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On intuitionistic fuzzy compact linear operators

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Abstract. The aim of this paper is to introduce the concept of intuitionistic fuzzy compact linear operators from one intuitionistic fuzzy n-normed linear space to another. Some interesting properties of intuitionistic fuzzy compact linear operators are also established.

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1. Introduction

Motivated by the theory of n-normed linear space [7, 8, 9, 10, 11, 12] and fuzzy normed linear space [1, 2, 3, 4, 5, 6] the notions of fuzzy n-normed linear space [13] and intuitionistic fuzzy n-normed linear space [14] have been developed. Also in [14] various types of continuities of operators and boundedness of linear operators over intuitionistic fuzzy n-normed linear spaces have been discussed.

In this paper we introduce the notion of intuitionistic fuzzy compact operators between intuitionistic fuzzy n-normed linear spaces and prove some results relating to these operators.

2. Preliminaries

In this section we recall some useful definitions and results.

Definition 2.1 ([13]). Let X be a linear space over a field $\mathbb{F}$. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if

(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \cdots, x_n, t) = 0$,
(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \cdots, x_n, t) = 1$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent,
(N3) $N(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$,

(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \cdots, cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in \mathbb{F}$,

(N5) For all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \cdots, x_n + x_n{}', s + t) \geq \min\{\mathrm{N}(x_1, x_2, \cdots, x_n, s), \mathrm{N}(x_1, x_2, \cdots, x_n{}', t)\},$$

(N6) $N(x_1, x_2, \cdots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} N(x_1, x_2, \cdots, x_n, t) = 1.$$

The pair (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Theorem 2.2 ([13]). *Let (X, N) be a f-n-NLS. Assume that*

(N7) $N(x_1, x_2, \cdots, x_n, t) > 0$ *for all* $t > 0$ *implies* $x_1, x_2, \cdots, x_n$ *are linearly dependent.*

Define

$$||x_1, x_2, \cdots, x_n||_\alpha = \inf\{t : N(x_1, x_2, \cdots, x_n, t) \geq \alpha\}, \ \ \alpha \in (0, 1).$$

Then $\{||\bullet, \cdots, \bullet||_\alpha : \alpha \in (0, 1)\}$ *is an ascending family of n-norms on X. We call these n-norms as α-n-norms on X corresponding to the fuzzy n-norm on X.*

Definition 2.3 ([14]). An intuitionistic fuzzy n-normed linear space or in short i-f-n-NLS is an object of the form

$$A = \{(X, N(x_1, x_2, \cdots, x_n, t), M(x_1, x_2, \cdots, x_n, t)) \mid (x_1, x_2, \cdots, x_n) \in X^n\}$$

where X is a linear space over a field $\mathbb{F}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$; N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, \cdots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions:

(1) $N(x_1, x_2, \cdots, x_n, t) + M(x_1, x_2, \cdots, x_n, t) \leq 1$
(2) $N(x_1, x_2, \cdots, x_n, t) > 0$
(3) $N(x_1, x_2, \cdots, x_n, t) = 1$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent
(4) $N(x_1, x_2, \cdots, x_n, t)$is invariant under any permutation of $x_1, x_2, \cdots, x_n$
(5) $N(x_1, x_2, \cdots, cx_n, t) = N(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$ if $c \neq 0$,$c \in \mathbb{F}$
(6) $N(x_1, x_2, \cdots, x_n, s) * N(x_1, x_2, \cdots, x_n', t) \leq N(x_1, x_2, \cdots, x_n + x_n', s + t)$
(7) $N(x_1, x_2, \cdots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t
(8) $M(x_1, x_2, \cdots, x_n, t) > 0$
(9) $M(x_1, x_2, \cdots, x_n, t) = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent
(10) $M(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$
(11) $M(x_1, x_2, \cdots, cx_n, t) = M(x_1, x_2, \cdots, x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in \mathbb{F}$
(12) $M(x_1, x_2, \cdots, x_n, s) \diamond M(x_1, x_2, \cdots, x_n', t) \geq M(x_1, x_2, \cdots, x_n + x_n', s + t)$
(13) $M(x_1, x_2, \cdots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

Definition 2.4 ([14]). Let A be an i-f-n-NLS. Assume that

(14) For all $t > 0$, $N(x_1, x_2, \cdots, x_n, t) > 0$ implies $x_1, x_2, \cdots, x_n$ are linearly dependent.

Define $||x_1, x_2, \cdots, x_n||_\alpha = \inf\{t : N(x_1, x_2, \cdots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$. Then

$$\{||\bullet, \cdots, \bullet||_\alpha : \alpha \in (0, 1)\}$$

is an ascending family of n-norms on X. We call these n-norms as α-n-norms on X corresponding to the i-f-n-NLS A. Further we assume that

(15) For $x_1, x_2, \cdots, x_n$ linearly independent, $N(x_1, x_2, \cdots, x_n, \centerdot)$ is a continuous function on $\mathbb{R}$ and strictly increasing on the subset

$$\{t : 0 < N(x_1, x_2, \cdots, x_n, t) < 1\}$$

of $\mathbb{R}$.

Definition 2.5 ([14])**.** A sequence $\{x_n\}$ in an i-f-n-NLS A is said to converge to x if given $r \in (0, 1), t > 0$ there exists an integer $n_0 \in \mathbb{N}$ such that

$$N(x_1, x_2, \cdots, x_n - x, t) > 1 - r \text{ and } M(x_1, x_2, \cdots, x_n - x, t) < r$$

for all $n \geq n_0$.

Definition 2.6 ([14])**.** Let A and B be two i-f-n-NLS. A mapping $T : A \to B$ is said to be intuitionistic fuzzy continuous at $z \in X$ if for given $\epsilon > 0$, $\alpha \in (0, 1)$, there exist $\delta = \delta(\alpha, \epsilon) > 0$ and $\beta = \beta(\alpha, \epsilon) > 0$ such that for all $x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$,

$$\begin{aligned} &N_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta) > \beta \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta) < 1 - \beta \\ &\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) > \alpha \text{ and} \\ &\qquad\qquad M_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) < 1 - \alpha. \end{aligned}$$

If T is intuitionistic fuzzy continuous at each point of X, then T is intuitionistic fuzzy continuous on X^n.

Definition 2.7 ([14])**.** Let A and B be two i-f-n-NLS. A mapping $T : A \to B$ is said to be strongly intuitionistic fuzzy continuous at $z \in X$ if for given $\epsilon > 0,\ \exists\ \delta > 0$ such that $\forall\ x_1, x_2, \cdots, x_{n-1}, x \in X, y_1, y_2, \cdots, y_{n-1} \in Y$,

$$N_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta)$$

and

$$M_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) \leq M_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta).$$

If T is strongly intuitionistic fuzzy continuous at each point of X, then T is said to be strongly intuitionistic fuzzy continuous on X^n.

