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# A ranking function method for solving fuzzy multi-objective linear programming problem

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Abstract. The aim of this paper is to present a novel method in which the comparison of fuzzy numbers is used by a linear ordered function for solving multi-objective linear Programming (MOLP) where the parameters and objectives are fuzzy.

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Keywords: Multi-Objective Linear Programming (MOLP), Fuzzy Multi-Objective Linear Programming (FMOLP), Fuzzy Numbers, Linear Ordered Function, Trapezoidal Fuzzy Numbers.

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## 1. Introduction

The most common approach to treat the challenge of solving fuzzy linear programming problems is to change the fuzzy linear program into the corresponding deterministic linear program. But, some methods based on comparison of fuzzy numbers have been suggested in the literature [8]. One of these methods is A. Ebrahimnejad, S. H. Nasseri [5] Using Complementary Slackness Property to Solve Linear Programming with Fuzzy Parameters. Moreover, most real word problems are inherently characterized by multiple, conflicting and incommensurate aspects of evaluation. These axes of evaluation are generally operationalized by objective functions to be optimized in framework of multiple objective linear programming models. Furthermore, when addressing real world problems, frequently the parameters are nonrandom uncertainness or imprecise numerical quantities. Fuzzy numbers are very adequate for modeling these situations. Bellman and Zadeh [1] introduced the concept of fuzzy quantities, Fuzzy numbers and the arithmetic operations on them. Also, they introduced the concept of fuzzy decision as intersection of fuzzy goals and fuzzy constraints to modeling to uncertainty in decision making environment. In this paper, we focus on Fuzzy Multi Objective Linear Programming (FMOLP) problems and give a method for solving it by ranking function method. Fuzzy ranking is a topic that has been studied by many researchers [2, 3, 6, 10-13].

This paper is organized as follows: In Section 2, we give some necessary concepts of fuzzy set theory, ranking functions and a review of the multi objective linear programming problems with crisp parameters. Solving FMOLP is given in Section 3 and finally in section 4 explain it by an illustrative example.

## 2. Preliminaries

In this section we review some preliminaries which are needed in the next section. For more details see [8, 9].

Let $X$ is a given set of possible alternatives which contains the solution of a decision making problem under consideration. A fuzzy goal $G$ and a fuzzy constraint $C$ are the fuzzy sets on $X$ which we characterized them by its membership functions $\mu_G : X \to [0, 1]$ , $\mu_C : X \to [0, 1]$ respectively. Bellman and Zadeh [1] defined the fuzzy decision $D$ resulting from the fuzzy goal $G$ and fuzzy constraint $C$ as the intersection of $G$ and $C$. To be more explicit, the fuzzy decision of Bellman and Zadeh is the fuzzy set $D$ on $X$ defined as $D = G \bigcap C$ and is characterized by its membership function $\mu_D(x) = \min(\mu_G(x), \mu_C(x))$. The optimal decision is then defined as $\max_{x\in X} \mu_D(x) = \max_{x\in X} \min(\mu_G(x), \mu_C(x))$. More generally, the fuzzy decision $D$ resulting from $k$ fuzzy goals $G_1, ..., G_k$ and $m$ fuzzy constraints $C_1, ..., C_m$ is defined by $D = G_1 \bigcap \ldots \bigcap G_k \bigcap C_1 \bigcap \ldots \bigcap C_m$. Among many applications of fuzziness in real word applications and mathematics, we consider fuzzy multi-objective linear programming (FMOLP) in which the objectives and parameters are fuzzy.

## 3. Multi-objective linear programming

The problem to optimize multiple conflicting objective functions simultaneously under given constraints is called multi-objective linear programming problem and can be formulated as the following vector maximization problem:

$$\begin{array}{ll} \max & f(x) = (f_1(x), f_2(x), ..., f_k(x))^T \\ s.t: & x \in X = \{x \in R^n | g_j(x) \leq 0, j = 1, 2, ..., m\} \end{array} \qquad (1)$$

where $f_1(x), f_2(x), ..., f_k(x)$ are k distinct objective functions of the decision vector and $X$ is the feasible set of constrained decision. If we directly apply the notion of optimality for single-objective linear programming to this multi-objective programming, we arrive at the following notion of a complete optimal solution.

**Definition 3.1.** $x^*$ is said to be a complete optimal solution for (1) if there exists $x^* \in X$ such that $f_i(x^*) \geq f_i(x)$, $i = 1, 2, \ldots, k$ for all $x \in X$.

