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Intuitionistic fuzzy topology on function spaces

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Abstract. The purpose of this paper is to give a topology on a given function space in the category of intuitionistic fuzzy topological spaces and investigate series of its properties.

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1. Introduction

The notion of function space was formulated around the turn of the century and plays a vital in several subdisciplines of analysis, general topology, algebraic topology and other branches of mathematics. In applied mathematics, for example, search of existence of minimum of a given functional, or a problem of existence and unique of solution of a differential equation. The study of topologies on function spaces and their properties is an important area of research. Thus the introduction of fuzzy topologies on a given function space is the most important for the further development of fuzzy mathematics. Ever since the introduction of fuzzy set by Zadeh [17] and fuzzy topological space by Chang [3], many authors investigated various aspects of fuzzy topology. On the otherhand various generalizations of the concept of fuzzy sets have been done by many authors. Atanassov [2] introduced the idea of intuitionistic fuzzy sets. Recently much work have been done with these concepts. Coker [5] introduced the idea of topology of intuitionistic fuzzy sets. Chattopadhyay [4] et al gave a definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets. They constructed connections between level Chang fuzzy topology and the new fuzzy topological space. Mondal [11] et al. defined the category of intuitionistic fuzzy topological spaces (briefly) and also established connections between a descending family of inclusive bitopologies of fuzzy subsets on and intuitionistic fuzzy topological spaces. Since firstly this definition was given as independently by Kubiak [10] and Šostak [14], these spaces are called Šostak fuzzy topological spaces.

The novelty of our paper is that pointwise intuitionistic fuzzy topology is introduced on a given function space in the category of intuitionistic fuzzy topological spaces and investigated series of its properties. First effort in fuzzy direction was made by Peng [15], who defined pointwise convergent topology and a version of compact open topology on a family of functions from a fuzzy topological space to another. Later, Alderton [1] considered the problem from a categorical view point. Dang and Behera [6] gave the concept of fuzzy compact-open topology (in case of Chang) and investigated some characterizations of this topology. Kohli and Prasannan [9] introduced three different fuzzy topologies on a given function space which are pointwise fuzzy topology, N-compact open fuzzy topology, jointly fuzzy continuous fuzzy topology, and examined some of their properties. In this paper, we introduce intuitionistic fuzzy topology on function spaces. Different topologies can be given on function spaces. We formulate pointwise intuitionistic fuzzy topology on function spaces. For this definition, firstly we give appropriate topology on function spaces of fuzzy bitopological spaces. We point out interrelations among some function spaces.

2. Preliminaries

In this paper X will denote a nonempty set; $I = [0,1]$, the closed unit interval of the real line; $I_0 = (0,1]$. All other notations are standard notations of fuzzy set theory. Next, we give some definitions and theorems which will be used in the sequel.

Definition 2.1 ([4])**.** Let X be a nonempty set and $\tau : I^x \to I$ be a mapping satisfying the following properties:

$$\begin{array}{ll}(O1) & \tau(\underline{0}) = \tau(\underline{1}) = 1,\\ (O2) & \tau(\lambda_1 \bigcap \lambda_2) \geq \tau(\lambda_1) \bigcap \tau(\lambda_2),\\ (O3) & \tau(\bigcup_{i\in I} \lambda_1) \geq \bigwedge_{i\in I} \tau(\lambda_i)\end{array}$$

where $0 < r \leq 1$; then τ is called a gradation of openness on X.

Definition 2.2 ([11])**.** Let X be a non-empty set. An IGO of fuzzy subsets of X is an ordered pair (τ, τ^*) of functions from I^X to I such that

(IGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1, \forall \lambda \in I^X$,

(IGO2) $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,

(IGO3) $\tau(\lambda_1 \bigcap \lambda_2) \geq \tau(\lambda_1) \bigwedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \bigcap \lambda_2) \leq \tau^*(\lambda_1) \bigvee \tau^*(\lambda_2)$, $\lambda_i \in I^X$, $i = 1,2$,

(IGO4) $\tau(\bigcup_{i\in\triangle} \lambda_i) \geq \bigwedge_{i\in\triangle} \tau(\lambda_i)$ and $\tau^*(\bigcup_{i\in\triangle} \lambda_i) \leq \bigvee_{i\in\triangle} \tau^*(\lambda_i)$, $\lambda_i \in I^X$, $i \in \triangle$.

The triplet (X, τ, τ^*) is called an $IFTS$. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.3 ([11])**.** Let (X, τ, τ^*) and $(Y, \sigma, \sigma*)$ be two $IFTS$s and $f : X \to Y$ be a mapping. Then f is called a gp-map if for each $\mu \in I^Y$,

$$\sigma(\mu) \leq \tau(f^{-1}(\mu)) \text{ and } \sigma^*(\mu) \geq \tau^*(f^{-1}(\mu)).$$

Theorem 2.4 ([11])**.** *Let (X, τ, τ^*) be an $IFTSs$. Then $\{\tau_r\}_{r\in I_0}$ and $\{\tau_r^*\}_{r\in I_0}$ are two descending families of topologies of fuzzy subsets on X such that*

(a) $\tau_r \subset \tau_r^*$

(b) $\tau_r = \bigcap_{s<r} \tau_s$; $\tau_r^* = \bigcap_{s<r} \tau_s^*$.
Here $\tau = \tau^{-1}[r, 1]$, $\tau^* = (\tau^*)^{-1}[0, 1-r]$.