Definition 2.8 ([14])**.** Let A and B be two i-f-n-NLS. A mapping $T : A \to B$ is said to be weakly intuitionistic fuzzy continuous at $z \in X$, if for given $\epsilon > 0$, $\alpha \in (0, 1)$, there exists $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$,

$$\begin{aligned} &N_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x - z, \delta) \leq 1 - \alpha \\ &\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) \geq \alpha \text{ and} \\ &\qquad\qquad M_2(y_1, y_2, \cdots, y_{n-1}, Tx - Tz, \epsilon) \leq 1 - \alpha. \end{aligned}$$

If T is weakly intuitionistic fuzzy continuous at each point of X, then T is weakly intuitionistic fuzzy continuous on X^n.

Definition 2.9 ([14]). Let A and B be two i-f-n-NLS. A mapping $T : A \to B$ is said to be sequentially intuitionistic fuzzy continuous at $z \in X$, if for any sequence $\{x_k\}$ in X with $x_k \to z$ implies $Tx_k \to Tz, k \in \mathbb{N}$. In other words, for all $t > 0$,

$$\lim_{k\to\infty} N_1(x_1, x_2, \cdots, x_{n-1}, x_k - z, t) = 1 \text{ and } \lim_{k\to\infty} M_1(x_1, x_2, \cdots, x_{n-1}, x_k - z, t) = 0$$
$$\Rightarrow \lim_{k\to\infty} N_2(y_1, y_2, \cdots, y_{n-1}, Tx_k - Tz, t) = 1 \text{ and}$$
$$\lim_{k\to\infty} M_2(y_1, y_2, \cdots, y_{n-1}, Tx_k - Tz, t) = 0,$$

for all $x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$. If T is sequentially intuitionistic fuzzy continuous at each point of X, then T is said to be sequentially intuitionistic fuzzy continuous on X^n.

Theorem 2.10 ([14]). *Let A and B be two i-f-n-NLS and $T : A \to B$ be a mapping. If T is strongly intuitionistic fuzzy continuous then it is sequentially intuitionistic fuzzy continuous.*

3. Intuitionistic fuzzy bounded linear operators

Definition 3.1 ([14]). Let A and B be two i-f-n-NLS and $T : A \to B$ be a linear operator. T is said to be strongly intuitionistic fuzzy bounded on X^n if and only if $\exists$ a positive number L such that $\forall\ x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$ and $s > 0$, $N_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \frac{s}{L})$ and

$$M_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \leq M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{s}{L}).$$

Let us denote the set of all strongly intuitionistic fuzzy bounded linear operators from A to B by $F(A, B)$.

Theorem 3.2. *$F(A, B)$ is a linear space.*

Proof. For $T_1, T_2 \in F(A, B)$ and $x \in X$, we have

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \text{ and } (\lambda T_1)(x) = \lambda T_1(x)$$

for any scalar λ. Since T_1 and T_2 are strongly intuitionistic fuzzy bounded, there exist positive numbers L_1 and L_2 such that

$$\begin{aligned}
&N_2(y_1, y_2, \cdots, y_{n-1}, T_1x, t) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_1})\\
&M_2(y_1, y_2, \cdots, y_{n-1}, T_1x, t) \leq M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_1})\\
&N_2(y_1, y_2, \cdots, y_{n-1}, T_2x, t) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_2})\\
&M_2(y_1, y_2, \cdots, y_{n-1}, T_2x, t) \leq M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_2})
\end{aligned}$$

for all $x_1, x_2, \cdots, x_{n-1},\ x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$ and for every $t \in \mathbb{R}$. Now for any two scalars α, β and for all $x \in X$ we have

$$\begin{aligned}
&N_2(y_1, y_2, \cdots, y_{n-1}, (\alpha T_1 + \beta T_2)(x), t)\\
&= N_2(y_1, y_2, \cdots, y_{n-1}, \alpha T_1(x) + \beta T_2(x), t)\\
&\geq \min\{N_2(y_1, y_2, \cdots, y_{n-1}, T_1(\alpha x), \tfrac{t}{2}), N_2(y_1, y_2, \cdots, y_{n-1}, T_2(\beta x), \tfrac{t}{2})\} \qquad (3.1)\\
&\geq \min\{N_1(x_1, x_2, \cdots, x_{n-1}, \alpha x, \tfrac{t}{2L_1}), N_1(x_1, x_2, \cdots, x_{n-1}, \beta x, \tfrac{t}{2L_2})\}\\
&\geq \min\{N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\alpha|L_1}), N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\beta|L_2})\}
\end{aligned}$$

Choose $L = \max\{2|\alpha|L_1, 2|\beta|L_2\} + 1$. Thus $L \geq 2|\alpha|L_1$ and $L \geq 2|\beta|L_2$ and this shows that $\frac{t}{2|\alpha|L_1} \geq \frac{t}{L}$ and $\frac{t}{2|\beta|L_2} \geq \frac{t}{L}$ $\forall\, t \geq 0$. Hence

$$N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\alpha|L_1}) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L})$$

and

$$N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\beta|L_2}) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L})$$

which implies

$$\min\{N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\alpha|L_1}), N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{2|\beta|L_2})\}$$
$$\geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L}).$$

Now from (3.1) we get

$$N_2(y_1, y_2, \cdots, y_{n-1}, (\alpha T_1 + \beta T_2)(x), t) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L}) \;\forall t \geq 0.$$

If $t < 0$, the relation is obvious. Thus $\exists\, L > 0$ such that

$$N_2(y_1, y_2, \cdots, y_{n-1}, (\alpha T_1 + \beta T_2)(x), t) \geq N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L})$$

for all $x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$ and for every $t \in \mathbb{R}$. Similarly we can prove that

$$M_2(y_1, y_2, \cdots, y_{n-1}, (\alpha T_1 + \beta T_2)(x), t) \leq M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L}) \;\forall\, t \in \mathbb{R}.$$

This implies that $\alpha T_1 + \beta T_2 \in F(A, B)$. Hence $F(A, B)$ is a linear space. □

Definition 3.3 ([14])**.** Let A and B be two i-f-n-NLS and $T : A \to B$ be a linear operator. T is said to be weakly intuitionistic fuzzy bounded on X^n if for any $\alpha \in (0, 1)$, $\exists\, L_\alpha > 0$ such that $\forall\, x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$ and $t > 0$,

$$N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \leq 1 - \alpha$$
$$\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1 - \alpha.$$

Let us denote the set of all weakly intuitionistic fuzzy bounded linear operators from A to B by $F^{'}(A, B)$.