However, in general, such a complete optimal solution that simultaneously maximize all of the multiple objective functions does not always exist when the objective function conflict with each other. Thus, instead of a complete optimal solution, a new solution concept, called Pareto optimality, is introduced in multi-objective programming.

**Definition 3.2.** $x^* \in X$ is said to be a Pareto optimal solution for (1) if there does not exist another $x \in X$ such that $f_i(x^*) \leq f_i(x)$ for all $i = 1, 2, \ldots, k$, and $f_j(x^*) < f_j(x)$ for at least one $j \in \{1, 2, \ldots, k\}$.

As can be seen from the definition, a Pareto optimal solution consists of an infinite number of points. A Pareto optimal solution is sometimes called as a no inferior solution since it is not inferior to other feasible solutions. In addition to Pareto optimality, the following weak Pareto optimality is defined as a slightly weaker solution concept than Pareto optimality.

**Definition 3.3.** $^* \in X$ is said to be a weak Pareto optimal solution if there does not exist another $x \in X$ such that $f_i(x^*) < f_i(x)$ , $i = 1, 2, \ldots, k$.

## 4. Ranking function for fuzzy numbers

**Definition 4.1.** Let $A$ be a fuzzy number, whose membership function can generally be defined as

$$\mu_A(x) = \begin{cases} {\mu_A}^L(x) & a^1 \le x \le a^2 \\ 1 & a^2 \le x \le a^3 \\ {\mu_A}^R(x) & a^3 \le x \le a^4 \\ 0 & otherwise \end{cases}$$

where ${\mu_A}^L(x)\ : [\ a^1, a^2] \to [0, 1]$ and ${\mu_A}^R(x)\ : [\ a^3, a^4] \to [0, 1]$ are strictly monotonic and continuous mappings. Then it is consider as a left-right fuzzy number. If the membership function $\mu_A(x)$ is piecewise linear, then is referred to as a trapezoidal fuzzy number and is usually denoted by $A = (a^1, a^2, a^3, a^4)$. if $a^2 = a^3$ the trapezoidal fuzzy number is turned into a triangular fuzzy number $A = (a^1, a^3, a^4)$.

A fuzzy number $A = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 1 & x = b \\ \frac{x-b}{b-c} & c \le x \le d \\ 0 & otherwise \end{cases}$$

Assume that $R : F(I\!R) \to I\!R$ be a linear ordered function that maps each fuzzy number into the real number, in which $F(I\!R)$ denotes the whole fuzzy numbers. Accordingly, for any two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ we have

$$\begin{gathered} \tilde{a} \underset{R}{\ge} \tilde{b}\ \ iff\ \ R(\tilde{a}) \ge R(\tilde{b}) \\ \tilde{a} \underset{R}{>} \tilde{b}\ \ iff\ \ R(\tilde{a}) > R(\tilde{b}) \\ \tilde{a} \underset{R}{=} \tilde{b}\ \ iff\ \ R(\tilde{a}) = R(\tilde{b}) \end{gathered}$$

We restrict our attention to linear ranking functions, that is, a ranking function $R$ such that $R(k\tilde{a} + \tilde{b}) = kR(\tilde{a}) + R(\tilde{b})$ for any $\tilde{a}$ and $\tilde{b}$ in $F(I\!R)$ and any $k \in I\!R$.

**Example 4.2.** The ranking function proposed by Roubenes is defined [6] by

$$R(\tilde{a}) = \frac{1}{2} \int_0^1 (\inf \tilde{a}_\alpha + \sup \tilde{a}_\alpha) d\alpha$$

which reduces to

$$R(\tilde{a}) = \frac{1}{2}(a^L + a^U + \frac{1}{2}(\beta - \alpha))$$

for a trapezoidal fuzzy number

$$\tilde{a} = (a^L - \alpha, a^L, a^U, a^U + \beta)$$

**Example 4.3.** Ranking function proposed by Compose and Munoz's [3] for a trapezoidal fuzzy number $\tilde{a} = (a^1, a^2, a^3, a^4)$ is

$$\begin{aligned} CM_1^{\lambda}(\tilde{a}) &= \int_0^1 (\lambda \inf(\tilde{a})_\alpha + (1-\lambda)\sup(\tilde{a})_\alpha) d\alpha \\ &= a^2 + \lambda((a^3 - a^2) + \tfrac{(a^4-a^3)+(a^2-a^1)}{2}) - \tfrac{a^2-a^1}{2} \end{aligned}$$