Theorem 2.5 ([11]). *Let* $\{(T_r, T_r^*) : r \in I_0\}$ *be a descending family of inclusive bitopologies of fuzzy subsets on* X. *Define* $\tau, \tau^* : I^X \to I$ *by*

$$\tau(\lambda) = \bigvee\{r : \lambda \in T_r\}$$

and

$$\tau^*(\lambda) = \bigwedge\{1 - r : \lambda \in T_r^*\}.$$

Then

(a) (τ, τ^*) *is an IGO on* X,
(b) $\tau_r = T_r$ *iff* $\bigcap_{s<r} T_s = T_r$, $\forall r \in I_0$,
(c) $\tau_r^* = T_r^*$ *iff* $\bigcap_{s<r} T_s^* = T_r^*$, $\forall r \in I_0$.

Definition 2.6 ([11]). Let $\{(X_i, \tau_i, \tau_i^*)\}_{i\in\triangle}$ be a family of IFTs and $X = \pi_{i\in\triangle}X_i$ and $p_i : X \to X_i$, $i \in \triangle$ be the projection mapping. Then the initial IGO on X generated by the family $\{p_i : X \to (X_i, \tau_i, \tau_i^*)\}_{i\in\triangle}$ is called the product IGO on X and is denoted by $(\pi_{i\in\triangle}\tau_i, \pi_{i\in\triangle}\tau_i^*)$. The triplet $(X, \pi_{i\in\triangle}\tau_i, \pi_{i\in\triangle}\tau_i^*)$ is called the product $IFTS$ of the family $\{(X_i, \tau_i, \tau_i^*)\}_{i\in\triangle}$.

The above definition is done for a gradation of openness τ on X in [4].

Definition 2.7 ([12, 13]). Let $\{(X_t, \tau_t)\}_{t\in T}$ be a family of pairwise disjoint Lfts's, i.e., $X_{t_1} \bigcap X_{t_2} = \varnothing$ for $t_1 \neq t_2$. Consider the set $X = \bigcup_{t\in T} X_t$ and $\forall t \in T$, $j_t : X_t \to X$ is the usual inclusion mapping (i.e., $\forall x \in X_t, j_t(x) = x$). Define

$$\forall B \in L^X, \sigma(B) = \bigwedge_{t\in T} \tau_t(j_t \leftarrow (B)).$$

Then σ is called the L-fuzzy sum topology of $\{\tau_t\}_{t\in T}$ and is denoted by $\sum_{t\in T} \tau_t$, briefly $\sum \tau_t$.

Definition 2.8 ([6]). Let X and Y be two fts's and let

$$Y^X = \{f : X \to Y | f \text{ is fuzzy continuous}\}.$$

Fuzzy compact-open topology on Y^X is given as follows: Let

$$\mathbf{K} = \{K \in I^X : K \text{ is fuzzy compact in } X\}.$$

and

$$\mathbf{V} = \{V \in I^Y : V \text{ is fuzzy open in } Y\}.$$

For any K$\in \mathbf{K}$ and V$\in \mathbf{V}$, let

$$N_{K,V} = \{f \in Y^X : f(K) \le V\}.$$

The collection $\{N_{K,V} : K \in \mathbf{K}, V \in \mathbf{V}\}$ can be used as a fuzzy subbase to generate a fuzzy topology on the class Y^X, called the fuzzy compact-open topology. The class Y^X with this topology is called a fuzzy compact-open topological space.

Definition 2.9 ([6]). The mapping $e : Y^X \times X \to Y$ defined by $e(f, x_t) = f(x_t)$ for each $f-$point $x_t \in X$ and $f \in Y^X$ is called the fuzzy evaluation map.

Let X be a set and let Y be a fts. Let Y^X denote the collection of all functions from X to Y. Let ξ be any non-empty subcollection of Y^X (see [9]).

Definition 2.10 ([9]). For each point $x \in X$ define a map $e_x : \xi \to Y$ by $e_x(f) = f(x)$. The mapping e_x is called evaluation map at the point $x \in X$. The initial fuzzy topology τ_p generated by the collection of maps $\{e_x : x \in X\}$ is called the pointwise fuzzy topology on ξ. The pair (ξ, τ_p) is referred to as the pointwise fuzzy function space.

3. The pointwise intuitionistic fuzzy topology on function spaces

Let (X, τ, τ^*) and (Y, γ, γ^*) be two $IFTS$s. Let Y^X denote the collection of all gp-maps from X to Y. In this section, we introduce the notion of pointwise intuitionistic fuzzy topology on Y^X and show that Y^X is a strong Hausdorff space if Y is a strong Hausdorff space. For introduce pointwise intuitionistic fuzzy topology, we firstly give function space for fuzzy bitopological spaces.