Remark 3.4. $F^{'}(A, B)$ is a linear space.

Theorem 3.5 ([14])**.** *Let A and B be two i-f-n-NLS and $T : A \to B$ be a linear operator. If T is strongly intuitionistic fuzzy bounded then it is weakly intuitionistic fuzzy bounded, but not conversely.*

The following example establishes that the converse does not hold.

Example 3.6. Let $(X, ||\bullet, \cdots, \bullet||)$ be an n-normed linear space. Define

$$a * b = \min\{a, b\} \text{ and } a \diamond b = \max\{a, b\} \text{ for all } a, b \in [0, 1],$$

$$N_1(x_1, x_2, \cdots, x_n, t) = \frac{t^2 - k||x_1, x_2, \cdots, x_n||^2}{t^2 + k||x_1, x_2, \cdots, x_n||^2},\ M_1(x_1, x_2, \cdots, x_n, t) = \frac{2k||x_1, x_2, \cdots, x_n||^2}{t^2 + k||x_1, x_2, \cdots, x_n||^2}$$

if $t > ||x_1, x_2, \cdots, x_n||$ and $N_1 = M_1 = 0$ if $t \leq ||x_1, x_2, \cdots, x_n||$,

$$N_2(x_1, x_2, \cdots, x_n, t) = \frac{t}{t + k||x_1, x_2, \cdots, x_n||},\ M_2(x_1, x_2, \cdots, x_n, t) = \frac{k||x_1, x_2, \cdots, x_n||}{t + k||x_1, x_2, \cdots, x_n||}$$

if $t > 0$ and N_1, N_2, M_1 and M_2 are defined to be zero for $t \leq 0$. (X, N_1) and (X, N_2) are fuzzy n-normed linear spaces. Now we define a linear operator

$$T : (X, N_1) \to (X, N_2)$$

by $T(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)\ \forall\, x_1, x_2, \cdots, x_n \in X$. We choose

$$L_\alpha = \tfrac{1}{1-\alpha}\ \ \forall \alpha \in (0, 1).$$

Then for $t > ||x_1, x_2, \cdots, x_n||$,

$$N_1(x_1, x_2, \cdots, x_n, \tfrac{t}{L_\alpha}) \geq \alpha$$
$$\Rightarrow \tfrac{t^2(1-\alpha)^2 - k||x_1,x_2,\cdots,x_n||^2}{t^2(1-\alpha)^2 + k||x_1,x_2,\cdots,x_n||^2} \geq \alpha$$
$$\Rightarrow t^2(1-\alpha)^2 - k||x_1, x_2, \cdots, x_n||^2 \geq t^2\alpha(1-\alpha)^2 + k\alpha||x_1, x_2, \cdots, x_n||^2$$
$$\Rightarrow t + k||x_1, x_2, \cdots, x_n|| \leq t\left[\tfrac{\sqrt{1+\alpha}+\sqrt{k}(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}}\right]$$
$$\Rightarrow \tfrac{t}{t+k||x_1,x_2,\cdots,x_n||} \geq \tfrac{\sqrt{1+\alpha}}{\sqrt{1+\alpha}+\sqrt{k}(1-\alpha)\sqrt{1-\alpha}}. \tag{3.2}$$

Now

$$\tfrac{\sqrt{1+\alpha}}{\sqrt{1+\alpha}+\sqrt{k}(1-\alpha)\sqrt{1-\alpha}} \geq \alpha$$
$$\Leftrightarrow \sqrt{1+\alpha} \geq \alpha\sqrt{1+\alpha} + \sqrt{k}\alpha(1-\alpha)\sqrt{1-\alpha}$$
$$\Leftrightarrow 1 + \alpha + k\alpha^3 \geq k\alpha^2.$$

This is true for all $\alpha \in (0, 1)$. Thus from (3.2) we get

$$N_2(T(x_1, x_2, \cdots, x_n), t) \geq \alpha \ \text{ if } \ t > ||x_1, x_2, \cdots, x_n||.$$

Again since for $t \leq ||x_1, x_2, \cdots, x_n||$, $\frac{t^2 - k||x_1,x_2,\cdots,x_n||^2}{t^2 + k||x_1,x_2,\cdots,x_n||^2} = 0$, it follows that

$$N_1(x_1, x_2, \cdots, x_n, \tfrac{t}{L_\alpha}) \geq \alpha \Rightarrow N_2(T(x_1, x_2, \cdots, x_n), t) \geq \alpha\ \forall\ \alpha \in (0, 1).$$

Thus in any case we get

$$N_1(x_1, x_2, \cdots, x_n, \tfrac{t}{L_\alpha}) \geq \alpha \Rightarrow N_2(T(x_1, x_2, \cdots, x_n), t) \geq \alpha\ \forall\ \alpha \in (0, 1).$$

Similarly, $M_1(x_1, x_2, \cdots, x_n, \frac{t}{L_\alpha}) \leq 1 - \alpha \Rightarrow M_2(T(x_1, x_2, \cdots, x_n), t) \leq 1 - \alpha$. Hence T is weakly intuitionistic fuzzy bounded. Now for $t > ||x_1, x_2, \cdots, x_n||$, $N_2(T(x_1, x_2, \cdots, x_n), t) \geq N_1(x_1, x_2, \cdots, x_n, \frac{t}{L_\alpha})$ and $M_2(T(x_1, x_2, \cdots, x_n), t) \leq M_1(x_1, x_2, \cdots, x_n, \frac{t}{L_\alpha})$

$$\Leftrightarrow \tfrac{t}{t+k||x_1,x_2,\cdots,x_n||} \geq \tfrac{t^2 - L^2k||x_1,x_2,\cdots,x_n||^2}{t^2 + L^2k||x_1,x_2,\cdots,x_n||^2}$$
$$\Leftrightarrow L \geq \tfrac{t}{\sqrt{2t||x_1,x_2,\cdots,x_n|| + k||x_1,x_2,\cdots,x_n||^2}}$$
$$\Leftrightarrow L \to \infty \text{ as } t \to \infty.$$

Hence T is not strongly intuitionistic fuzzy bounded.

