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Common fixed point theorems in fuzzy metric spaces

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Abstract. In this paper, we prove common fixed point theorems in fuzzy metric spaces. We also discuss result related to R-weakly commuting type mappings.

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1. Introduction

In 1965, Zadeh [9] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. Kramosil and Michalek [3] introduced and studied the concept of fuzzy metric space and later there has been much progress in the study of fuzzy metric spaces by many authors [1-8]. In this paper, we prove common fixed point theorems in fuzzy metric spaces. We also discuss result related to R-weakly commuting type mappings.

2. Preliminaries

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Example 2.2. $a * b = \min\{a, b\}$ and $a * b = a \cdot b$ are t-norms.

Kramosil and Michalek [3] introduced the concept of fuzzy metric spaces as follows:

Definition 2.3. The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in X^2 $\times$ [0,∞) satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x.t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$, (Triangular inequality)
(FM-5) $M(x, y, \cdot) : [0, 1) \to [0, 1]$ is left continuous

for all $x, y, z \in X$ and $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way: Let (X, d) be a metric space. Define $a * b = a + b$ for all $a, b \in X$. Define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric. Consider M to be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Definition 2.4. Let $(X, M, *)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to
  (i) be a Cauchy sequence if
  $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $n, p \in N$,
  (ii) be convergent to a point $x \in X$ if
  $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(b) X is said to be complete if every Cauchy sequence in X converges to some point in X.

Example 2.5. Let $X = \{\frac{1}{n} : n = 1, 2, 3, ...\} \cup \{0\}$ and let $*$ be the continuous t-norm defined by $a * b = ab$ for all $a, b \in [0,1]$. For each $t \in (0,\infty)$ and $x, y \in X$, define

$$M(x, y, t) = \begin{cases} \left(\frac{t}{t+|x-y|}\right) & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

Clearly, $(X, M, *)$ is complete fuzzy metric space.

Definition 2.6. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be commuting if $M(fgx, gfx, t) = 1$ for all $x \in X$.

Definition 2.7. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all x in X and $t > 0$.

In 1994, Mishra et al. [4] introduced the concept of compatible mapping in FM-space akin to concept of compatible mapping in metric space as follows:

Definition 2.8. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

In 1994, Pant [5] introduced the notion of R-weakly commuting maps in metric spaces. Later on, Vasuki [8] initiated the concept of non compatibility of mappings in fuzzy metric spaces, by introducing the notion of R-weakly commuting mappings in fuzzy metric spaces and proved some common fixed point theorems for these maps.

Definition 2.9. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, \tfrac{t}{R})$$

for all $x \in X$ and $t > 0$.

Later on, Pathak et al. [7] improved the notion of R-weakly commuting mappings in metric spaces by introducing the notions of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f). In 2008, Imdad and Ali [1] embarked the notion of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f) in fuzzy metric with inspiration from Pathak et al. [7] and they further introduced the notion of R-weakly commuting mappings of type (P) in fuzzy metric spaces.

Definition 2.10. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting of type (A_f) if there exists some $R > 0$ such that $M(ggx, fgx, t) \geq M(gx, fx, \frac{t}{R})$ for all $x \in X$ and $t > 0$.

Definition 2.11. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting of type (A_g) if there exists some $R > 0$ such that $M(ffx, gfx, t) \geq M(fx, gx, \frac{t}{R})$ for all $x \in x$ and $t > 0$.

Definition 2.12. Let $(X, M, *)$ be a fuzzy metric space. Suppose f and g be self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.

Definition 2.13. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.14. A pair of self mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be R-weakly commuting of type (P) if there exists some $R > 0$ such that $M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R})$ for all $x \in X$ and $t > 0$.

Now, we establish the example which illustrates the independency of R-weakly commuting mappings of different kinds:

Example 2.15. Let $(X, M, *)$ be a fuzzy metric space with $X = [0, 1]$, t-norm $*$ defined by $a * b = \min\{a, b\}$ where $a, b \in [0, 1]$, respectively. Let M be the fuzzy set on $X^2 \times (0, \infty)$, defined by

$$M(x, y, t) = \begin{cases} \left(exp\left(\frac{|x-y|}{t}\right)\right)^{-1} & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

Then, it is well known that $(X, M, *)$ is fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$. Then by a straightforward calculation, one can show that

$$\begin{aligned} M(fgx, gfx, t) &= \begin{cases} \left(exp\left(\frac{2|x-y|^2}{t}\right)\right)^{-1} & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases} \\ &= M(fx, gx, t/2) \end{aligned}$$

It is clear that for $R = 2$, the pair (f, g) is R-weakly commuting but not weakly commuting as strict increasing property of the exponential function.