Let (X, T, T^*) and (Y, K, K^*) be two fuzzy bitopological spaces. For each fuzzy point $x_\alpha \in X$, define a map $e_{x_\alpha} : Y^X \to Y$ by $e_{x_\alpha}(f) = f(x_\alpha)$. Here e_{x_α} is the evaluation map at the point x_α. Since (Y, K, K^*) is a fuzzy bitopological space, we can define two fuzzy topology on the fuzzy set Y^X. Subbases of these topologies are $\{e_{x_\alpha}^{-1}(\mu) : x_\alpha \in X,\ \mu \in K\}$ and $\{e_{x_\alpha}^{-1}(\mu) : x_\alpha \in X,\ \mu \in K^*\}$, respectively. Pointwise fuzzy topologies generated by these subbases are denoted as K_p and K_p^*, respectively. Since $K \subset K^*$, $K_p \subset K_p^*$ is satisfied. Thus the triplet (Y^X, K_p, K_p^*) is a fuzzy bitopological space.

Definition 3.1. (Y^X, K_p, K_p^*) is called the pointwise fuzzy bitopological function space.

Remark 3.2. It is clear that the evaluation map $e_{x_\alpha} : Y^X \to Y$ is fuzzy continuous.

Now, we pass function spaces of intuitionistic fuzzy topological spaces. Let (X, τ, τ^*) and (Y, γ, γ^*) be two $IFTS$s. For each $r \in I_0$, the set $(Y, \gamma^r, \gamma^{*r})^{(X, \tau^r, \tau^{*r})}$ is the collection of all fuzzy continuous functions of fuzzy bitopological spaces. Then we obtain pointwise fuzzy bitopological function spaces $(Y^X, \gamma_p^r, \gamma_p^{*r})$, for each $r \in I_0$.

If $r' \succ r \in I_0$, then $\gamma^{r'} \subset \gamma^r$ and $\gamma^{*r'} \subset \gamma^{*r}$ are satisfied. Hence

$$\gamma_p^{r'} = \{e_{x_\alpha}^{-1}(\mu) : x_\alpha \in X,\ \mu \in \gamma^{r'}\} \subset \{e_{x_\alpha}^{-1}(\nu) : x_\alpha \in X,\ \nu \in \gamma^r\} = \gamma_p^r$$

and

$$\gamma_p^{*r'} = \{e_{x_\alpha}^{-1}(\mu) : x_\alpha \in X,\ \mu \in \gamma^{*r'}\} \subset \{e_{x_\alpha}^{-1}(\nu) : x_\alpha \in X,\ \nu \in \gamma^{*r}\} = \gamma_p^{*r}$$

are obtained. Thus $\{\gamma_p^r\}_{r \in I_0}$ and $\{\gamma_p^{*r}\}_{r \in I_0}$ are descending families of fuzzy topologies on the fuzzy set Y^X. By using these families, IGOs on Y^X

$$\gamma_p : Y^X \to I, \quad \gamma_p^* : Y^X \to I$$

are defined by

$$\gamma_p(\mu) = \vee\{r : \mu \in \gamma_p^r\}, \quad \gamma_p^*(\mu) = \wedge\{1 - r : \mu \in \gamma_p^{*r}\} \text{ (see [11])}.$$

Definition 3.3. The triplet $(Y^X, \gamma_p, \gamma_p^*)$ is called the pointwise intuitionistic fuzzy function space (briefly $PIFFS$).

It is clear from the Definition 3.3 that properties of $IGO(\gamma_p, \gamma_p^*)$ depend on the properties of $IGO(\gamma, \gamma^*)$, and the properties $IGO(\tau, \tau^*)$ play essentially no role.

Remark 3.4. The evaluation map $e_{x_\alpha} : (Y^X, \gamma_p, \gamma_p^*) \to (Y, \gamma, \gamma^*)$ is a gp-map for each fuzzy point $x_\alpha \in X$.

In this study, we use the product operation as defined by Mondal et al. [11]. The following theorems are easily obtained from [8, 9, 16].

Proposition 3.5. *The $PIFFS$ $(Y^X, \gamma_p, \gamma_p^*)$ is intuitionistic fuzzy homeomorphic to the product space $\prod_{x \in X}(Y, \gamma, \gamma^*)$.*

Proposition 3.6. *A map $g : (Z, \eta, \eta^*) \to (Y^X, \gamma_p, \gamma_p^*)$, where (Z, η, η^*) is an $IFTS$, is a gp-map if and only if the map $e_{x_\alpha} \circ g : (Z, \eta, \eta^*) \to (Y, \gamma, \gamma^*)$ is a gp-map for each $x_\alpha \in X$.*

Definition 3.7. Let (X, τ, τ^*) be an $IFTS$.

(a) The space (X, τ, τ^*) is called $r-$Hausdorff if and only if for each fuzzy points $x_\alpha, y_\beta \in X$, there exist fuzzy sets λ, μ such that
$x_\alpha \leq \lambda,\ y_\beta \leq \mu,\ \lambda \wedge \mu = \underline{0}$ and $\tau(\lambda), \tau(\mu) \geq r;\ \tau^*(\lambda), \tau^*(\mu) \leq 1 - r$.