Theorem 3.7 ([14])**.** *Let A and B be two i-f-n-NLS and $T : A \to B$ be a linear operator. Then T is strongly intuitionistic fuzzy continuous if and only if T is strongly intuitionistic fuzzy bounded.*

Theorem 3.8 ([14])**.** *Let A and B be two i-f-n-NLS and $T : A \to B$ be a linear operator. Then T is weakly intuitionistic fuzzy continuous if and only if T is weakly intuitionistic fuzzy bounded.*

Theorem 3.9. *Let A and B be i-f-n-NLS satisfying* (14) *and* (15)*. Let $T : A \to B$ be a linear operator. Then T is weakly intuitionistic fuzzy bounded if and only if T is bounded with respect to α-n-norms of N_1 and N_2, $\alpha \in (0,1)$.*

Proof. First we suppose that T is weakly intuitionistic fuzzy bounded. Thus $\forall\, \alpha \in (0,1)$, $\exists\, L_\alpha > 0$ such that $\forall\, x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$ and $t > 0$ we have

$$N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \leq 1-\alpha$$
$$\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1-\alpha,$$

i.e.,

$$N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \leq 1-\alpha$$
$$\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1-\alpha,$$

i.e.,

$$\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} \geq \alpha$$
$$\Rightarrow \sup\{\beta \in (0,1) : ||y_1, y_2, \cdots, y_{n-1}, Tx||^2_\beta \leq t\} \geq \alpha.$$

Now we show that

$$\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} \geq \alpha$$
$$\Leftrightarrow ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha \leq t.$$

If $x = 0$ then the relation is obvious. Suppose $x \neq 0$. If

$$\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} > \alpha$$

then

$$||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha \leq t. \tag{3.3}$$

If $\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} = \alpha$, then there exists an increasing sequence $\{\alpha_n\}$ in $(0,1)$ such that $\alpha_n \uparrow \alpha$ and $||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_{\alpha_n} \leq t$. Then we have

$$||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha \leq t. \tag{3.4}$$

Thus from (3.3) and (3.4) we get

$$\begin{aligned} &\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} \geq \alpha \\ &\Rightarrow ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha \leq t. \end{aligned} \tag{3.5}$$

Next suppose that $||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha \leq t$. If $||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\alpha < t$ then

$$N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \leq 1-\alpha,$$

i.e.,

$$\sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||^1_\beta \leq t\} \geq \alpha. \tag{3.6}$$

If $||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\alpha^1 = t$, i.e., $\inf\{s : N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \geq \alpha$ and $M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \leq 1 - \alpha\} = t$, then $\exists$ a decreasing sequence $\{s_k\}$ in $\mathbb{R}$ such that $s_k \downarrow t$ and

$$\begin{aligned}
& N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s_k) \geq \alpha \text{ and} \\
& \qquad M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s_k) \leq 1 - \alpha \\
& \Rightarrow \lim_{k\to\infty} N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s_k) \geq \alpha \text{ and} \\
& \qquad \lim_{k\to\infty} M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s_k) \leq 1 - \alpha \\
& \Rightarrow N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, \lim_{k\to\infty} s_k) \geq \alpha \text{ and} \\
& \qquad M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, \lim_{k\to\infty} s_k) \leq 1 - \alpha \\
& \Rightarrow N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \geq \alpha \text{ and} \\
& \qquad M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \leq 1 - \alpha \\
& \Rightarrow \sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\beta^1 \leq t\} \geq \alpha.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\begin{aligned}
& ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\alpha^1 \leq t \\
& \Rightarrow \sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\beta^1 \leq t\} \geq \alpha.
\end{aligned} \tag{3.8}$$

From (3.5)and (3.8) we have

$$\begin{aligned}
& \sup\{\beta \in (0,1) : ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\beta^1 \leq t\} \geq \alpha \\
& \Leftrightarrow ||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\alpha^1 \leq t.
\end{aligned} \tag{3.9}$$

In a similar way we can show that

$$\begin{aligned}
& \sup\{\beta \in (0,1) : ||y_1, y_2, \cdots, y_{n-1}, Tx||_\beta^2 \leq t\} \geq \alpha \\
& \Leftrightarrow ||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq t.
\end{aligned} \tag{3.10}$$

Therefore from (3.9) and (3.10) we have

$$\begin{aligned}
& N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \geq \alpha \text{ and} \\
& \qquad M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \leq 1 - \alpha \\
& \Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and} \\
& \qquad M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1 - \alpha.
\end{aligned}$$

Then $||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\alpha^1 \leq t \Rightarrow ||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq t$. This implies that $||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq L_\alpha ||x_1, x_2, \cdots, x_{n-1}, x||_\alpha^1 \; \forall \, \alpha \in (0,1)$.

Conversely suppose that $\forall \alpha \in (0,1)$, $\exists \, L_\alpha > 0$ such that

$$||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq L_\alpha ||x_1, x_2, \cdots, x_{n-1}, x||_\alpha^1$$

for every $x_1, x_2, \cdots, x_{n-1}, x \in X$, $y_1, y_2, \cdots, y_{n-1} \in Y$. Then for $x \neq 0$,

$$||x_1, x_2, \cdots, x_{n-1}, L_\alpha x||_\alpha^1 \leq t \Rightarrow ||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq t \; \forall \, t > 0,$$

i.e.,

$$\inf\{s : N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \leq 1 - \alpha\} \leq t$$
$$\Rightarrow \inf\{s : N_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \leq 1 - \alpha\} \leq t.$$

In a similar way as above we can show that

$$\inf\{s : N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, s) \leq 1 - \alpha\} \leq t$$
$$\Leftrightarrow N_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, L_\alpha x, t) \leq 1 - \alpha,$$

and

$$\inf\{s : N_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, s) \leq 1 - \alpha\} \leq t$$
$$\Leftrightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1 - \alpha.$$

Thus we have

$$N_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \geq \alpha \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x, \tfrac{t}{L_\alpha}) \leq 1 - \alpha$$
$$\Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \geq \alpha \text{ and } M_2(y_1, y_2, \cdots, y_{n-1}, Tx, t) \leq 1 - \alpha$$

for all $x_1, x_2, \cdots, x_{n-1}, x \in X,\ y_1, y_2, \cdots, y_{n-1} \in Y$. If $x \neq 0, t \leq 0$ and $x = 0, t > 0$ then the above relation is obvious. Hence the theorem follows. □

Theorem 3.10. *Let A and B be i-f-n-NLS satisfying* (14) *and* (15) *and $T : A \to B$ be a linear operator. If X is of finite dimension then T is weakly intuitionistic fuzzy bounded.*

Proof. Since A and B satisfy (14), we may suppose that $||\bullet, \cdots, \bullet||^1_\alpha$ and $||\bullet, \cdots, \bullet||^2_\alpha$ are the α-n-norms of N_1 and N_2 respectively. Since X is of finite dimension, $T : A \to B$ is a bounded linear operator for each $\alpha \in (0, 1)$. Thus by Theorem 3.9, it follows that T is weakly intuitionistic fuzzy bounded. □

4. Intuitionistic fuzzy compact linear operators

Definition 4.1. Let A be an i-f-n-NLS. A subset D of X is said to be the closure of $K \subset X$ if for any $x \in D$, there exists a sequence $\{x_k\}$ in K such that $\lim\limits_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 1$ and $\lim\limits_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 0$ for all $t > 0$.We denote the set D by $\overline{K}$.

