) = \{a, b, d\}$ .It is clear that each $F(x)$ is a ternary subsemigroup of S for all $x \in S$. Hence, (F, S) is a soft ternary semigroup over S. It is clear that not every soft set over a ternary semigroup S, gives us a soft ternary semigroup over S. Let $G : S \to P(S)$ defined by $G(x) = \{y \in S : y = x\}$. Then (G, S) is a soft set over S, but not a soft ternary semigroup over S, because $G(c) = \{c\}$ is not a ternary subsemigroup of S.

Definition 4.6. A soft set (F, A) over a ternary semigroup S is called a soft left (right, lateral) ideal over S if $(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)$ $((F, A)\hat{o}(S, E)\hat{o}(S, E) \subseteq (F, A)$, $(S, E)\hat{o}(F, A)\hat{o}(S, E) \subseteq (F, A))$, respectively. A soft set (F, A) over S is called a soft ideal over S, if it is a soft left, soft right and a soft lateral ideal over S.

Proposition 4.7. *A soft set (F, A) over S is a soft left (right, lateral) ideal over S if and only if for all $a \in A$, $F(a) \neq \varnothing$ is a left (right, lateral) ideal of S.*

Proof. Suppose that (F, A) is a soft left ideal over S. We show that $F(a) \neq \varnothing$ is a left ideal of S. By definition

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (H, E \cap E \cap A) = (H, A)$$

$\Rightarrow$

$$S(a)S(a)F(a) = H(a), \quad \text{for all } a \in A$$

That is,

$$SSF(a) = H(a).$$

As

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A).$$

$\Rightarrow$

$$(H, A) \subseteq (F, A).$$

Thus,

$$H(a) \subseteq F(a) \ \text{ for all } a \in A.$$

Therefore,

$$SSF(a) \subseteq F(a).$$

This shows that $F(a)$ is a left ideal of S.

Conversely, assume that $F(a) \neq \varnothing$ is a left ideal of S. We show that (F, A) is a soft left ideal over S. By definition

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (H, E \cap E \cap A) = (H, A)$$

$\Rightarrow$

$$S(a)S(a)F(a) = H(a), \text{ for all } a \in A$$

That is,

$$SSF(a) = H(a).$$

But

$$SSF(a) \subseteq F(a)$$

$\Rightarrow$

$$H(a) \subseteq F(a)$$

$\Rightarrow$

$$(H, A) \subseteq (F, A).$$

So

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A).$$

Hence, (F, A) is a soft left ideal over S. □

Definition 4.8. Let (F, A), (G, B) and (H, C) be three soft sets over a ternary semigroup $(S,*)$. Then the ternary operation $*$ (extended product) for soft sets is defined as $(K, D) = (F, A)*(G, B)*(H, C)$. Where $D = A\times B\times C$ and $K(a, b, c) = F(a) * G(b) * H(c)$, $a \in A$, $b \in B$, $c \in C$, where $A \times B \times C$ is the Cartesian product of the sets A, B and C. If there does not arise any ambiguity then we can write $(F, A)(G, B)(H, C)$ instead of $(F, A) * (G, B) * (H, C)$ and $F(a)G(b)H(c)$ for $F(a) * G(b) * H(c)$.

Proposition 4.9. *Let (F, A) and (G, B) be two soft ternary semigroups over S, such that $A \cap B \neq \varnothing$. Then $(F, A) \cap_R (G, B)$ is a soft ternary semigroup over S.*

Proof. By definition $(H, C) = (F, A) \cap_R (G, B)$, where $C = A \cap B \neq \varnothing$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. As $F(c)$ and $G(c)$ both are ternary subsemigroups of S, so $H(c)$ is either empty or a ternary subsemigroup of S. Consequently, (H, C) is a soft ternary semigroup over S. □

Proposition 4.10. *Let (F, A) and (G, B) be two soft ternary semigroups over S such that $A \cap B = \varnothing$. Then $(F, A) \cup_E (G, B)$ is a soft ternary semigroup over S.*

Proof. By definition $(H, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ and $A \cap B = \varnothing$. Then for all $c \in C$ either $c \in A - B$ or $c \in B - A$. If $c \in A - B$, then $H(c) = F(c)$ and if $c \in B - A$, then $H(c) = G(c)$, in both cases $H(c)$ is a ternary subsemigroup of S. Therefore, (H, C) is a soft ternary semigroup over S. □

Proposition 4.11. *Let (F, A) and (G, B) be two soft ternary semigroups over S. Then $(F, A) \wedge (G, B)$ is a soft ternary semigroup over S.*

Proof. By definition $(H, C) = (F, A) \wedge (G, B)$, where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$. As $F(a)$ and $G(b)$ are ternary subsemigroups of S, therefore either $F(a) \cap G(b) = \varnothing$ or $F(a) \cap G(b)$ is a ternary subsemigroup of S. Consequently, (H, C) is a soft ternary semigroup over S. □

Proposition 4.12. *Let (F, A), (G, B) and (H, C) be any three soft ternary semigroups over a commutative ternary semigroup S. Then $(F, A) * (G, B) * (H, C)$ is a soft ternary semigroup over S.*

Proof. By definition $(K, A \times B \times C) = (F, A) * (G, B) * (H, C)$, where

$$K(a, b, c) = F(a)G(b)H(c) \text{ for all } (a, b, c) \in A \times B \times C.$$

We show that $(F, A)*(G, B)*(H, C)$ is a soft ternary semigroup over S. It is enough to show that $K(a, b, c) = F(a)G(b)H(c)$ is a ternary subsemigroup of S. Since S is commutative, so

$$\begin{aligned}
&(K(a,b,c)K(a,b,c)K(a,b,c))\\
&=((F(a)G(b)H(c))(F(a)G(b)H(c))(F(a)G(b)H(c)))\\
&=((F(a)G(b)H(c))F(a)(G(b)H(c)(F(a))G(b)H(c))\\
&=(F(a)(G(b)H(c)F(a))(F(a)H(c)G(b))G(b)H(c))\\
&=(F(a)F(a)(H(c)G(b)F(a))(H(c)G(b)G(b))H(c))\\
&=((F(a)F(a)F(a))G(b)(H(c)H(c)G(b))G(b)H(c))\\
&=((F(a)F(a)F(a))G(b)G(b)(H(c)H(c)G(b))H(c))\\
&=((F(a)F(a)F(a))(G(b)G(b)G(b))(H(c)H(c)H(c)))\\
&\subseteq(F(a)G(b)H(c))=K(a,b,c)
\end{aligned}$$

because $F(a)$, $G(b)$ and $H(c)$ are ternary subsemigroups. This implies that

$$K(a,b,c)K(a,b,c)K(a,b,c)\subseteq K(a,b,c).$$

Thus $K(a,b,c)$ is a ternary subsemigroup of S. Hence $(F,A)*(G,B)*(H,C)$ is a soft ternary semigroup over S. □

If S is non-commutative, then $(F,A)*(G,B)*(H,C)$ is not necessarily a soft ternary semigroup over S. The following example shows that in a non-commutative ternary semigroup the above Proposition does not hold.