(b) If (X, τ, τ^*) is $r-$ Hausdorff for each $r \in I_0$, then this space is called the strong Hausdorff space.

Lemma 3.8. *(X, τ, τ^*) is a strong Hausdorff space if and only if for each $r \in I_0$, (X, τ^r, τ^{*r}) is a Hausdorff space.*

Proof. It is clear that if (X, τ, τ^*) is a strong Hausdorff space, then the fuzzy topological space (X, τ^r) is a fuzzy Hausdorff space, for each $r \in I_0$. Since $\tau^r \subset \tau^{*r}$, (X, τ^{*r}) is a fuzzy Hausdorff space. Thus the fuzzy bitopological space (X, τ^r, τ^{*r}) is a fuzzy Hausdorff space with respect to τ^r and τ^{*r}.

Similarly if the fuzzy bitopological space (X, τ^r, τ^{*r}) is fuzzy Hausdorff space, for each $r \in I_0$, then $IFTS$ (X, τ, τ^*) generated from bitopologies $\{(\tau^r, \tau^{*r})\}$ is a strong Hausdorff space. □

Theorem 3.9. *If $IFTS$ (Y, γ, γ^*) is a strong Hausdorff space, then $PIFFS$*

$$(Y^X, \gamma_p, \gamma_p^*)$$

is a strong Hausdorff space for each $IFTS$ (X, τ, τ^*).

Proof. Since (Y, γ, γ^*) is a strong Hausdorff space, the fuzzy bitopological space $(Y, \gamma^r, \gamma^{*r})$ is a fuzzy Hausdorff space for each $r \in I_0$. Then fuzzy bitopological space $(Y^X, \gamma_p^r, \gamma_p^{*r})$ is a fuzzy Hausdorff space for fuzzy bitopological space (X, τ^r, τ^{*r}) [9]. Thus for each $r \in I_0$, since the fuzzy bitopological space $(Y^X, \gamma_p^r, \gamma_p^{*r})$ is a fuzzy Hausdorff space and IGOs generated by the family of the fuzzy bitopologies $\{(\gamma_p^r, \gamma_p^{*r})\}$ are equal to IGOs (γ_p, γ_p^*), $PIFFS$ $(Y^X, \gamma_p, \gamma_p^*)$ is a strong Hausdorff space. □

4. Interrelations between function spaces

In this section, we construct relationships between some function spaces in the category $IFTS$ as similar to an ordinary topology. For this, we firsty give necassary operations in the category of $IFTS$. Subspace operation is defined in [11]. If there is a fuzzy topology τ on X, then sum fuzzy topological space, quotient space, subbase, base and product operation are defined [8, 16].

Now we introduce an internal form of sum of a family of $IFTS$.

Definition 4.1. Let $\{(X_t, \tau_t, \tau_t^*)\}_{t\in T}$ be a family of pairwise disjoint $IFTS$s, i.e., $X_{t_1} \bigcap X_{t_2} = \varnothing$ for $t_1 \neq t_2$. Consider the set $X = \bigcup_{t\in T} X_t$ and each $t \in T$, $j_t : X_t \to X$ is the usual inclusion mapping. Define

$$\forall B \in X,\ \sigma(B) = \bigwedge_{t\in T} \tau_t(j_t^{-1}(B)),\ \sigma^*(B) = \bigvee_{t\in T} \tau_t^*(j_t^{-1}(B)).$$

Then (σ, σ^*) is an IGO on the fuzzy set X, denoted by $\sum_{t\in T}(\tau_t, \tau_t^*)$. The $IFTS$ $(X, \sum_{t\in T}(\tau_t, \tau_t^*))$ is called intuitionistic fuzzy sum topological spaces and written as briefly $\sum_{t\in T}(X_t, \tau_t, \tau_t^*)$.

It is clear that for each $t \in T$, $j_t : (X_t, \tau_t, \tau_t^*) \to \sum(X_t, \tau_t, \tau_t^*)$ is a gp-map.

Proposition 4.2. *Let* $(X, \sigma, \sigma^*) = \sum_{t\in T}(X_t, \tau_t, \tau_t^*)$. *Then*

(1) $\forall A \in X,\ \sigma(A) = \bigwedge_{t\in T} \tau_t(A|_{X_t}),\quad \sigma^*(A) = \bigvee_{t\in T} \tau_t^*(A|_{X_t})$.

(2) *Let* $Y_t \subset X_t$ *for each* $t \in T$. *Then*

$$\sigma|_Y = \sum(\tau_t|_{Y_t}) \text{ and } \sigma^*|_Y = \sum(\tau_t^*|_{Y_t}),$$

where $Y = \bigcup_{t\in T} Y_t$.