Definition 4.2. Let A be an i-f-n-NLS. A subset D of X is said to be intuitionistic fuzzy bounded if and if there exists $t > 0$ and $0 < r < 1$ such that $N(x_1, x_2, \cdots, x_n, t) \geq 1 - r$ and $M(x_1, x_2, \cdots, x_n, t) \leq r$ for all $x_1, x_2, \cdots, x_n \in D$.

Definition 4.3. Let A be an i-f-n-NLS. A subset D of X is said to be compact if every sequence $\{x_k\}$ in D has a subsequence converging to an element of D.

Definition 4.4. Let A and B be i-f-n-NLS. A linear operator $T : A \to B$ is called intuitionistic fuzzy compact if for every bounded subset K of X the subset $T(K) \subset Y$ is relatively compact. i.e. the intuitionistic fuzzy closure of $T(K)$ is an intuitionistic fuzzy compact set.

Example 4.5. Let $(X, ||\bullet, \cdots, \bullet||_1)$ and $(Y, ||\bullet, \cdots, \bullet||_2)$ be two ordinary n-normed linear spaces and $T : A \to B$ be a compact operator. Then $T : A \to B$ is an intuitionistic fuzzy compact operator, where N_1 and N_2 are standard fuzzy norms induced by ordinary norms $||\bullet, \cdots, \bullet||_1$ and $||\bullet, \cdots, \bullet||_2$ respectively. i.e. If $t > 0, t \in \mathbb{R}$,

$$N_1(x_1, x_2, \cdots, x_n, t) = \frac{t}{t+||x_1, x_2, \cdots, x_n||_1},$$
$$M_1(x_1, x_2, \cdots, x_n, t) = \frac{||x_1, x_2, \cdots, x_n||_1}{t+||x_1, x_2, \cdots, x_n||_1},$$
$$N_2(y_1, y_2, \cdots, y_n, t) = \frac{t}{t+||y_1, y_2, \cdots, y_n||_2},$$
$$M_2(y_1, y_2, \cdots, y_n, t) = \frac{||y_1, y_2, \cdots, y_n||_2}{t+||y_1, y_2, \cdots, y_n||_2},$$

and N_1, N_2, M_1 and M_2 are defined to be zero for $t \leq 0$.

Theorem 4.6. *Let A and B be i-f-n-NLS and $T : A \to B$ be a linear operator. Then T is intuitionistic fuzzy compact if and only if it maps every bounded sequence $\{x_k\}$ in X onto a sequence $\{Tx_k\}$ in Y which has an intuitionistic fuzzy convergent subsequence.*

Proof. Suppose that T be an intuitionistic fuzzy compact operator and $\{x_k\}$ be an intuitionistic fuzzy bounded sequence in X. The intuitionistic fuzzy closure of $\{Tx_k : k \in \mathbb{N}\}$ is an intuitionistic fuzzy compact set. So $\{Tx_k\}$ has an intuitionistic fuzzy convergent subsequence by definition.

Conversely, let D be an intuitionistic fuzzy bounded subset of X. We show that the intuitionistic fuzzy closure of $T(D)$ is intuitionistic fuzzy compact. Let $\{Tx_k\}$ be a sequence in the closure of $T(D)$. For given $\epsilon > 0, k \in \mathbb{N}$ and $t > 0$, $\exists \{Tz_k\}$ in $T(D)$ such that $N_2(y_1, y_2, \cdots, y_{n-1}, Tx_k - Tz_k, \frac{t}{2}) \geq 1 - \epsilon$ and

$$M_2(y_1, y_2, \cdots, y_{n-1}, Tx_k - Tz_k, \tfrac{t}{2}) \leq \epsilon.$$

$\{Tz_k\}$ has an intuitionistic fuzzy convergent subsequence $\{Tz_{k_i}\}$. Let $\{Tz_{k_i} \to y\}$ for some $y \in Y$. Hence

$$N_2(y_1, y_2, \cdots, y_{n-1}, Tz_{k_i} - y, \tfrac{t}{2}) \geq 1 - \epsilon$$

and

$$M_2(y_1, y_2, \cdots, y_{n-1}, Tz_{k_i} - y, \tfrac{t}{2}) \leq \epsilon$$

for all $k_i > n_0$. Now,

$$\begin{aligned} &N_2(y_1, y_2, \cdots, y_{n-1}, Tx_{k_i} - y, t) \\ &\geq \min\{N_2(y_1, y_2, \cdots, y_{n-1}, Tx_{k_i} - Tz_{k_i}, \tfrac{t}{2}), N_2(y_1, y_2, \cdots, y_{n-1}, Tz_{k_i} - y, \tfrac{t}{2})\} \\ &\geq 1 - \epsilon, \end{aligned}$$

$$\begin{aligned} &M_2(y_1, y_2, \cdots, y_{n-1}, Tx_{k_i} - y, t) \\ &\leq \max\{M_2(y_1, y_2, \cdots, y_{n-1}, Tx_{k_i} - Tz_{k_i}, \tfrac{t}{2}), M_2(y_1, y_2, \cdots, y_{n-1}, Tz_{k_i} - y, \tfrac{t}{2})\} \\ &\leq \epsilon \end{aligned}$$

for all $k_i > n_0$. Hence $\{Tx_{k_i}\}$ is an intuitionistic fuzzy convergent subsequence of $\{Tx_k\}$. Thus the intuitionistic fuzzy closure of $T(D)$ is a fuzzy compact set. □

Lemma 4.7. *Let A be an i-f-n-NLS satisfying* (14) *and $\{x_k\}$ be a sequence in X. Then* $\lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 1$ *and*

$$\lim_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 0\, \forall\, t > 0$$
$$\Leftrightarrow \lim_{k\to\infty} ||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha = 0\, \forall \alpha \in (0,1)$$

where $||\bullet, \cdots, \bullet||_\alpha$ *denotes the corresponding α-n-norm* $(0 < \alpha < 1)$ *of N.*

Proof. Let $\{x_k\}$ be a sequence in X such that $x_k \to x$ (w.r.t N). Thus

$$\lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 1$$

and

$$\lim_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) = 0$$

for all $t > 0$. Choose $0 < \alpha < 1$. Then $\exists$ a positive integer $k_0(\alpha, t)$ such that

$$\begin{aligned} &N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) \geq \alpha \text{ and} \\ &M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) \leq 1 - \alpha \end{aligned} \tag{4.1}$$

for all $k \geq k_0(\alpha, t)$. Now we have

$$\begin{aligned} &||x_1, x_2, \cdots, x_{n-1}, x_k||_\alpha \\ &= \inf\{t > 0 : N(x_1, x_2, \cdots, x_{n-1}, x_k, t) \geq \alpha \text{ and} \\ &\qquad\qquad M(x_1, x_2, \cdots, x_{n-1}, x_k, t) \leq 1 - \alpha\}. \end{aligned} \tag{4.2}$$