Example 4.13. Let $S=\{a,b,c,d\}$ be a semigroup under the operation ( ), given below :

( )	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	d

Define the ternary operation [ ], as $[xyz]=x\,(yz)=(xy)\,z$. Then $(S,[\ \ ])$ is a ternary semigroup. We define soft ternary semigroups (F,A), (G,B) and (H,C), as $A=\{a,b,c,d\}$, $B=\{a,d\}$ and $C=\{a,c,d\}$ such that $F(a)=\{a\}$, $F(b)=\{c\}$, $F(c)=\{a\}$, and $F(d)=\{b\}$, $G(a)=\{a,d\}$, $G(d)=\{a,c\}$, $H(a)=\{a,b,c\}$, $H(c)=\{b,c\}$, $H(d)=\{d\}$. Now, $(F,A)*(G,B)*(H,C)=(K,A\times B\times C)$ and $K(x,y,z)=F(x)G(y)H(z)$ for all $(x,y,z)\in A\times B\times C$. As $K(a,d,d)=F(a)G(d)H(d)=\{c,d\}$, which is not a ternary subsemigroup of S. Hence, $(F,A)*(G,B)*(H,C)$ is not a soft ternary semigroup over S.

Proposition 4.14. *Let (F,A) and (G,B) be any two soft ideals over a ternary semigroup S, with $A\cap B\neq\varnothing$. Then $(F,A)\cap_R(G,B)$ is a soft ideal over S contained in both (F,A) and (G,B).*

Proof. Straightforward. □

Proposition 4.15. *Let (F,A) and (G,B) be any two soft ideals over a ternary semigroup S. Then $(F,A)\cup_E(G,B)$ is a soft ideal over S containing both (F,A) and (G,B).*

Proof. By definition $(H, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ for all $c \in C = A \cup B$, either $c \in A - B$ or $c \in B - A$ or $c \in A \cap B$. If $c \in A - B$, then $H(c) = F(c)$, if $c \in B - A$, then $H(c) = G(c)$, and if $c \in A \cap B$, then $H(c) = F(c) \cup G(c)$, in all the cases $H(c)$ is an ideal of S. Hence, (H, C) is soft ideal over S. As $A \subseteq A \cup B$, $B \subseteq A \cup B$ and $F(c) \subseteq H(c)$, $G(c) \subseteq H(c)$ for all $c \in C$. Therefore, by definition of soft subsets $(F, A) \subseteq (H, C)$ and $(G, B) \subseteq (H, C)$. □

Proposition 4.16. *Let (F, A) and (G, B) be any two soft ideals over a ternary semigroup S. Then $(F, A) \wedge (G, B)$ is a soft ideal over S.*

Proof. Straightforward. □

Proposition 4.17. *Let (F, A) and (G, B) be any two soft ideals over a ternary semigroup S. Then $(F, A) \vee (G, B)$ is a soft ideal over S.*

Proof. Straightforward. □

5. Soft ternary subsemigroups of soft ternary semigroups

Definition 5.1. Let (G, B) be a soft subset of a soft ternary semigroup (F, A) over S. Then (G, B) is called a soft ternary subsemigroup (ideal) of (F, A) if $G(b)$ is a ternary subsemigroup (ideal) of $F(b)$ for all $b \in B$.

The soft ideal of a soft ternary semigroup defined above is different from the soft ideal over S. The following example is to depict this fact.

Example 5.2. Let $S = \{0, a, b, c\}$ be a semigroup under the operation $(\ )$, given below :

$(\ )$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Define the ternary operation $[\ ]$, as $[xyz] = x\,(yz) = (xy)\,z$. Then $(S, [\ ])$ is a ternary semigroup. Consider the soft set (F, S), where $F : S \to P(S)$ is defined as $F(0) = \{0\}$, $F(a) = \{0, a\}$, $F(b) = \{0, a, b\}$, $F(c) = \{0, a, b, c\}$. It is clear that (F, S) is a soft ternary semigroup over S. Now, consider the soft set $(G, \{b\})$ in which $G : \{b\} \to P(S)$ is defined as $G(b) = \{0, c\}$. As $\{b\} \subseteq S$ and $G(b)$ is an ideal of $F(b)$, therefore $(G, \{b\})$ is a soft ideal of (F, S). But $G(b) = \{0, c\}$ is not an ideal over S, so $(G, \{b\})$ is not a soft ideal over S.

Theorem 5.3. *Let (F, A) be a soft ternary semigroup over S and $\{(H_i, B_i) : i \in I\}$ be a non-empty family of soft ternary subsemigroups of (F, A). Then*

(1) $\cap_{Ri \in I}(H_i, B_i)$ *is a soft ternary subsemigroup of* (F, A).
(2) $\wedge_{i \in I}(H_i, B_i)$ *is a soft ternary subsemigroup of* $\wedge_{i \in I}(F, A)$.
(3) *If* $B_i \cap B_j = \varnothing$ *for all different* $i, j \in I$, *then* $\cup_{Ei \in I}(H_i, B_i)$ *is a soft ternary subsemigroup of* (F, A).

Proof. Straightforward. □

Theorem 5.4. *Let (F, A) be a soft ternary semigroup over S and $\{(H_i, B_i) : i \in I\}$ be a non-empty family of soft ideals of (F, A). Then*

(1) $\cap_{Ri\in I}(H_i, B_i)$ *is a soft ideal of* (F, A).
(2) $\wedge_{i\in I}(H_i, B_i)$ *is a soft ideal of* $\wedge_{i\in I}(F, A)$.
(3) $\cup_{Ri\in I}(H_i, B_i)$ *is a soft ideal of* (F, A).
(4) $\vee_{i\in I}(H_i, B_i)$ *is a soft ideal of* $\vee_{i\in I}(F, A)$.

Proof. Straightforward. □

6. Soft quasi-ideals over ternary semigroups

Definition 6.1. A soft set (F, A) over a ternary semigroup S is called a soft quasi-ideal over S if

(1) $(F, A)\hat{o}(S, E)\hat{o}(S, E)\cap_R(S, E)\hat{o}(F, A)\hat{o}(S, E)\cap_R(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)$,
(2) $(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\hat{o}(S, E)\hat{o}(S, E)\cap_R$
$(S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A)$,

where (S, E) is the absolute soft set over S.