Proof. The proof of proposition is immediately obtained from [8]. □

Definition 4.3. Let $\{(X_t, \tau_t, \tau_t^*)\}_{t\in T}$ and $\{(Y_t, \gamma_t, \gamma_t^*)\}_{t\in T}$ be two families of pairwise disjoint $IFTS$s and for each $t \in T$, $f_t : (X_t, \tau_t, \tau_t^*) \to (Y_t, \gamma_t, \gamma_t^*)$ be a gp-map. Define

$$\forall x_\alpha \in \bigcup_{t\in T} X_t, \quad f(x_\alpha) = f_{t_0}(x_\alpha),$$

where x_α belongs to a unique X_{t_0}. Then

$$f : \sum(X_t, \tau_t, \tau_t^*) \to \sum(Y_t, \gamma_t, \gamma_t^*)$$

is called the sum of family $\{f_t\}_{t\in T}$ and denoted by $\sum f_t$.

Proposition 4.4. (a) $\sum f_t : \sum(X_t, \tau_t, \tau_t^*) \to \sum(Y_t, \gamma_t, \gamma_t^*)$ *is a gp-map.*

(b) $g : \sum(X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ *is a gp-map if and only if*

$$g \circ j_t : (X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$$

is a gp-map for each $t \in T$.

Proof. (a) For each $\mu \in \sum(Y_t, \gamma_t, \gamma_t^*)$,

$$\begin{aligned}(\sum \tau_t)(f^{-1}(\mu)) &= (\sum \tau_t)((\sum f_t)^{-1}(\mu)) = (\sum \tau_t)(\sum f_t^{-1}(\mu|_{Y_t}))\\ &= \bigwedge_t \tau_t(j_t^{-1}(\sum f_t^{-1}(\mu|_{Y_t}))) = \bigwedge_t \tau_t(j_t^{-1} f_t^{-1}(\mu|_{Y_t}))\\ &= \bigwedge_t \tau_t((f_t j_t)^{-1}(\mu|_{Y_t})) \geq \bigwedge_t \gamma_t(\mu|_{Y_t}) = (\sum \gamma_t)(\mu),\end{aligned}$$

$$(\sum \tau_t^*)(f^{-1}(\mu)) = (\sum \tau_t^*)((\sum f_t)^{-1}(\mu)) = \bigwedge_t \tau_t^*)(j_t^{-1}(\sum f_t^{-1}(\mu|_{Y_t})))$$
$$= \bigwedge_t \tau_t^*(j_t^{-1} f_t^{-1}(\mu|_{Y_t})) = \bigwedge_t \tau_t^*((f_t j_t)^{-1}(\mu|_{Y_t})) \leq \bigwedge_t \gamma_t^*(\mu|_{Y_t}) = (\sum \gamma_t^*)(\mu).$$

Then $f = \sum f_t$ is a gp-map.

(b) Let $g : \sum(X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ is a gp-map. Then $g \circ j_t$ being the composition of gp-maps is itself gp-map.

Conversely, suppose that $g \circ j_t$ is a gp-map. For each $\mu \in Y$,

$$(\sum \tau_t)(g^{-1}(\mu)) = \bigwedge_t \tau_t(j_t^{-1}(g^{-1}(\mu))) = \bigwedge_t \tau_t((g \circ j_t)^{-1}(\mu))$$
$$\geq \bigwedge_t \gamma_t(\mu) = \gamma(\mu),$$

$$(\sum \tau_t^*)(g^{-1}(\mu)) = \bigvee_t \tau_t^*(j_t^{-1}(g^{-1}(\mu))) = \bigvee_t \tau_t^*((g \circ j_t)^{-1}(\mu))$$
$$\leq \bigvee_t \gamma_t^*(\mu) = \gamma^*(\mu),$$

i.e., g is a gp-map. □

Remark 4.5. $\sum \prod IFTS \to IFTS$ is a functor.

Definition 4.6. Let $\{(X_t, \tau_t, \tau_t^*)\}_{t \in T}$ be a family of pairwise disjoint $IFTS$s and (Y, γ, γ^*) be an $IFTS$ and $f_t : (X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ be a gp-map ,for each $t \in T$. Define

$$\nabla f_t : \sum(X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$$

such that

$$(\nabla f_t)(x_\alpha) = f_{t_0}(x_\alpha), \quad \forall x_\alpha \in \sum X_t$$

where x_α belongs to a unique X_{t_0}. The mapping ∇f_t is called the combination of the family $\{f_t\}_{t \in T}$.

Proposition 4.7. *$f_t : (X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ is a gp-map, for each $t \in T$, if and only if $\nabla f_t : \sum(X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ is gp-map, too.*

Proof. From Proposition 4.4, the mapping ∇f_t is a gp-map if and only if $(\nabla f_t) \circ j_t : (X_t, \tau_t, \tau_t^*) \to (Y, \gamma, \gamma^*)$ is a gp-map for each $t \in T$. Since $(\nabla f_t) \circ j_t = f_t$, the proof is completed. □

Definition 4.8. Let $\{(X_t, \tau_t, \tau_t^*)\}_{t \in T}$ be a family of pairwise disjoint $IFTS$s, (Y, γ, γ^*) be an $IFTS$ and $\prod({Y^X}_t, \gamma_p^t, \gamma_p^{*t})$, $\sum(X_t, \tau_t, \tau_t^*)$ be product and sum of $PIFFS$, respectively. Define

$$\nabla : \prod({Y^X}_t, \gamma_p^t, \gamma_p^{*t}) \to (Y^{\sum X_t}, \gamma_p^{\sum \tau_t}, {\gamma_p^*}^{\sum \tau_t^*})$$

such that

$$\forall \{f_t\} \in \prod {Y^X}_t, \quad \forall x_\alpha \in \sum X_t, \quad \nabla(\{f_t\})(x_\alpha) = f_{t_0}(x_\alpha),$$

where x_α belongs to a unique X_{t_0}.