## 5. Solving fuzzy multi-objective linear programming (FMOLP)

A fuzzy multi-objective linear programming problem is defined as follows

$$\begin{array}{lll} \max \quad \tilde{z}_r = \sum_j \tilde{c}_{rj} x_j & r = 1,2,...,q & \\ s.t: \quad \sum_j \tilde{a}_{ij} x_j \lessapprox \tilde{b}_i & i = 1,2,...,m & (2) \\ \qquad x_j \geq 0 & & \end{array}$$

where $\tilde{a}_{ij}$ and $\tilde{c}_{rj}$ in the above mentioned relation are in the trapezoidal form as

$$\begin{aligned} \tilde{a}_{ij} &= ({a_{ij}}^1, {a_{ij}}^2, {a_{ij}}^3, {a_{ij}}^4) \\ \tilde{c}_{rj} &= ({c_{rj}}^1, {c_{rj}}^2, {c_{rj}}^3, {c_{rj}}^4) \end{aligned}$$

**Definition 5.1.** $x \in X$ is said to be a feasible solution to the FMOLP problem (2) if it satisfy in constraints of (2).

**Definition 5.2.** $x^* \in X$ is said to be a Pareto optimal solution to the FMOLP problem (2) if there does not exist another $x \in X$ such that $\tilde{z}_i(x) \geq \tilde{z}_i(x^*)$ for all $i = 1, 2, \ldots, q$ and $\tilde{z}_j(x) > \tilde{z}_j(x^*)$ for at least one j.

Now, the FMOLP can be easily transformed to a classic form of a MOLP by considering $R$ as a linear ranking function. By implementing the $R$ on the above model, we obtain the classical form of MOLP problem:

$$\begin{array}{ll} \max \quad R(\tilde{z}_r) = \sum_j R(\tilde{c}_{rj}) x_j & r = 1,2,...,q \\ s.t: \quad \sum_j R(\tilde{a}_{ij}) x_j \leq R(\tilde{b}_i) & i = 1,2,...,m \\ \qquad x_j \geq 0 & \end{array}$$

So we have

$$\begin{array}{lll} \max \quad z'_r = \sum_j c'_{rj} x_j & r = 1,2,...,q & \\ s.t: \quad \sum_j a'_{ij} x_j \leq b'_i & i = 1,2,...,m & (3) \\ \qquad x_j \geq 0 & & \end{array}$$

where $a'_{ij}, b'_i$, $c'_j$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}, \tilde{b}_i$, $\tilde{c}_j$ with respect to linear ranking function $R$, respectively.

**Lemma 5.3.** *The Pareto optimal solutions set of (2) and (3) are equivalent.*

*Proof.* Let $M_1, M_2$ be sets of all feasible solutions of (2) and (3), respectively. Then $x \in M_1$ if and only if

$$\sum_j \tilde{a}_{ij} x_j \lessapprox \tilde{b}_i \qquad i = 1, 2, ..., m$$

by considering $R$ as a linear ranking function, we have

$$R\,(\sum_j \tilde{a}_j x_{ij}) \leq R(\tilde{b}_i) \qquad i = 1, 2, ..., m$$

hence

$$\sum_j R(\tilde{a}_{ij}) x_j \leq R(\tilde{b}_i) \qquad i = 1, 2, ..., m$$

so

$$\sum_j a'_{ij} x_j \leq b'_i \qquad i = 1, 2, ..., m$$

hence $x \in M_2$ . Hence $M_1 = M_2$ .
Now suppose that $x^*$ is a arbitrary Pareto optimal solution for (2), then there does not exist another $x \in X$ such that $\tilde{z}_i(x) \geq \tilde{z}_i(x^*)$ for all $i = 1, 2, ..., q$ and $\tilde{z}_l(x) > \tilde{z}_l(x^*)$ for at least one l. By considering a linear ranking function $R$ we conclude that there does not exist another $x \in X$ such that $R\,(\sum\limits_j \tilde{c}_{ij} x_j^*) \geq R\,(\sum\limits_j \tilde{c}_{ij} x_j)$ for all $i = 1, 2, ..., q$ and $R\,(\sum\limits_j \tilde{c}_{lj} x_j^*) \geq R\,(\sum\limits_j \tilde{c}_{lj} x_j)$ for at least one l. Equivalently, there does not exist another $x \in X$ such that $\sum\limits_j R(\tilde{c}_{ij}) x_j^* \geq \sum\limits_j R(\tilde{c}_{ij}) x_j$ for all $i = 1, 2, ..., q$ and $\sum\limits_j R(\tilde{c}_{lj}) x_j^* \geq \sum\limits_j R(\tilde{c}_{lj}) x_j$ for at least one l. Finally we have, there does not exist another $x \in X$ such that $\sum\limits_j c'_{ij} x_j^* \geq \sum\limits_j c'_{ij} x_j$ for all $i = 1, 2, ..., q$ and $\sum\limits_j c'_{lj} x_j^* \geq \sum\limits_j c'_{lj} x_j$ for at least one l. This shown that is a Pareto optimal solution of (3). □