Proposition 6.2. *A soft set* (F, A) *over a ternary semigroup* S *is a soft quasi-ideal over* S *if and only if for all* $a \in A$, $F(a) \neq \varnothing$ *is a quasi-ideal of* S.

Proof. Suppose that a soft set (F, A) over S is a soft quasi-ideal over S. We show that $F(a)$ is a quasi-ideal of S. By definition of restricted product,

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) = (G, A \cap E \cap E) = (G, A) \tag{6.1}$$

$$(S, E)\hat{o}(F, A)\hat{o}(S, E) = (H, E \cap A \cap E) = (H, A) \tag{6.2}$$

$$(S, E)\hat{o}(S, E)\hat{o}(F, A) = (I, E \cap E \cap A) = (I, A) \tag{6.3}$$

$$(S, E)\hat{o}(S, E)\hat{o}(F, A)\hat{o}(S, E)\hat{o}(S, E) = (J, E \cap E \cap A \cap E \cap E) = (J, A) \tag{6.4}$$

Eq. (6.1), (6.2) and (6.3) imply that

$$\begin{aligned}&(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\\ &= (G, A) \cap_R (H, A) \cap_R (I, A)\end{aligned} \tag{6.5}$$

$$\begin{aligned}&(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A)\\ &= (K, A)\end{aligned} \tag{6.6}$$

But (F, A) is a soft quasi-ideal over S. Thus

$$(F, A)\hat{o}(S, E)\hat{o}(S, E) \cap_R (S, E)\hat{o}(F, A)\hat{o}(S, E) \cap_R (S, E)\hat{o}(S, E)\hat{o}(F, A) \subseteq (F, A).$$

From Eq. (6.6), $(K, A) \subseteq (F, A)$, that is, $K(a) \subseteq F(a)$ for all $a \in A$. Again Eq. (6.6) implies that

$$F(a)S(a)S(a) \cap S(a)F(a)S(a) \cap S(a)S(a)F(a) = K(a) \text{ for all } a \in A.$$

This implies that

$$F(a)S(a)S(a) \cap S(a)F(a)S(a) \cap S(a)S(a)F(a) \subseteq F(a). \tag{6.7}$$

Similarly, from Eq. (6.1), (6.3) and (6.4),

(6.8)
$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A) = (P,A).$$

As (F,A) is a soft quasi-ideal over S, so

$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A) \subseteq (F,A).$$

Therefore $(P,A)\subseteq(F,A)$, that is, $P(a)\subseteq F(a)$ for all $a\in A$. From Eq. (6.8)

$$F(a)S(a)S(a)\cap S(a)S(a)F(a)S(a)S(a)\cap S(a)S(a)F(a)=P(a)$$

for all $a\in A$. This implies that

$$F(a)S(a)S(a)\cap S(a)S(a)F(a)S(a)S(a)\cap S(a)S(a)F(a)\subseteq F(a) \tag{6.9}$$

From Eq. (6.7) and (6.9), it is clear that $F(a)$ is a quasi-ideal of S.

Conversely, let $F(a)\neq\varnothing$ be a quasi-ideal of S for all $a\in A$. We show that (F,A) is a soft quasi-ideal over S. From Eq. (6.1), (6.2) and (6.3) we have

$$\begin{aligned}&(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(F,A)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\\ &=(G,A)\cap_R(H,A)\cap_R(I,A)=(K,A)\,.\end{aligned}$$

By definition

$$F(a)S(a)S(a)\cap S(a)F(a)S(a)\cap S(a)S(a)F(a)=K(a)$$

for all $a\in A$. But $F(a)$ is a quasi-ideal of S. Therefore

$$K(a)=F(a)S(a)S(a)\cap S(a)F(a)S(a)\cap S(a)S(a)F(a)\subseteq F(a),$$

and so $(K,A)\subseteq(F,A)$. Thus

(6.10)
$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(F,A)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\subseteq(F,A)$$

Now, from Eq (6.1), (6.3) and (6.4)

$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A) =(G,A)\cap_R(I,A)\cap_R(J,A)=(P,A)\,,$$

$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A) =(P,A)\,,$$

$$F(a)S(a)S(a)\cap S(a)S(a)F(a)S(a)S(a)\cap S(a)S(a)F(a)=P(a)$$

for all $a\in A$. Since $F(a)$ is a quasi-ideal of S, so

$$P(a)=F(a)S(a)S(a)\cap S(a)S(a)F(a)S(a)S(a)\cap S(a)S(a)F(a)\subseteq F(a),$$

and thus $(P,A)\subseteq(F,A)$. Hence

(6.11)
$$(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A)\hat{o}(S,E)\hat{o}(S,E)\cap_R(S,E)\hat{o}(S,E)\hat{o}(F,A) \subseteq(F,A)$$

From Eq. (6.10) and (6.11), (F,A) is a soft quasi-ideal over S. □

Proposition 6.3. *Let $(R, A), (L, B)$ and (M, C) be soft right, soft left and soft lateral ideals over S, respectively. Then $(R, A) \cap_R (M, C) \cap_R (L, B)$ is a soft quasi-ideal over S.*

Proof. Straightforward. □

Proposition 6.4. *Let $(R, A), (L, B)$ and (M, C) be the soft right, soft left and soft lateral ideals over S, respectively, such that $A \cap B \cap C = \varnothing$. Then $(R, A) \cap_E (M, C) \cap_E (L, B)$ is a soft quasi-ideal over S.*

Proof. By definition $(H, D) = (R, A) \cap_E (M, C) \cap_E (L, B)$, where $D = A \cup B \cup C$, $A \cap B \cap C = \varnothing$, and

$$H(d) = \begin{cases} R(d), & \text{if } d \in A - B \cap C \\ M(d), & \text{if } d \in C - A \cap B \\ L(d), & \text{if } d \in B - A \cap C \end{cases}$$

for any $d \in D$. In each case $H(d)$ is a quasi-ideal of S. As every left, right and a lateral ideal of a ternary semigroup S is a quasi-ideal of S, thus, by definition, $(H, D) = (R, A) \cap_E (M, C) \cap_E (L, B)$ is a soft quasi-ideal over S. □

Proposition 6.5. *Every soft left (right, lateral) ideal over a ternary semigroup S is a soft quasi-ideal over S.*

Proof. Let (L, A) be a soft left ideal over S. Then $L(a)$ is a left ideal of S. As each left deal of S is a quasi-ideal of S, therefore $L(a)$ is a quasi-ideal of S. Hence (L, A) is a soft quasi-ideal over S. □

Remark 6.6. The converse of Proposition 6.5 is not true in general as seen in the following example.