Theorem 4.9. *The mapping*

$$\nabla : \prod (Y^X{}_t, \gamma_p^t, \gamma_p^{*t}) \to (Y^{\sum X_t}, \gamma_p^{\sum \tau_t}, \gamma_p^{*\sum \tau_t^*})$$

is an intuitionistic fuzzy homeomorphism in the pointwise intuitionistic fuzzy topology.

Proof. ∇ is bijective mapping [7]. Let us consider the following mapping of fuzzy bitopological spaces

$$\nabla : \prod (Y^X{}_t, (\gamma_p^t)^r, (\gamma_p^{*t})^r) \to (Y^{\sum X_t}, (\gamma_p^{\sum \tau_t})^r, (\gamma_p^{*\sum \tau_t^*})^r)$$

for each $r \in I_0$. We can verify that ∇ is a fuzzy continuous mapping in the pointwise intuitionistic fuzzy topology. For this, we need to show that inverse image of each fuzzy set $e^{-1}_{(x_\alpha)}(\mu)$ is a fuzzy open set, where $e^{-1}_{(x_\alpha)}(\mu)$ belongs to subbase of fuzzy bitopological space $Y^{\sum X_t}$. Here $\mu \in (\gamma^{\sum \tau_t})^r$ and $\mu \in (\gamma^{*\sum \tau_t^*})^r$. Let fuzzy point x_α belongs to a unique set X_{t_0}. Then

$$e^{-1}_{x_\alpha}(\mu) = \{f : \sum X_t \to Y | f(x_\alpha) \leq \mu\} = \{f_{t_0} : X_{t_0} \to Y | f(x_\alpha) \leq \mu\}.$$

Since

$$\nabla^{-1}(e^{-1}_{x_\alpha}(\mu)) = \nabla^{-1}(\{f_{t_0} : X_{t_0} \to Y | f_{t_0}(x_\alpha) \leq \mu\})$$
$$= \{f_{t_0} : X_{t_0} \to Y | f_{t_0}(x_\alpha) \leq \mu\} \times \prod (Y^X{}_t, (\gamma_p^t)^r, (\gamma_p^{*t})^r),$$

∇ is a fuzzy continuous mapping on the fuzzy bitopological spaces.

Now we prove that the mapping

$$\nabla^{-1} : Y^{\sum X_t} \to \prod Y^{X_t}, \quad \nabla^{-1}(f) = \{f|_{X_t}\}$$

is a fuzzy continuous mapping on the fuzzy bitopological spaces. Indeed, for each the set $e^{-1}_{(x_\alpha)_{t_0}}(\mu) \times \prod_{t \neq t_0} Y^{X_t}$ belonging to subbase of the bitopological space $\prod_{t \neq t_0} (Y^X{}_t, (\gamma_p^t)^r, (\gamma_p^{*t})^r)$,

$$e^{-1}_{(x_\alpha)_{t_0}}(\mu) \times \prod_{t \neq t_0} Y^{X_t} = \left\{ \{f_t\} \in \prod_{t \neq t_0} Y^{X_t} : f_{t_0}(x_\alpha) \leq \mu \right\}$$

is satisfied. Since the set

$$(\nabla^{-1})^{-1} \left(e^{-1}_{(x_\alpha)_{t_0}}(\mu) \times \prod_{t \neq t_0} Y^{X_t} \right) = \nabla \left(e^{-1}_{(x_\alpha)_{t_0}}(\mu) \times \prod_{t \neq t_0} Y^{X_t} \right)$$
$$= \{\nabla f_t : f_{t_0}(x_\alpha) \leq \mu\}$$

belongs to subbase of the fuzzy bitopological space $Y^{\sum X_t}$ with respect to $(\gamma^{\sum \tau_t})^r$ and $(\gamma^{*\sum \tau_t^*})^r$, the mapping ∇^{-1} is fuzzy continuous mapping. Thus since the mapping

$$\nabla : \prod \left(Y^{X_t}, (\gamma_p^t)^r, (\gamma_p^{*t})^r\right) \to \left(Y^{\sum X_t}, (\gamma_p^{\sum \tau_t})^r, \ (\gamma_p^{*\sum \tau_t^*})^r\right)$$

is a fuzzy homeomorphism of fuzzy bitopological spaces for each $r \in I_0$, the mapping

$$\nabla : \prod \left(Y^{X_t}, \gamma_p^t, \gamma_p^{*t}\right) \to \left(Y^{\sum X_t}, \gamma_p^{\sum \tau_t}, \ \gamma_p^{*\sum \tau_t^*}\right)$$

is an intuitionistic fuzzy homeomorphism.