Several indices [2], such as Delgado et al. index, Liou and Wang index or Fortemps and Roubens index can be used as linear ranking function $R$. Here, we used the index of Delgado et al. as linear ranking function $R$ (see [4]). Delgado et al. presented the following scheme to comparing the fuzzy numbers by introducing two important parameters of Value and Ambiguity. Let be a fuzzy number with r-cut representation $(L(r), R(r))$, then the Value and Ambiguity of $\tilde{a}$ are defined as:

$$V(\tilde{a}) = \int_0^1 r(L(r) + R(r))dr$$
$$A(\tilde{a}) = \int_0^1 r(L(r) - R(r))dr$$

where $L$ and $R$ are the left and right shape functions, respectively.
Now, we consider the following form of (2):

$$\begin{array}{lll} \max & \tilde{z}_r = \tilde{c}_r x & r = 1, 2, ..., q \\ s.t: & \tilde{a}_i x \lessapprox \tilde{b}_i & i = 1, 2, ..., m \\ & x_j \geq 0 & \end{array}$$

where $\tilde{c}_r$ and $\tilde{a}_i$ are vector of trapezoidal fuzzy numbers

$$\begin{array}{c} \tilde{c}_{rj} = (c^1_{rj}, c^2_{rj}, c^3_{rj}, c^4_{rj}) \\ \tilde{a}_{ij} = (a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}) \end{array}$$

where $j = 1, 2, ..., n$. Now, because $x \geq 0$ , then from $\tilde{a}_i x \tilde{\lessapprox} b_i$ and $\tilde{c}_r x \gtrapprox \tilde{z}_r$ we have

$$\begin{array}{ll} (\sum\limits_j \tilde{a}^1_{ij}x_j, \sum\limits_j \tilde{a}^2_{ij}x_j, \sum\limits_j \tilde{a}^3_{ij}x_j, \sum\limits_j \tilde{a}^4_{ij}x_j) \lessapprox \tilde{b}_i & i = 1, 2, ..., m \\ (\sum\limits_j \tilde{c}^1_{rj}x_j, \sum\limits_j \tilde{c}^2_{rj}x_j, \sum\limits_j \tilde{c}^3_{rj}x_j, \sum\limits_j \tilde{c}^4_{rj}x_j) \gtrapprox \tilde{z}_r & r = 1, 2, ..., q \end{array}$$

Now, let $z_r$ is an aspiration level for $r^{th}$ objective function and $d_i$ and $t_r$ be the tolerance for $\tilde{b}_i$ and $\tilde{z}_r$ respectively. Also let for $r = 1, 2, ..., q$ and $i = 1, 2, ..., m$, $\mu_r(\tilde{c}_r x), \mu_i(\tilde{a}_i x)$ be the membership functions of objective functions and constraints. Hence, by considering a linear ranking function R we have the following membership functions for objective functions and constraints:

$$\mu_r(\tilde{c}_r x) = \begin{cases} 1 & R(\tilde{c}_r x) \geq R(\tilde{z}_r) \\ 1 - \frac{R(\tilde{c}_r x) - R(\tilde{z}_r) - t_r}{t_r} & R(\tilde{z}_r) - t_r \leq R(\tilde{c}_r x) \leq R(\tilde{z}_r) \quad r = 1, 2, ..., q \\ 0 & otherwise \end{cases}$$
$$\mu_i(\tilde{a}_i x) = \begin{cases} 1 & R(\tilde{a}_i x) \leq R(\tilde{b}_i) \\ 1 - \frac{R(\tilde{a}_i x) - R(\tilde{b}_i)}{d_i} & R(\tilde{b}_i) \leq R(\tilde{a}_i x) \leq R(\tilde{b}_i) + d_i \quad i = 1, 2, ..., m \\ 0 & otherwise \end{cases}$$