Example 6.7. Let

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be the ternary semigroup under matrix multiplication defined as

$$ABC = (AB)C = A(BC)$$

for all $A, B, C \in S$. Let $B = \{\alpha\}$ and $G : B \to P(S)$ defined by

$$G(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then (G, B) is a soft quasi-ideal over S. But it is not a soft left, soft right and a soft lateral ideal over S.

Proposition 6.8. *Every soft left (right, lateral) ideal over S is a soft ternary semigroup over S.*

Proof. Straightforward. □

Proposition 6.9. *Every soft quasi-ideal is a soft ternary semigroup over S.*

Proof. Straightforward. □

Proposition 6.10. *Let $(R, A), (L, B)$ and (M, C) be soft right, soft left and soft lateral ideals over S, respectively. Then $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-ideal over S.*

Proof. By definition $(H, D) = (R, A) \wedge (M, C) \wedge (L, B)$ where $D = A \times C \times B$, and for any $(a, c, b) \in A \times C \times B$, $H(a, c, b) = R(a) \cap M(c) \cap L(b)$ is a quasi-ideal of S. Since the intersection of a left, right and a lateral ideal is a quasi-ideal of S, thus, $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-ideal over S. □

Proposition 6.11. *Let (F, A) and (G, B) be two soft quasi-ideals over a ternary semigroup S. Then the following statements hold.*

(1) *$(F, A) \cap_R (G, B)$ is a soft quasi-ideal over S.*
(2) *$(F, A) \cap_E (G, B)$ is a soft quasi-ideal over S.*
(3) *$(F, A) \wedge (G, B)$ is a soft quasi-ideal over S.*
(4) *$(F, A) \cup_E (G, B)$ is a soft quasi-ideal over S, whenever $A \cap B = \varnothing$.*

Proof. Straightforward. □

Proposition 6.12. *Let (F, A) be a soft quasi-ideal and (G, B) a soft ternary semigroup over S. Then $(F, A) \cap_R (G, B)$ is a soft quasi-ideal of (G, B).*

Proof. By definition $(H, C) = (F, A) \cap_R (G, B)$, where $C = A \cap B \neq \varnothing$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$, as $H(c) \subseteq F(c)$ and $H(c) \subseteq G(c)$. We show that $H(c)$ is a quasi-ideal of $G(c)$. Since $H(c) \subseteq G(c)$,

$$\begin{aligned} & H(c)\,G(c)\,G(c) \cap G(c)\,H(c)\,G(c) \cap G(c)\,G(c)\,H(c) \\ & \subseteq G(c)\,G(c)\,G(c) \cap G(c)\,G(c)\,G(c) \cap G(c)\,G(c)\,G(c) \\ & \subseteq G(c)\,G(c)\,G(c) \subseteq G(c) \end{aligned}$$

because $G(c)$ is a ternary subsemigroup of S. This implies that

$$H(c)\,G(c)\,G(c) \cap G(c)\,H(c)\,G(c) \cap G(c)\,G(c)\,H(c) \subseteq G(c). \tag{6.12}$$

Also $H(c) \subseteq F(c)$. So

$$\begin{aligned} & H(c)\,G(c)\,G(c) \cap G(c)\,H(c)\,G(c) \cap G(c)\,G(c)\,H(c) \\ & \subseteq F(c)\,G(c)\,G(c) \cap G(c)\,F(c)\,G(c) \cap G(c)\,G(c)\,F(c) \\ & \subseteq F(c)\,S(c)\,S(c) \cap S(c)\,F(c)\,S(c) \cap S(c)\,S(c)\,F(c) \subseteq F(c) \end{aligned}$$

because $F(c)$ is a quasi-ideal of S. Thus

$$H(c)\,G(c)\,G(c) \cap G(c)\,H(c)\,G(c) \cap G(c)\,G(c)\,H(c) \subseteq F(c). \tag{6.13}$$

From Eq (6.12) and (6.13), we have

$$\begin{aligned} & H(c)\,G(c)\,G(c) \cap G(c)\,H(c)\,G(c) \cap G(c)\,G(c)\,H(c) \\ & \subseteq F(c) \cap G(c) = H(c). \end{aligned} \tag{6.14}$$

Similarly, we can show that

$$H(c)\,G(c)\,G(c) \cap G(c)\,G(c)\,H(c)\,G(c)\,G(c) \cap G(c)\,G(c)\,H(c) \subseteq H(c). \tag{6.15}$$

From Eq (6.14) and (6.15), $H(c)$ is a quasi-ideal of $G(c)$. Thus $(F, A) \cap_R (G, B)$ is a soft quasi-ideal of (G, B). □

7. Soft bi-ideals over ternary semigroups

Definition 7.1. A soft set (F, A) over a ternary semigroup S is called a soft bi-ideal over S if

(1) (F, A) is a soft ternary semigroup over S,
(2) $(F, A)\hat{o}\,(S, E)\,\hat{o}(F, A)\hat{o}\,(S, E)\,\hat{o}(F, A) \subseteq (F, A)$ where (S, E) is the absolute soft set over S.

Proposition 7.2. *A soft set (F, A) over a ternary semigroup S is a soft bi-ideal over S if and only if for all $a \in A$, $F(a) \neq \varnothing$ is a bi-ideal of S.*

Proof. Let (F, A) be a soft bi-ideal over a ternary semigroup S. Then by definition (F, A) is a soft ternary semigroup over S. By Proposition 4.4, for any $a \in A$, $F(a) \neq \varnothing$ is a ternary subsemigroup of S. Moreover, since (F, A) is a soft bi-ideal over S, we have $(F, A)\hat{o}\,(S, E)\,\hat{o}\,(F, A)\hat{o}\,(S, E)\,\hat{o}(F, A) \subseteq (F, A)$ where (S, E) is the absolute soft set over S. It follows that $F(a)SF(a)SF(a) \subseteq F(a)$, which shows that $F(a)$ is a bi-ideal of S.