Let (X,τ,τ^*) be an $IFTS$ and $\{(Y_t,\gamma_t,\gamma_t^*)\}_{t\in T}$ be a family of $IFTS$S. We define the mapping

$$\Delta:\prod_t (Y_t^X,\gamma_p^t,\gamma_p^{*t}) \to \left(\left(\prod_t Y_t\right)^X, \left(\prod_t \gamma^t\right)_p, \left(\prod_t \gamma^{*t}\right)_p\right)$$

by the rule $\forall x_\alpha \in X, (\Delta(\{f_t\}))(x_\alpha)=\{f_t(x_\alpha)\}$ (see [7]).

The inverse of Δ is defined by the rule

$$\forall f: X\to \prod_{t\in T} Y_t, \quad \Delta^{-1}(f)=\{P_t\circ f\}_{t\in T},$$

where $P_t:\prod Y_t \to Y_t$ are projections for each $t\in T$. □

Theorem 4.10. *The mapping*

$$\Delta:\prod_t (Y_t^X,\gamma_p^t,\gamma_p^{*t}) \to \left(\left(\prod_t Y_t\right)^X, \left(\prod_t \gamma^t\right)_p, \left(\prod_t \gamma^{*t}\right)_p\right)$$

is an intuitionistic fuzzy homeomorphism in the pointwise intuitionistic fuzzy topology.

Proof. Δ is bijective mapping [7]. For each $r\in I_0$,

$$\Delta:\prod_t (Y_t^X,(\gamma_p^t)^r,(\gamma_p^{*t})^r) \to \left(\left(\prod_t Y_t\right)^X, \left(\prod_t \gamma^t\right)_p^r, \left(\prod_t \gamma^{*t}\right)_p^r\right)$$

is a fuzzy mapping of fuzzy bitopological spaces. Now let us show that this mapping is fuzzy open on each fuzzy topologies. Let us take arbitrary a set

$$e^{-1}_{(x^1_{\alpha_1})_{t_1}}(\mu_{t_1})\times\ldots\times e^{-1}_{(x^k_{\alpha_k})_{t_k}(\mu_{t_k})}\times \prod_{t\neq t_1,\ldots,t_k} Y_t^X$$

belonging to base of each topologies in the fuzzy bitopological space

$$\prod_t (Y_t^X,(\gamma_p^t)^r,(\gamma_p^{*t})^r).$$

$$\begin{aligned}&\Delta\left(e^{-1}_{(x^1_{\alpha_1})_{t_1}}(\mu_{t_1})\times\ldots\times e^{-1}_{(x^k_{\alpha_k})_{t_k}(\mu_{t_k})}\times \prod_{t\neq t_1,\ldots,t_k} Y_t^X\right)\\ &=\left\{\{f_t\}|f_{t_1}(x^1_{\alpha_1})\le\mu_{t_1},\ldots,f_{t_k}(x^k_{\alpha_k})\le\mu_{t_k}\right\}\\ &=M\left(\{x^1_{\alpha_1},\ldots,x^k_{\alpha_k}\},\mu_{t_1}\times\ldots\times\mu_{t_k}\times\prod_{t\neq t_1,\ldots,t_k} Y_t^X\right)\end{aligned}$$

is a fuzzy open set in the suitable topology of the fuzzy bitopological space $\left(\prod Y_t\right)^X$.

Similarly, it can be proven that Δ^{-1} is a fuzzy open mapping. Indeed for each fuzzy open set

$$e_{x_\alpha}^{-1}\left(\mu_{t_1}\times\ldots\times\mu_{t_k}\times\prod_{t\neq t_1,\ldots,t_k}Y_t\right),$$

$$\Delta^{-1}\left(e_{x_\alpha}^{-1}\left(\mu_{t_1}\times\ldots\times\mu_{t_k}\times\prod_{t\neq t_1,\ldots,t_k}Y_t\right)\right)$$

$$=\Delta^{-1}\left(\left\{f:X\to\prod_{t\in T}Y_t|f(x_\alpha)\in\mu_{t_1}\times\ldots\times\mu_{t_k}\times\prod_{t\neq t_1,\ldots,t_k}Y_t^X\right\}\right)$$

$$=\left\{p_t\circ f|f(x_\alpha)\in\mu_{t_1}\times\ldots\times\mu_{t_k}\times\prod_{t\neq t_1,\ldots,t_k}Y_t^X\right\}$$

$$=\{p_t\circ f|p_{t_1}\circ f(x_\alpha)\in\mu_{t_1},\ldots,p_{t_k}\circ f(x_\alpha)\in\mu_{t_k}\}$$

$$=e_{x_\alpha}^{-1}\left(p_{t_1}^{-1}(\mu_{t_1})\cap\ldots\cap p_{t_k}^{-1}(\mu_{t_k})\right).$$

Hence this set is a fuzzy open. Now, let (X,τ,τ^*), (Y,γ,γ^*) and (Z,η,η^*) be $IFTS$s and $f:(Z,\eta,\eta^*)\times(X,\tau,\tau^*)\to(Y,\gamma,\gamma^*)$ be a mapping of intuitionistic fuzzy sets. Then the induced map $\hat{f}: X\to Y^Z$ is defined by $\left(\hat{f}(x_\alpha)\right)(z_\beta)=f(z_\beta,x_\alpha)$ for fuzzy point $x_\alpha\in X$ and $z_\beta\in Z$ [6]. We define exponential law $E:Y^{Z\times X}\to(Y^Z)^X$ by using induced maps

$$E(f)=\hat{f}$$

i.e., $E(f)(x_\alpha)(z_\beta)=f(z_\beta,x_\alpha)=\hat{f}(x_\alpha)(z_\beta)$.