Thus from (4.1) it follows that $||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha \leq t\, \forall\, k \geq k_0(\alpha, t)$. Since $t > 0$ is arbitrary it follows that

$$||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha \to 0 \text{ as } k \to \infty\, \forall\, \alpha \in (0,1).$$

Conversely, suppose that, $||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha \to 0$ as $k \to \infty\, \forall\, \alpha \in (0,1)$. Then for $\alpha \in (0,1)$ and $\epsilon > 0, \exists$ a positive integer $k_0(\alpha, \epsilon)$ such that

$$||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha < \epsilon\, \forall\, k \geq k_0(\alpha, \epsilon).$$

From (4.2) we have

$$\begin{aligned}
\epsilon &> ||x_1, x_2, \cdots, x_{n-1}, x_k - x||_\alpha \\
&= \inf\{t > 0 : N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) \geq \alpha \text{ and} \\
&\qquad M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t) \leq 1 - \alpha\} \ \forall\, k \geq k_0(\alpha, \epsilon) \\
&\Rightarrow \exists\, t_n > 0, \epsilon > t_n > 0 \text{ such that } N(x_1, x_2, \cdots, x_{n-1}, x_k - x, t_n) \geq \alpha \\
&\qquad \text{and } M(x_1, x_2, \cdots, x_{n-1}, x_k - x, t_n) \leq 1 - \alpha \ \forall\, k \geq k_0(\alpha, \epsilon) \\
&\Rightarrow N(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) \geq \alpha \text{ and} \\
&\qquad M(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) \leq 1 - \alpha \ \forall\, k \geq k_0(\alpha, \epsilon) \\
&\Rightarrow \lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) \geq \alpha \text{ and} \\
&\qquad \lim_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) \leq 1 - \alpha \\
&\Rightarrow \lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) = 1 \text{ and} \\
&\qquad \lim_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x_k - x, \epsilon) = 0 \\
&\qquad\qquad (\text{since } \alpha \in (0,1) \text{ is arbitrary}) \\
&\Rightarrow x_k \to x \ (\text{w.r.t } N).
\end{aligned}$$

This completes the proof. □

Definition 4.8. Let A be an i-f-n-NLS. We define the following subset of X.

$$B(x, \alpha, t) = \{y : N(x_1, x_2, \cdots, x_{n-1}, x - y, t) \geq \alpha \text{ and}$$
$$M(x_1, x_2, \cdots, x_{n-1}, x - y, t) \leq 1 - \alpha\}$$

where $x_1, x_2, \cdots, x_{n-1}, x \in X, \alpha \in (0, 1)$.

Lemma 4.9. *Let* $A(x, \alpha, t) = \{y \in X : ||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha \leq t\}$ *where* $\alpha \in (0, 1)$. *Then* $B(x, \alpha, t) = A(x, \alpha, t)$.

Proof. If $y \in B(x, \alpha, t)$ then
$N(x_1, x_2, \cdots, x_{n-1}, x - y, t) \geq \alpha$ and $M(x_1, x_2, \cdots, x_{n-1}, x - y, t) \leq 1 - \alpha$

$$\begin{aligned}
&\Rightarrow \inf\{t : N(x_1, x_2, \cdots, x_{n-1}, x - y, t) \geq \alpha \text{ and} \\
&\qquad M(x_1, x_2, \cdots, x_{n-1}, x - y, t) \leq 1 - \alpha\} \\
&\Rightarrow ||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha \leq t \\
&\Rightarrow y \in A(x, \alpha, t).
\end{aligned}$$

Now if $y \in A(x, \alpha, t)$ then $||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha \leq t$. If

$$||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha < t$$

then $N(x_1, x_2, \cdots, x_{n-1}, x - y, t) \geq \alpha$ and $M(x_1, x_2, \cdots, x_{n-1}, x - y, t) \leq 1 - \alpha$. If $||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha = t$, i.e.,

$$\inf\{s : N(x_1, x_2, \cdots, x_{n-1}, x - y, s) \geq \alpha \text{ and}$$
$$M(x_1, x_2, \cdots, x_{n-1}, x - y, s) \leq 1 - \alpha\} = t$$

then $\exists$ a decreasing sequence $\{s_k\}$ in $\mathbb{R}$ such that $s_k \downarrow t$ and
$N(x_1, x_2, \cdots, x_{n-1}, x - y, s_k) \geq \alpha$ and

$$M(x_1, x_2, \cdots, x_{n-1}, x - y, s_k) \leq 1 - \alpha$$

$$\Rightarrow \lim_{k\to\infty} N(x_1, x_2, \cdots, x_{n-1}, x - y, s_k) \geq \alpha \text{ and}$$

$$\lim_{k\to\infty} M(x_1, x_2, \cdots, x_{n-1}, x - y, s_k) \leq 1 - \alpha$$

$$\Rightarrow N(x_1, x_2, \cdots, x_{n-1}, x - y, \lim_{k\to\infty} s_k) \geq \alpha \text{ and}$$

$$M(x_1, x_2, \cdots, x_{n-1}, x - y, \lim_{k\to\infty} s_k) \leq 1 - \alpha$$

$$\Rightarrow N(x_1, x_2, \cdots, x_{n-1}, x - y, t) \geq \alpha \text{ and}$$

$$M(x_1, x_2, \cdots, x_{n-1}, x - y, t) \leq 1 - \alpha.$$

Thus in both the cases we find that $y \in B(x, \alpha, t)$. Therefore $B(x, \alpha, t) = A(x, \alpha, t)$. □

Theorem 4.10. *Let A be an i-f-n-NLS satisfying* (14) *and $N(x_1, x_2, \cdots, x_n, .)$ be a continuous function on R. Then X is a finite dimensional if and only if $B(x, \alpha, t)$ is an intuitionistic fuzzy compact set in X, for each $\alpha \in (0, 1)$.*

Proof. Let $A(x, \alpha, t) = \{y \in X : ||x_1, x_2, \cdots, x_{n-1}, x - y||_\alpha \leq t\}$ where $\alpha \in (0, 1)$. By Lemma 4.9 $B(x, \alpha, t) = A(x, \alpha, t)$. Now suppose that

$$\dim X < \infty, x_1, x_2, \cdots, x_{n-1}, x \in X \text{ and } t > 0.$$

Choose the sequence $\{x_k\}$ in $B(x, \alpha, t)$. Clearly, $A(x, \alpha, t)$ is a compact subset of $(X, ||\bullet, \cdots, \bullet||_\alpha)$. Hence $\exists$ a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $v \in A(x, \alpha, t)$ such that $x_{k_i} \to v$ (w.r.t $||\bullet, \cdots, \bullet||_\alpha$). Thus by Lemma 4.7 $x_{k_i} \to v$ (w.r.t N). Since $B(x, \alpha, t) = A(x, \alpha, t)$, we have $v \in B(x, \alpha, t)$. Thus $B(x, \alpha, t)$ is intuitionistic fuzzy compact.