Conversely, suppose that (F, A) is a soft set over S such that for all $a \in A$, $F(a)$ is a bi-ideal of S, whenever $F(a) \neq \varnothing$. Then it is clear that each $F(a) \neq \varnothing$ is a ternary subsemigroup of S. Hence, by Proposition 4.4, (F, A) is a soft ternary semigroup over S. Furthermore, since $F(a) \neq \varnothing$ is a bi-ideal of S, so $F(a)SF(a)SF(a) \subseteq F(a)$ for all $a \in A$. Hence we conclude that $(F, A)\hat{o}\,(S, E)\,\hat{o}(F, A)\hat{o}\,(S, E)\,\hat{o}(F, A) \subseteq (F, A)$. This shows that (F, A) is a soft bi-ideal over S. □

Proposition 7.3. *Every soft quasi-ideal over a ternary semigroup S is a soft bi-ideal over S.*

Proof. Straightforward. □

Proposition 7.4. *Let (F, A) and (G, B) be two non-empty soft sets over a ternary semigroup S. Then the soft set $(H, C) = (F, A)\hat{o}\,(S, E)\,\hat{o}\,(G, B)$ is a soft bi-ideal over S.*

Proof. By definition

$$
\begin{aligned}
&(H, C)\,\hat{o}\,(H, C)\,\hat{o}\,(H, C)\\
&= ((F, A)\hat{o}\,(S, E)\,\hat{o}\,(G, B))\,\hat{o}\,((F, A)\hat{o}\,(S, E)\,\hat{o}\,(G, B))\,\hat{o}\,((F, A)\hat{o}\,(S, E)\,\hat{o}\,(G, B))\\
&= (F, A)\hat{o}\,((S, E)\,\hat{o}\,(G, B)\,\hat{o}(F, A))\,\hat{o}\,((S, E)\,\hat{o}\,(G, B)\,\hat{o}(F, A))\,\hat{o}\,(S, E)\,\hat{o}\,(G, B)\\
&\subseteq (F, A)\hat{o}\,((S, E)\,\hat{o}\,(S, E)\,\hat{o}\,(S, E))\,\hat{o}\,((S, E)\,\hat{o}\,(S, E)\,\hat{o}\,(S, E))\,\hat{o}\,(S, E)\,\hat{o}\,(G, B)\\
&\subseteq (F, A)\hat{o}\,((S, E)\,\hat{o}\,(S, E)\,\hat{o}\,(S, E))\,\hat{o}\,(G, B) \subseteq (F, A)\hat{o}\,(S, E)\,\hat{o}\,(G, B) = (H, C)\,.
\end{aligned}
$$

This implies that$(H, C)\,\hat{o}\,(H, C)\,\hat{o}\,(H, C) \subseteq (H, C)$. Thus (H, C) is a soft ternary semigroup over S. Also,

$$
\begin{aligned}
&(H,C)\,\hat{o}\,(S,E)\,\hat{o}\,(H,C)\,\hat{o}\,(S,E)\,\hat{o}\,(H,C)\\
=\ &((F,A)\hat{o}\,(S,E)\,\hat{o}\,(G,B))\,\hat{o}\,(S,E)\,\hat{o}\,((F,A)\hat{o}\,(S,E)\,\hat{o}\,(G,B))\,\hat{o}\,(S,E)\,\hat{o}\\
&\qquad ((F,A)\hat{o}\,(S,E)\,\hat{o}\,(G,B))\\
=\ &(F,A)\hat{o}\,((S,E)\,\hat{o}\,(G,B)\,\hat{o}\,(S,E))\,\hat{o}\,((F,A)\hat{o}\,(S,E)\,\hat{o}\,(G,B))\\
&\qquad \hat{o}\,((S,E)\,\hat{o}(F,A)\hat{o}\,(S,E))\,\hat{o}\,(G,B)\\
\subseteq\ &(F,A)\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\\
&\qquad \hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}\,(G,B)\\
\subseteq\ &(F,A)\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}\,(G,B)\subseteq(F,A)\hat{o}\,(S,E)\,\hat{o}\,(G,B)=(H,C)\,.
\end{aligned}
$$

This implies that $(H,C)\,\hat{o}\,(S,E)\,\hat{o}\,(H,C)\,\hat{o}\,(S,E)\,\hat{o}\,(H,C)\subseteq(H,C)$. Hence (H,C) is a soft bi-ideal over S. □

Proposition 7.5. *Let (F,A) be a soft bi-ideal over S and (G,B) a soft ternary semigroup over S. Then $(F,A)\cap_R(G,B)$ is a soft bi-ideal of (G,B).*

Proof. By definition $(H,C)=(F,A)\cap_R(G,B)$ where $C=A\cap B\neq\varnothing$, and H is defined by $H(c)=F(c)\cap G(c)$ for all $c\in C$. We show that (H,C) is a soft bi-ideal of (G,B). Now

$$
\begin{aligned}
&(H,C)\,\hat{o}\,(H,C)\,\hat{o}\,(H,C)\\
&=((F,A)\cap_R(G,B))\,\hat{o}\,((F,A)\cap_R(G,B))\,\hat{o}\,((F,A)\cap_R(G,B))\\
&\subseteq(F,A)\hat{o}(F,A)\hat{o}(F,A)\subseteq(F,A)
\end{aligned}
$$

because (F,A) is a soft bi-ideal over S. This implies that

$$(H,C)\,\hat{o}\,(H,C)\,\hat{o}\,(H,C)\subseteq(F,A). \tag{7.1}$$

Also

$$
\begin{aligned}
&(H,C)\,\hat{o}\,(H,C)\,\hat{o}\,(H,C)\\
&=((F,A)\cap_R(G,B))\,\hat{o}\,((F,A)\cap_R(G,B))\,\hat{o}\,((F,A)\cap_R(G,B))\\
&\subseteq(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\subseteq(G,B)
\end{aligned}
$$

because (G,B) is a soft ternary semigroup over S. This implies that

$$(H,C)\,\hat{o}\,(H,C)\,\hat{o}\,(H,C)\subseteq(G,B)\,. \tag{7.2}$$

From Eq (7.1) and (7.2),

$$(H,C)\,\hat{o}\,(H,C)\,\hat{o}\,(H,C)\subseteq(F,A)\cap_R(G,B)=(H,C)\,.$$

This implies that (H,C) is a soft ternary semigroup over S. Also

$$
\begin{aligned}
&(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\\
&=((F,A)\cap_R(G,B))\,\hat{o}\,(G,B)\,\hat{o}((F,A)\cap_R(G,B))\hat{o}\,(G,B)\,\hat{o}\,((F,A)\cap_R(G,B))\\
&\subseteq(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\\
&\subseteq(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\subseteq(G,B)\,,
\end{aligned}
$$

and so

$$(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\subseteq(G,B)\,. \tag{7.3}$$

Again

$$\begin{aligned}
&(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\\
&= ((F,A)\cap_R (G,B))\,\hat{o}\,(G,B)\,\hat{o}((F,A)\cap_R (G,B))\hat{o}\,(G,B)\,\hat{o}\,((F,A)\cap_R (G,B))\\
&\subseteq (F,A)\hat{o}\,(S,E)\,\hat{o}(F,A)\hat{o}\,(S,E)\,\hat{o}(F,A)\\
&\subseteq (F,A)
\end{aligned}$$

because (F,A) is a soft bi-ideal over S. This implies that

$$(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C) \subseteq (F,A). \tag{7.4}$$