By considering a linear ranking function $R$ and using the Bellman-Zadeh fuzzy decision, FMOLP (2) is transformed to the following model

$$\begin{array}{ll} \max & \min\{\mu_1(R(\tilde{c}_1 x)), ..., \mu_q(R(\tilde{c}_q x)), \mu_1(R(\tilde{a}_1 x)), ..., \mu_m(R(\tilde{a}_m x))\} \\ s.t: & R(\tilde{a}_i x) \leq R(\tilde{b}_i) + d_i \qquad i = 1, 2, ..., m \\ & R(\tilde{z}_r) - t_r \leq R(\tilde{c}_r x) \quad r = 1, 2, ..., q \\ & x \geq 0 \end{array}$$

this is equivalent to

$$\begin{array}{lll} \max & \lambda & \\ s.t: & \lambda \leq \mu_i(R(\tilde{a}_i x)) & i = 1, 2, ..., m \\ & \lambda \leq \mu_r R(\tilde{c}_r x)) & r = 1, 2, ..., q \\ & R(\tilde{a}_i x) \leq R(\tilde{b}_i) + d_i & i = 1, 2, ..., m \\ & R(\tilde{z}_r) - t_r \leq R(\tilde{c}_r x) & r = 1, 2, ..., q \\ & x \geq 0 & \end{array}$$

## 6. Numerical example

In this section we explain the above method by the following example:

$$\begin{aligned} &Find\ \ (x_1, x_2, x_3) \\ &s.t:\quad \tilde{23}x_1 + \tilde{8}x_2 + \tilde{15}x_3 \gtrsim 95 \\ &\qquad \tilde{3}x_1 + \tilde{1}x_2 + \tilde{2}x_3 \lesssim 10.5 \\ &\qquad \tilde{15}x_1 + \tilde{5}x_2 + \tilde{5}x_3 \lesssim 50 \\ &\qquad x_1, x_2, x_3 \leqslant 5 \\ &\qquad x_1, x_2, x_3 \geqslant 1 \end{aligned}$$

The uncertain parameters are estimated by the following (triangular or trapezoidal) fuzzy numbers:

$$\begin{aligned} &\tilde{23} = (22.5, 22.8, 23.4, 23.98),\ \tilde{8} = (7.9, 8, 8.2),\ \tilde{15} = (14.7, 14.9, 15.1, 15.4), \\ &\tilde{3} = (2.9, 3, 3.1),\ \tilde{1} = (0.95, 1, 1.1),\ \tilde{2} = (1.9, 2, 2.5),\ \tilde{15} = (14.5, 15, 15.3), \\ &\tilde{5} = (4.8, 5, 5.1),\ \tilde{5} = (4.9, 5, 5.1). \end{aligned}$$

By considering Delgado et al. indices as linear ranking function and the above method we have the following classical linear programming problem:

$$\begin{aligned} &\max \quad \lambda \\ &s.t:\quad \lambda \leqslant (\frac{1}{20})(22.98x_1 + 8.02x_2 + 15.05x_3 - 70) \\ &\qquad \lambda \leqslant (\frac{1}{2.5})(13 - 3x_1 - 1.01x_2 - 1.99x_3) \\ &\qquad \lambda \leqslant (\frac{1}{4})(56 - 14.97x_1 - 4.98x_2 - 5x_3) \\ &\qquad 22.05x_1 + 7.9x_2 + 14.7x_3 \geqslant 75 \\ &\qquad 3.1x_1 + 1.1x_2 + 2.05x_3 \leqslant 13 \\ &\qquad 15.3x_1 + 5.1x_2 + 5.1x_3 \leqslant 54 \\ &\qquad x_1, x_2, x_3 \leqslant 5 \\ &\qquad x_1, x_2, x_3 \geqslant 1 \end{aligned}$$

The optimal solution of this model which is a Pareto optimal solution of the above FMOLP is $x_1 = 1.41, x_2 = 5, x_3 = 1, \lambda = 0.7$ (see [7]).

## 7. Conclusions

In this paper we propose a method that applies ranking function to solving fuzzy multi-objective linear programming problems. The use of linear ranking function allow us to use nonlinear fuzzy numbers without losing information or increasing the complexity.

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