Conversely, let $B(x, \alpha, t)$ be intuitionistic fuzzy compact. To show that X is finite dimensional, it suffices to prove that $A(x, \alpha, t)$ is compact w.r.t α-n-norm. Choose a sequence $\{x_k\}$ in $A(x, \alpha, t)$. Since $B(x, \alpha, t)$ is intuitionistic fuzzy compact, it has an intuitionistic fuzzy convergent subsequence $\{x_{k_i}\}$. Lemma 4.7 implies that $\{x_{k_i}\}$ is convergent under $||\bullet, \cdots, \bullet||_\alpha$. Thus $A(x, \alpha, t)$ is compact w.r.t α-n-norm which shows that X is finite dimensional. □

Lemma 4.11. *Let A and B be i-f-n-NLS satisfying* (14) *and* (15) *and $T : A \to B$ be an intuitionistic fuzzy compact operator. Then*

$$T : (X, ||\bullet, \cdots, \bullet||^1_\alpha) \to (Y, ||\bullet, \cdots, \bullet||^2_\alpha)$$

is an ordinary compact operator for all $\alpha \in (0, 1)$.

Proof. We shall show that for each bounded sequence $\{x_k\}$ in $(X, ||\bullet, \cdots, \bullet||^1_\alpha)$, the sequence $\{Tx_k\}$ has a convergent subsequence in $(Y, ||\bullet, \cdots, \bullet||^2_\alpha)$. Then there exists $p > 0$ such that $||x_1, x_2, \cdots, x_{n-1}, x_k||^1_\alpha < p$, $\forall\, k \in \mathbb{N}$. Hence

$$N(x_1, x_2, \cdots, x_{n-1}, x_k, p) \geq \alpha \text{ and } M(x_1, x_2, \cdots, x_{n-1}, x_k, p) \leq 1 - \alpha$$

for all k, i.e., $\{x_k\}$ is intuitionistic fuzzy bounded. Also $T : A \to B$ is an intuitionistic fuzzy compact operator. Hence $\{Tx_k\}$ has an intuitionistic fuzzy convergent subsequence $\{Tx_{k_i}\}$. By Lemma 4.7 $\{Tx_{k_i}\}$ is convergent under $||\bullet, \cdots, \bullet||^2_\alpha$. □

Theorem 4.12. *Let A and B be i-f-n-NLS satisfying* (14) *and* (15). *Then*

(a) *Every intuitionistic fuzzy compact operator $T : A \to B$ is weakly intuitionistic fuzzy continuous.*

(b) *If $\dim X = \infty$ then the identity operator $I : A \to A$ is not an intuitionistic fuzzy compact operator.*

Proof. (a) Choose $\alpha \in (0, 1)$. Let $||\bullet, \cdots, \bullet||_\alpha^1$ and $||\bullet, \cdots, \bullet||_\alpha^2$ be the α-n-norms on X and Y corresponding to the fuzzy n-norms N_1 and N_2 respectively. By Lemma 4.11, $T : (X, ||\bullet, \cdots, \bullet||_\alpha^1) \to (Y, ||\bullet, \cdots, \bullet||_\alpha^2)$ is a compact operator. Since compact operator is bounded $\exists\, L_\alpha > 0$ such that

$$||y_1, y_2, \cdots, y_{n-1}, Tx||_\alpha^2 \leq L_\alpha ||x_1, x_2, \cdots, x_{n-1}, x||_\alpha^1.$$

Hence by Theorem 3.9, T is weakly intuitionistic fuzzy bounded. Now Theorem 3.8 implies that T is weakly intuitionistic fuzzy continuous.

(b) The identity operator I maps $B(x, \alpha, t)$ to itself. Suppose on the contrary that I is an intuitionistic fuzzy compact operator. Then $\overline{B}(x, \alpha, t)$ is intuitionistic fuzzy compact for all $\alpha \in (0, 1)$. Now $\overline{B}(x, \alpha, t) \subset A(x, \alpha, t) = B(x, \alpha, t)$, implies that $B(x, \alpha, t)$ is closed and so intuitionistic fuzzy compact. Thus by Theorem 4.10, X is finite dimensional which is a contradiction. □

Theorem 4.13. *Let A and B be i-f-n-NLS. Then the set of all intuitionistic fuzzy compact operators from $A \to B$ is a linear subspace of $F^{'}(A, B)$.*

Proof. Suppose that T_1 and T_2 are intuitionistic fuzzy compact linear operators from A to B and $\{x_k\}$ be any intuitionistic fuzzy bounded sequence in X. Then the sequence $\{T_1x_k\}$ has an intuitionistic fuzzy convergent subsequence $\{T_1x_{k_i}\}$. The sequence $\{T_2x_k\}$ also has an intuitionistic fuzzy convergent subsequence $\{T_2x_{k_i}\}$. Hence $\{T_1x_{k_i}\}$ and $\{T_2x_{k_i}\}$ are intuitionistic fuzzy convergent sequences.
Let $T_1x_{k_i} \to y$ and $T_2x_{k_i} \to z$. If $t > 0$, we have

$$\lim_{k_i \to \infty} N_2(y_1, y_2, \cdots, y_{n-1}, (T_1 + T_2)x_{k_i} - y - z, t)$$
$$\geq \lim_{k_i \to \infty} \{\min N_2(y_1, y_2, \cdots, y_{n-1}, T_1x_{k_i} - y, \tfrac{t}{2}), N_2(y_1, y_2, \cdots, y_{n-1}, T_2x_{k_i} - z, \tfrac{t}{2})\}$$

and

$$\lim_{k_i \to \infty} M_2(y_1, y_2, \cdots, y_{n-1}, (T_1 + T_2)x_{k_i} - y - z, t)$$
$$\leq \lim_{k_i \to \infty} \{\max M_2(y_1, y_2, \cdots, y_{n-1}, T_1x_{k_i} - y, \tfrac{t}{2}), M_2(y_1, y_2, \cdots, y_{n-1}, T_2x_{k_i} - z, \tfrac{t}{2})\}$$

for all $y_1, y_2, \cdots, y_{n-1}, y, z \in Y$. Thus

$$\lim_{k_i \to \infty} N_2(y_1, y_2, \cdots, y_{n-1}, (T_1 + T_2)x_{k_i} - y - z, t) = 1$$