From Eq (7.3) and (7.4), we have

$$(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C) \subseteq (F,A)\cap_R (G,B) = (H,C)\,.$$

Hence (H,C) is a soft bi-ideal of (G,B). □

Definition 7.6. A soft ideal (F,A) over a ternary semigroup S is soft idempotent if $(F,A)\,\hat{o}\,(F,A)\,\hat{o}\,(F,A) = (F,A)\,.$

Theorem 7.7. *Let (F,A) be a soft bi-ideal over S and (G,B) a soft bi-ideal of (F,A) such that $(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B) = (G,B)\,.$ Then (G,B) is a soft bi-ideal over S.*

Proof. By given condition, $(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B) \subseteq (G,B)$. This implies that (G,B) is a soft ternary semigroup over S. Since $(G,B)\subseteq (F,A)$ and

$$(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B) = (G,B)\,,$$

so we have

$$\begin{aligned}
&(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\\
&= ((G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B))\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,((G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B))\\
&= (G,B)\,\hat{o}\,(G,B)\,\hat{o}\,((G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B))\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\\
&\subseteq (G,B)\,\hat{o}\,(G,B)\,\hat{o}\,((F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A))\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\\
&\subseteq (G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(F,A)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B) \text{ since } (F,A) \text{ is a soft bi-ideal over } S.\\
&= (G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(F,A)\,\hat{o}\,(G,B)\,\hat{o}\,((G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B))\\
&= (G,B)\,\hat{o}\,((G,B)\,\hat{o}\,(F,A)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B))\,\hat{o}\,(G,B)\\
&\subseteq (G,B)\,\hat{o}\,((G,B)\,\hat{o}\,(F,A)\,\hat{o}\,(G,B)\,\hat{o}\,(F,A)\,\hat{o}\,(G,B))\,\hat{o}\,(G,B)\\
&\subseteq (G,B)\,\hat{o}\,(G,B)\,\hat{o}\,(G,B) = (G,B) \text{ because } (G,B) \text{ is a soft bi-ideal of } (F,A)\,.
\end{aligned}$$

This implies that $(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\,\hat{o}\,(S,E)\,\hat{o}\,(G,B)\subseteq (G,B)\,.$ Hence (G,B) is a soft bi-ideal over S. □

Proposition 7.8. *Let (F,A) be a non-empty soft subset over a ternary semigroup S. If (R,B) is a soft right ideal, (M,C) is a soft lateral ideal and (L,D) is a soft left ideal over S such that*

$$(R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D)\subseteq (F,A)\subseteq (R,B)\cap_R (M,C)\cap_R (L,D)\,,$$

then (F,A) is a soft bi-ideal over S.

Proof. By definition

$$(F,A)\,\hat{o}\,(F,A)\,\hat{o}\,(F,A) \subseteq ((R,B)\cap_R(M,C)\cap_R(L,D))\,\hat{o}$$
$$((R,B)\cap_R(M,C)\cap_R(L,D))\,\hat{o}\,((R,B)\cap_R(M,C)\cap_R(L,D))$$
$$\subseteq (R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D) \subseteq (F,A)\,.$$

This implies that $(F,A)\,\hat{o}\,(F,A)\,\hat{o}\,(F,A) \subseteq (F,A)$. Thus (F,A) is a soft ternary semigroup over S. Again consider

$$(F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A) \subseteq ((R,B)\cap_R(M,C)\cap_R(L,D))\,\hat{o}\,(S,E)\,\hat{o}$$
$$((R,B)\cap_R(M,C)\cap_R(L,D))\hat{o}\,(S,E)\,\hat{o}((R,B)\cap_R(M,C)\cap_R(L,D))$$
$$\subseteq (R,B)\,\hat{o}((S,E)\,\hat{o}\,(M,C)\,\hat{o}\,(S,E))\hat{o}\,(L,D)$$
$$\subseteq (R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D) \subseteq (F,A) \text{ because } (M,C) \text{ is a soft lateral ideal over } S.$$

This implies that $(F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A)\,\hat{o}\,(S,E)\,\hat{o}\,(F,A) \subseteq (F,A)$. Hence (F,A) is a soft bi-ideal over S. □

8. Regular Ternary Semigroups

Recall that an element a in a ternary semigroup S is called regular if there exist $x \in S$ such that $a = axa$, i.e. $a \in aSa$. A ternary semigroup S is called regular if every element of S is regular.

Definition 8.1. A soft ternary semigroup (F,A) over a ternary semigroup S is called a soft regular ternary semigroup if for each $a \in A$, $F(a)$ is regular.

Lemma 8.2. *Every soft lateral ideal over a regular ternary semigroup S is a soft regular ternary semigroup.*

Proof. Let (F,A) be a soft lateral ideal over S. Then $F(a)$ is a lateral ideal of S. Let $x \in F(a)$. Then by the regularity of S there exist $y \in S$ such that $x = xyx = x(yxy)x = xbx$, where $b = yxy \in F(a)$ ($\because F(a)$ is a lateral ideal of S). This shows that the element $x \in F(a)$ is also regular in $F(a)$. Thus (F,A) is a soft regular ternary semigroup over S. □

Proposition 8.3. *A ternary semigroup S is regular if and only if*

$$(R,A)\cap_R(M,C)\cap_R(L,B) = (R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)$$

for every soft right ideal (R,A), soft left ideal (L,B) and a soft lateral ideal (M,C) over S.

Proof. Let S be a regular ternary semigroup and (R,A), (M,C) and (L,B) be the soft right, soft lateral and soft left ideals over S, respectively. Now, by definition $(H,D) = (R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)$, where $D = A\cap C\cap B$ and H is defined as $H(d) = R(d)\,M(d)\,L(d)$ for all $d \in D$. Also, $(G,O) = (R,A)\cap_R(M,C)\cap_R(L,B)$, where $O = A\cap C\cap B$ and G is defined as $G(d) = R(d)\cap M(d)\cap L(d)$ for all $d \in O$. Now

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B) \subseteq (R,A)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E) \subseteq (R,A)\,, \tag{8.1}$$

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B) \subseteq (S,E)\,\hat{o}\,(M,C)\,\hat{o}\,(S,E) \subseteq (M,C)\,, \tag{8.2}$$

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B) \subseteq (S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(L,B) \subseteq (L,B)\,. \tag{8.3}$$

From Eq. (8.1), (8.2) and (8.3), we have

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B) \subseteq (R,A)\cap_R (M,C)\cap_R (L,B). \tag{8.4}$$