However, various kinds of above mentioned R-weak commutativity notions are independent of one another and none implies the other. The earlier example can be utilized to demonstrate the inter-independence.

To illustrate the independency of R-weak commutativity with R-weak commutativity of type (A_f), one should consider

$$M(fgx, ggx, t) = \left(e^{\frac{|x^4-2x^2+1|}{t}}\right)^{-1} < \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, \tfrac{t}{R})$$

when $x > 1$ which shows that R-weak commutativity does not imply R-weak commutativity of type (A_f).

Secondly, in order to demonstrate the independence of R-weak commutativity with R-weak commutativity of type (P). We note that

$$\begin{aligned} M(ffx, ggx, t) &= \left(e^{\frac{|x^4-4x+3|}{t}}\right)^{-1} \\ &< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, \tfrac{t}{R}) \text{ for } x > 1 \end{aligned}$$

which implies that R-weak commutativity does not imply R-weak commutativity of type (P).

Finally, the pair (f, g) is R-weakly commuting of type (A_g) as

$$M(gfx, ffx, t) = \left(e^{\frac{|(2x-1)^2-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{4|x-1|^2}{t}}\right)^{-1} = M(fx, gx, \tfrac{t}{4})$$

which shows that (f, g) is R-weakly commuting of type (A_g) for $R = 4$. This situation may also be utilized to interpret that an R-weakly commuting pair of type (A_g) need not be R-weakly commuting pair of type (A_f) or type (P).

3. Main results

Theorem 3.1. *Let $(X, M, *)$ be a complete metric space. Let f and g be weakly compatible self maps of X satisfying*

(3.1) $M(gx, gy, kt) \geq M(fx, fy, t)$ *where* $0 < k < 1$,
(3.2) $g(X) \subseteq f(X)$.

If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose $x_1 \in X$ such that $g(x_0) = f(x_1)$. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Then by (3.1), we have

$$M(fx_n, fx_{n+1}, t) = M(gx_{n-1}, gx_n, t) \geq M(fx_{n-1}, fx_n, \tfrac{t}{k})$$
$$= M(gx_{n-2}, gx_{n-1}, \tfrac{t}{k}) \geq \cdots \geq M(fx_0, fx_1, \tfrac{t}{k^n}).$$

Therefore, for any p,

$$M(fx_n, fx_{n+p}, t) \geq M(fx_n, fx_{n+1}, \frac{t}{p}) \geq \cdots \geq M(fx_{n+p-1}, fx_{n+p}, \tfrac{t}{p})$$
$$\geq M(fx_0, fx_1, \tfrac{t}{pk^n}) \geq \cdots \geq M(fx_0, fx_1, \tfrac{t}{pk^{n+p-1}}).$$

As $n \to \infty$, $\{fx_n\} = \{y_n\}$ is a Cauchy sequence and so, by completeness of X, $\{y_n\} = \{fx_n\}$ is convergent. Call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. As $f(X)$ is complete, so there exist a point p in X such that $fp = z$. Now, from (3.1),

$M(gp, gx_n, kt) \geq M(fp, fx_n, t)$,
$n \to \infty$,
$M(gp, z, kt) \geq M(fp, z, t)$,
$M(gp, z, kt) \geq M(z, z, t)$,
$M(gp, z, kt) \geq 1$,
$M(gp, z, kt) = 1$,
$gp = z = fp$.

As f and g are weakly compatible, therefore $fgp = gfp$ i.e. $fz = gz$. Now, we show that z is fixed point of f and g. From (3.1),

$M(gz, gx_n, kt) \geq M(fz, fx_n, t)$,
$n \to \infty$,
$M(gz, z, kt) \geq M(fz, z, t)$,
$M(gz, z, kt) \geq M(gz, z, t)$,
$gz = z = fz$.