and

$$\lim_{k_i \to \infty} M_2(y_1, y_2, \cdots, y_{n-1}, (T_1 + T_2)x_{k_i} - y - z, t) = 0$$

for all $t > 0$. This implies that $T_1 + T_2$ is an intuitionistic fuzzy compact operator.
Now if $T_1x_{k_i} \to y$ then

$$\lim_{k_i \to \infty} N_2(y_1, y_2, \cdots, y_{n-1}, \alpha T_1x_{k_i} - \alpha y, t)$$
$$= \lim_{k_i \to \infty} N_2(y_1, y_2, \cdots, y_{n-1}, T_1x_{k_i} - y, \tfrac{t}{|\alpha|}) = 1,$$

$$\lim_{k_i\to\infty} M_2(y_1, y_2, \cdots, y_{n-1}, \alpha T_1 x_{k_i} - \alpha y, t)$$
$$= \lim_{k_i\to\infty} M_2(y_1, y_2, \cdots, y_{n-1}, T_1 x_{k_i} - y, \tfrac{t}{|\alpha|}) = 0$$

for all $\alpha \in R\backslash\{0\}$ and $t > 0$. Hence αT_1 is also an intuitionistic fuzzy compact operator which completes the proof. □

Theorem 4.14. *Let A be an i-f-n-NLS and $T : A \to A$ be an intuitionistic fuzzy compact linear operator and $S : A \to A$ be a strongly intuitionistic fuzzy continuous linear operator. Then ST and TS are intuitionistic fuzzy compact operators.*

Proof. Let $\{x_k\}$ be any intuitionistic fuzzy bounded sequence in X. Then the sequence $\{Tx_k\}$ has an intuitionistic fuzzy convergent subsequence $\{Tx_{k_i}\}$. Let $\{Tx_{k_i}\} \to y$ for some $y \in Y$. Since S is strongly intuitionistic fuzzy continuous, by Theorem 2.10 we have $STx_{k_i} \to Sy$. Hence $\{STx_k\}$ has an intuitionistic fuzzy convergent subsequence. This proves that ST is intuitionistic fuzzy compact. Next we show that TS is intuitionistic fuzzy compact. Choose any intuitionistic fuzzy bounded sequence $\{x_k\}$ in X. Then $\exists\, t_0 > 0$ and $r_0 \in (0, 1)$ such that

$$N_1(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \geq 1 - r_0$$

and

$$M_1(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \leq r_0$$

for all $k \geq 1$. By Theorem 3.7 S is strongly intuitionistic fuzzy bounded linear operator. Thus $\exists\, L_\alpha > 0$ such that $N_2(y_1, y_2, \cdots, y_{n-1}, Sx_k, t_0 L_\alpha) \geq 1 - r_0$ and

$$M_2(y_1, y_2, \cdots, y_{n-1}, Sx_k, t_0 L_\alpha) \leq r_0$$

for all k. It follows that $\{Sx_k\}$ is intuitionistic fuzzy bounded sequence in $S(X)$. Since T is intuitionistic fuzzy compact $\{TSx_k\}$ has an intuitionistic fuzzy convergent subsequence, which completes the proof. □

Lemma 4.15. *Let A be an i-f-n-NLS satisfying* (14) *and $N(x_1, x_2, \cdots, x_n, \centerdot)$ be a continuous function on R and* $\dim X < \infty$. *Then each intuitionistic fuzzy bounded sequence $\{x_k\}$ in X has an intuitionistic fuzzy convergent subsequence.*

Proof. Let $\{x_k\}$ be an intuitionistic fuzzy bounded sequence in X. $\exists\, t_0 > 0$ and $r_0 \in (0, 1)$ such that $N(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \geq r_0$ and

$$M(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \leq 1 - r_0$$

for all $k \in \mathbb{N}$. Hence $x_k \in B(0, r_0, t_0)$ for all $k \in \mathbb{N}$. By Theorem 4.10, $B(0, r_0, t_0)$ is an intuitionistic fuzzy compact set and so $\{x_k\}$ has an intuitionistic fuzzy convergent subsequence. □

Theorem 4.16. *Let A be an i-f-n-NLS and $T : A \to A$ be an intuitionistic fuzzy compact linear operator and $S : A \to A$ be a strongly intuitionistic fuzzy continuous linear operator. Then ST and TS are intuitionistic fuzzy compact operators.*

Proof. Follows from Theorem 3.10 and Theorem 3.8. □

Theorem 4.17. *Let A and B be two i-f-n-NLS satisfying (14), $N_2(y_1, y_2, \cdots, y_n, \centerdot)$ is a continuous function on R and $T : A \to B$ a linear operator. Then the following hold.*

(a) *If T is weakly intuitionistic fuzzy bounded and* $\dim T(X) < \infty$*, then T is an intuitionistic fuzzy compact operator.*
(b) *In addition if A and B satisfy (15) and* $\dim T(X) < \infty$*, then T is an intuitionistic fuzzy compact operator.*

Proof. (a) Let $\{x_k\}$ be an intuitionistic fuzzy bounded sequence in X. There exist $t_0 > 0$ and $r_0 \in (0, 1)$ such that

$$N_1(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \geq r_0 \text{ and } M_1(x_1, x_2, \cdots, x_{n-1}, x_k, t_0) \leq 1 - r_0$$

for all $k \in \mathbb{N}$. Since T is weakly intuitionistic fuzzy bounded, $\exists\ L_{r_0} > 0$ such that for all k

$$N_1(x_1, x_2, \cdots, x_{n-1}, x_k, \tfrac{t_0}{L_{r_0}}) \geq r_0 \Rightarrow N_2(y_1, y_2, \cdots, y_{n-1}, Tx_k, t_0) \geq r_0$$

and

$$M_1(x_1, x_2, \cdots, x_{n-1}, x_k, \tfrac{t_0}{L_{r_0}}) \leq 1 - r_0 \Rightarrow M_2(y_1, y_2, \cdots, y_{n-1}, Tx_k, t_0) \leq 1 - r_0.$$

It follows that $\{Tx_k\}$ is an intuitionistic fuzzy bounded sequence in $T(X)$. Since $\dim T(X) < \infty$, the sequence $\{Tx_k\}$ has an intuitionistic fuzzy convergent subsequence by Lemma 4.15. Hence T is intuitionistic fuzzy compact.

(b) T is weakly intuitionistic fuzzy continuous by Theorem 4.16. Furthermore Theorem 3.8 implies that T is weakly intuitionistic fuzzy bounded. Since $\dim T(X) < \infty$, by (a) we conclude that T is an intuitionistic fuzzy compact operator. □

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