Let $b \in R(a)\cap M(a)\cap L(a)$. Then $b \in R(a)$, $b \in M(a)$ and $b \in L(a)$. Since S is regular, so there exist $x \in S$ such that $b = bxb$, $b = b(xbx)b$ and $b = bcb \in R(a)\,M(a)\,L(a)$, where $c = xbx \in M(a)$. This implies that $b \in R(a)\,M(a)\,L(a)$, i.e., $R(a)\cap M(a)\cap L(a) \subseteq R(a)\,M(a)\,L(a)$. Thus

$$(R,A)\cap_R (M,C)\cap_R (L,B) \subseteq (R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B). \tag{8.5}$$

From Eq. (8.4) and (8.5), we have

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B) = (R,A)\cap_R (M,C)\cap_R (L,B).$$

Conversely, Suppose that $A = B = C = S$ and R is a function from A to $P(S)$. Define $R(a) = \{a\}\cup aSS$ for all $a \in S$ and let L be a function from B to $P(S)$ defined by $L(a) = \{a\}\cup SSa$ for all $a \in S$. Also let M be a function from C to $P(S)$ defined by $M(a) = \{a\}\cup SaS \cup SSaSS$ for all $a \in S$. Then (R,S) is a soft right ideal, (L,S) is a soft left ideal and (M,S) is a soft lateral ideal over S. Let $a \in S$. By hypothesis, $(R,S)\,\hat{o}\,(M,S)\,\hat{o}\,(L,S) = (R,S)\cap_R (M,S)\cap_R (L,S)$ and so $R(a)\,M(a)\,L(a) = R(a)\cap M(a)\cap L(a)$ for all $a \in S$. Now

$$\begin{aligned}
& a \in R(a)\cap M(a)\cap L(a) = R(a)\,M(a)\,L(a)\\
& = (\{a\}\cup aSS)\,(\{a\}\cup SaS\cup SSaSS)\,(\{a\}\cup SSa)\\
& \quad aaa \cup a\,(aSS)\,a \cup a\,(SaS)\,a \cup a\,(SaS)\,SSa \cup a\,((SSa)\,SS)\,a\cup\\
& \quad a((S\,(SaS)\,S)\,SS)a \cup a\,(SSa)\,a \cup a\,(S\,(SaS)\,S)\,a \cup a\,(S\,(SSa)\,S)\,a\cup\\
& \quad a((SS\,(SaS))\,SS))a \cup a\,(SS(S\,(SaS)\,S))a\cup\\
& \quad a\,((SSS)\,(SaS)\,(SSS))\,a\\
& \subseteq aSa.
\end{aligned}$$

Thus $a \in aSa$, which implies that $a = axa$ for some $x \in S$, that is, every element a of S is regular. Hence S is a regular ternary semigroup. □

Proposition 8.4. *Every soft bi-ideal over a regular ternary semigroup S is a soft quasi-ideal over S.*

Proof. Straightforward. □

Theorem 8.5. *A commutative ternary semigroup S is regular if and only if every soft ideal over S is soft idempotent.*

Proof. Let S be a regular ternary semigroup and (F,A) a soft ideal over S. By Proposition 8.3,

$$(F,A) = (F,A)\cap_R (F,A)\cap_R (F,A) = (F,A)\,\hat{o}\,(F,A)\,\hat{o}\,(F,A).$$

This implies that $(F,A) = (F,A)\,\hat{o}\,(F,A)\,\hat{o}\,(F,A)$. Thus (F,A) is soft idempotent.

Conversely, assume that (R,A), (M,C) and (L,B) are the soft right, soft lateral and soft left ideals over S, respectively. As S is commutative, so (R,A), (M,C) and

(L,B) are soft ideal over S. By hypothesis,

$$(R,A)\cap_R(M,C)\cap_R(L,B)=((R,A)\cap_R(M,C)\cap_R(L,B))\,\hat{o}((R,A)\cap_R(M,C)\cap_R(L,B))\hat{o}\,((R,A)\cap_R(M,C)\cap_R(L,B))$$
$$\subseteq(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)\,.$$

This implies that $(R,A)\cap_R(M,C)\cap_R(L,B)\subseteq(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)\,.$ But

$$(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)\subseteq(R,A)\cap_R(M,C)\cap_R(L,B)$$

always holds. Thus $(R,A)\cap_R(M,C)\cap_R(L,B)=(R,A)\,\hat{o}\,(M,C)\,\hat{o}\,(L,B)\,.$ Hence, by Proposition 8.3, S is regular. □

Theorem 8.6. *If $(Q,A)\,\hat{o}\,(Q,A)\,\hat{o}\,(Q,A)=(Q,A)$ for every soft quasi-ideal (Q,A) over S, then S is a regular ternary semigroup.*

Proof. Since the restricted intersection of a soft left, soft right and a soft lateral ideal is a soft quasi-ideal over S. Let $(R,B)\,,(M,C)$ and (L,D) be the soft right, soft lateral and soft left ideals over S, respectively. Then by the given condition

$$(R,B)\cap_R(M,C)\cap_R(L,D)=((R,B)\cap_R(M,C)\cap_R(L,D))\hat{o}((R,B)\cap_R(M,C)\cap_R(L,D))\hat{o}((R,B)\cap_R(M,C)\cap_R(L,D))$$
$$\subseteq(R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D)\,.$$

But $(R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D)\subseteq(R,B)\cap_R(M,C)\cap_R(L,D)$ always holds, and so

$$(R,B)\,\hat{o}\,(M,C)\,\hat{o}\,(L,D)=(R,B)\cap_R(M,C)\cap_R(L,D)\,.$$

Hence, by Proposition 8.3, S is a regular ternary semigroup. □

Theorem 8.7. *If $(F,B)\,\hat{o}\,(F,B)\,\hat{o}\,(F,B)=(F,B)$ for every soft bi-ideal (F,B) over S, then S is a regular ternary semigroup.*

Proof. The proof is similar to the proof of Theorem 8.6. □

Theorem 8.8. *The following assertions on a ternary semigroup S are equivalent:*

(1) *S is a regular ternary semigroup.*
(2) *$(F,B)=(F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B)$ for every soft bi-ideal (F,B) over S.*
(3) *$(G,Q)=(G,Q)\,\hat{o}\,(S,E)\,\hat{o}\,(G,Q)$ for every soft quasi-ideal (G,Q) over S.*

Proof. $(1)\Rightarrow(2)$: Suppose that S is a regular ternary semigroup and (F,B) be a soft bi-ideal over S. Then $F(b)$ is a bi-ideal of S. Let $x\in F(b)\,.$ Since S is regular, there exists $y\in S$ such that $x=xyx\in F(b)\,SF(b)\,.$ Hence

$$F(b)\subseteq F(b)\,SF(b)\,. \tag{8.6}$$

Let $z\in F(b)\,SF(b)$. Then $z=b_1sb_2$ for some $b_1,b_2\in F(b)$ and $s\in S$. Since S is regular, so b_1 can be written as $b_1=b_1tb_1$ for some $t\in S$. Thus $z=b_1tb_1sb_2\in F(b)\,SF(b)\,SF(b)\subseteq F(b)$ because $F(b)$ is a bi-ideal of S. Thus

$$F(b)\,SF(b)\subseteq F(b)\,. \tag{8.7}$$

Now, from Eq. (8.6) and (8.7), we have $F(b)\,SF(b)=F(b)\,.$ Hence $(F,B)=(F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B)$ for every soft bi-ideal (F,B) over S.