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, kt) \geq M(fz, fw, t)$, $M(z, w, kt) \geq M(z, w, t)$ and $z = w$. Therefore z is unique common fixed point of f and g. □

Example 3.2. Let $X = [0, 1]$. Define

$$M(x, y, t) = \begin{cases} \left(\frac{t}{t+|x-y|}\right) & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases}$$

Clearly, $(X, M, *)$ is complete fuzzy metric space. Define self maps f and g on X by $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{6}$. Then $g(X) \subseteq f(X)$ and for $\frac{1}{3} < k < 1$, condition (3.1) is satisfied. However, maps are weakly compatible at $x = 0$ and $x = 0$ is unique common fixed point of f and g.

Theorem 3.3. *Let $(X, M, *)$ be a fuzzy metric space. Let f and g be weakly compatible self maps of X satisfying condition (3.1) and (3.2). If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.*

Proof. From the proof of above theorem, we conclude that $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in X. Now, suppose that $f(X)$ is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in $f(X)$. Call it be u and $f(v) = u$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (3.1),

$$
\begin{aligned}
&M(gv, gx_n, kt) \geq M(fv, fx_n, t),\\
&n \to \infty,\\
&M(gv, u, kt) \geq M(fv, u, t),\\
&M(gv, u, kt) \geq M(u, u, t),\\
&M(gv, u, kt) \geq 1,\\
&M(gv, u, kt) = 1,\\
&gv = u = fv,
\end{aligned}
$$

which shows that pair (f, g) has a point of coincidence. Since, f and g are weakly compatible, $fgv = gfv$, i.e. $fu = gu$. Now, we show that u is a fixed point of f and g. From (3.1),

$$
\begin{aligned}
&M(gu, gx_n, kt) \geq M(fu, fx_n, t),\\
&n \to \infty,\\
&M(gu, u, kt) \geq M(fu, u, t),\\
&M(gu, u, kt) \geq M(gu, u, t),\\
&gu = u = fu.
\end{aligned}
$$

Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, kt) \geq M(fz, fw, t)$, $M(z, w, kt) \geq M(z, w, t)$ and $z = w$. Therefore z is unique common fixed point of f and g. □

Theorem 3.4. *Theorem 3.3 remains true if a weakly compatible property is replaced by any one of the following:*

(i) *R-weakly commuting property,*
(ii) *R-weakly commuting property of type (A_f),*
(iii) *R-weakly commuting property of type (A_g),*
(iv) *R-weakly commuting property of type (P),*
(v) *weakly commuting property.*

Proof. (i) Since all the conditions of Theorem 3.3 are satisfied, the existence of coincidence points for both the pairs are insured. Let x be an arbitrary point of coincidence for the pair (f, g). Then using R-weak commutativity, one gets

$$M(fgx, gfx, t) \geq M(fx, gx, \tfrac{t}{R}) = M(fx, fx, \tfrac{t}{R}) = 1 \text{ and } fgx = gfx.$$

Thus the pair (f, g) is weakly compatible. Now applying Theorem 3.3, one conclude that f and g have a unique common fixed point.

(ii) In case (f, g) is an R-weakly commuting pair of type (A_f), we have

$$\begin{aligned}
&M(ggx, fgx, t) \geq M(gx, fx, \tfrac{t}{R}) = M(fx, fx, \tfrac{t}{R}) = 1,\\
&fgx = ggx,\\
&M(fgx, gfx, t) \geq M(fgx, ggx, \tfrac{t}{2}) * M(ggx, gfx, \tfrac{t}{2})\\
&\qquad\qquad = M(fgx, fgx, \tfrac{t}{2}) * M(x, x, \tfrac{t}{2}) \geq 1 * 1 = 1,\\
&fgx = gfx.
\end{aligned}$$

(iii) In case (f, g) is an R-weakly commuting pair of type (A_g), we have

$$\begin{aligned}
&M(ffx, gfx, t) \geq M(fx, gx, \tfrac{t}{R}) = M(fx, fx, \tfrac{t}{R}) = 1,\\
&ffx = gfx,\\
&M(fgx, gfx, t) \geq M(fgx, ffx, \tfrac{t}{2}) * M(ffx, gfx, \tfrac{t}{2})\\
&\qquad\qquad = M(x, x, \tfrac{t}{2}) * M(ffx, ffx, \tfrac{t}{2}) \geq 1 * 1 = 1,\\
&fgx = gfx.
\end{aligned}$$

Similarly, if pair (f, g) is R-weakly commuting of type (P) or weakly commuting property then (f, g) also commutes at their point of coincidence. Now, in view of Theorem 3.3, in all five cases, f and g have a unique common fixed point. This completes the proof. □

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