We define the following mapping

$$E^{-1}:(Y^Z)^X\to Y^{Z\times X}$$

which is an inverse mapping E as follows:

$E^{-1}(\hat{f})=f$, $\quad E^{-1}(\hat{f})(z_\beta,x_\alpha)=\hat{f}(x_\alpha)(z_\beta)=f(z_\beta,x_\alpha)$ [4, 5]. □

Theorem 4.11. *Let (X,τ,τ^*), (Y,γ,γ^*) and (Z,η,η^*) be $IFTSs$ and $e:Y^Z\times Z\to Y$, $e(f,z)=f(z)$ be gp-map. Function space Y^X with pointwise intuitionistic fuzzy topology and for each gp-map $\hat{g}: X\to Y^Z$,*

$$E^{-1}(\hat{g}):Z\times X\to Y$$

is also gp-map.

Proof. By using the mapping

$$1_Z\times\hat{g}: Z\times X\to Z\times Y^Z,$$

we take

$$Z\times X\xrightarrow{1_Z\times\hat{g}}Z\times Y^Z\xrightarrow{t}Y^Z\times Z\xrightarrow{e}Y.$$

Hence $e \circ t \circ (1_Z \times \hat{g}) \in Y^{Z\times X}$, where t denotes switching map. Let us apply exponential law E to $e \circ t \circ 1_Z \times \hat{g}$. For each fuzzy points $x_\alpha \in X$ and $z_\beta \in Z$,

$$\left\{\left[E\left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right)\right](x_\alpha)\right\}(z_\beta) = \left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right)(z_\beta, x_\alpha)$$
$$= e \circ t\left(z_\beta, \hat{g}(x_\alpha)\right) = e\left(\hat{g}(x_\alpha), z_\beta\right) = \left(\hat{g}(x_\alpha)\right)(z_\beta).$$

Since $E\left(e \circ t \circ \left(1_Z \times \hat{g}\right)\right) = \hat{g}$, $E^{-1}\left(\hat{g}\right) = e \circ t \circ \left(1_Z \times \hat{g}\right)$. Hence since evaluation map e and switching map t are gp-maps, $E^{-1}\left(\hat{g}\right)$ is a gp-map. □

5. Conclusion

This paper summarized intuitionistic fuzzy topologies on function spaces. Later we construct interrelations among some function spaces. To extend this work, by using pointwise intuitionistic fuzzy function spaces, one could study different subdisciplines of mathematics. For example, algebraic topology and functional analysis etc.

References

[1] I. W. Alderton, Function spaces in fuzzy topology, Fuzzy Sets and Systems 32 (1989) 115–124.
[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1) (1986) 87–96.
[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[4] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness: fuzzy topology, Fuzzy Sets and Systems 49 (1992) 237–242.
[5] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and systems 88 (1997) 81–89.
[6] S. Dang and A. Behera, On fuzzy compact-open topology, Fuzzy Sets and Systems 80 (1996) 377–381.
[7] R. Engelking, General Topology, Polish Scientific Publishers, Warszawa 1977.
[8] J. Fang, Sums of L-fuzzy topological spaces, Fuzzy Sets and Systems 157 (2005) 739–754.
[9] J. K. Kohli and A. R. Prasannan, Fuzzy topologies on function spaces, Fuzzy Sets and Systems 116 (2000) 415–420.
[10] T. Kubiak, On fuzzy topologies, Ph. D. Thesis, Adam Mickiewicz, Ponzan, Poland 1985.
[11] T. K. Mondal and S. K. Samanta, On intuitionistic gradation of openness, Fuzzy Sets and Systems 131 (2002) 323–336.
[12] A. Šostak, Two decades of fuzzy topology: Basic ideas notions and results, Russian Math. Surveys 44 (1989) 125–186.
[13] A. Šostak, On the neighborhood structure of a fuzzy topological space, Zb. Radova Uni. Nis, Ser Math. 4 (1990) 7–14.
[14] A. Šostak, On a fuzzy topological structure, Rendiconti Ciecolo Matematico Palermo (Supp. Ser II) 11 (1985) 89–103.
[15] Y. W. Peng, Topological structure of a fuzzy function space - the pointwise convergent topology and compact open topology, Kexue Tongbao 29 (1984) 289–292.
[16] Y. Yue, Lattice-valued induced fuzzy topological spaces, Fuzzy Sets and Systems 158 (2007) 1461–1471.
[17] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

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