(2) $\Rightarrow$ (3) : Since every soft quasi-ideal over a ternary semigroup S is a soft bi-ideal over S, we have $(G,Q) = (G,Q)\,\hat{o}\,(S,E)\,\hat{o}\,(G,Q)$ for every soft quasi-ideal (G,Q) over S.

(3) $\Rightarrow$ (1) : By Theorem 8.6, S is a regular ternary semigroup. □

Proposition 8.9. *Let (F,B) be a soft bi-ideal over a regular ternary semigroup S, and $(G,C)\,,(H,D)$ be non-empty soft subsets over S. Then $(F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)\,,$ $(G,C)\,\hat{o}\,(F,B)\,\hat{o}\,(H,D)\,,$ and $(G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(F,B)$ are soft bi-ideals over S.*

Proof. Let S be a regular ternary semigroup, (F,B) a soft bi-ideal over S, and $(G,C)\,,(H,D)$ be non-empty soft subsets over S. Then

$$\begin{aligned}
&((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&= (F,B)\,\hat{o}((G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(F,B))\hat{o}\,(G,C)\,\hat{o}\,(H,D))\hat{o}((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\subseteq (F,B)\,\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\subseteq (F,B)\,\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\subseteq ((F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B))\hat{o}\,(G,C)\,\hat{o}\,(H,D) = (F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)\,.
\end{aligned}$$

Since $(F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B) = (F,B)$ for every soft bi-ideal (F,B) over S, we have

$$\begin{aligned}
&((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\subseteq (F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)\,.
\end{aligned}$$

This implies that $(F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)$ is a soft ternary semigroup over S. Now,

$$\begin{aligned}
&((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,(S,E)\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,(S,E)\,\hat{o}\\
&\qquad\qquad ((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&= ((F,B)\,\hat{o}\,((G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(S,E))\hat{o}\,(F,B))\,\hat{o}((G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(S,E))\hat{o}\\
&\qquad\qquad (F,B)\,\hat{o}\,(G,C))\hat{o}\,(H,D)\\
&\subseteq ((F,B)\,\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}\,(F,B))\hat{o}\,((S,E)\,\hat{o}\,(S,E)\,\hat{o}\,(S,E))\,\hat{o}\\
&\qquad\qquad ((F,B)\,\hat{o}\,(G,C))\hat{o}\,(H,D))\\
&\subseteq ((F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B)\,\hat{o}\,(S,E)\,\hat{o}\,(F,B))\,\hat{o}\,(G,C))\hat{o}\,(H,D)\\
&\subseteq (F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)\,.
\end{aligned}$$

This implies that

$$\begin{aligned}
&((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\,\hat{o}\,(S,E)\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\qquad\qquad \hat{o}\,(S,E)\,\hat{o}\,((F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D))\\
&\subseteq (F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)\,.
\end{aligned}$$

Hence $(F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)$ is a soft bi-ideal over S. Similarly, we can show that $(G,C)\,\hat{o}\,(F,B)\,\hat{o}\,(H,D)$ and $(G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(F,B)$ are soft bi-ideals over S. □

Corollary 8.10. *Let (F,B) be a soft bi-ideal over a regular ternary semigroup S, and (G,C), (H,D) be two soft ternary semigroups over S. Then $(F,B)\,\hat{o}\,(G,C)\,\hat{o}\,(H,D)$, $(G,C)\,\hat{o}\,(F,B)\,\hat{o}\,(H,D)$ and $(G,C)\,\hat{o}\,(H,D)\,\hat{o}\,(F,B)$ are soft bi-ideals over S.*

Corollary 8.11. *Let $(F,A)\,,(G,B)$ and (H,C) be three soft bi-ideals over a regular ternary semigroup S. Then $(F,A)\,\hat{o}\,(G,B)\,\hat{o}\,(H,C)$ is a soft bi-ideal over S.*

Corollary 8.12. *Let* $(F, A), (G, B)$ *and* (H, C) *be three soft quasi-ideals over a regular ternary semigroup* S*. Then* $(F, A)\,\hat{o}\,(G, B)\,\hat{o}\,(H, C)$ *is a soft bi-ideal over* S*.*

Corollary 8.13. *Let* $(F, A), (G, B)$ *and* (H, C) *be three soft quasi-ideals over a regular ternary semigroup* S*. Then* $(F, A)\,\hat{o}\,(G, B)\,\hat{o}\,(H, C)$ *is a soft quasi-ideal over* S*.*

Theorem 8.14. *In a regular ternary semigroup,*

$$(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q) = (F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)$$

for every soft quasi-ideal (F, Q) *over* S*.*

Proof. By Corollary 8.13, $(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)$ is a soft quasi-ideal over S. Since S is regular, Theorem 8.8 implies that

$$\begin{aligned}
&(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\\
&= ((F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q))\,\hat{o}\,(S, E)\,\hat{o}\,((F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q))\\
&= (F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,((F, Q)\,\hat{o}\,(S, E)\,\hat{o}\,(F, Q))\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\\
&= (F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)
\end{aligned}$$

since $(F, Q) = (F, Q)\,\hat{o}\,(S, E)\,\hat{o}\,(F, Q)$ in a regular ternary semigroup. Thus

$$(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q) = (F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)\,\hat{o}\,(F, Q)$$

for every soft quasi-ideal (F, Q) over S. □

9. Conclusions

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainty. We applied the algebraic properties of soft sets in ternary semigroups. We introduced the notions of soft ternary semigroups, soft left (right, lateral, quasi, bi) ideal of ternary semigroups, soft ternary subsemigroup and soft ideal of soft ternary semigroups are defined and proved with the help of an example that soft ideal of a soft ternary semigroup is different from the soft ideal over a ternary semigroup. By using soft set theory, one may consider further algebraic structures of soft ternary semigroups.

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M. Shabir (mshabirbhatti@yahoo.co.uk) – Department of Mathematics Quaid-i-Azam University Islamabad-45320, Pakistan.

A. Ahmad (ali.ahmad200873@yahoo.com) – Department of Mathematics Quaid-i-Azam University Islamabad-45320, Pakistan.