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Yao's generalized rough approximations in left almost semigroups

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Abstract. In this paper, we shall introduce the notion of generalized rough left [right, two-sided, bi-, interior, quasi and prime] ideals in left almost semigroups, which is an extended notion of a left [right, two-sided, bi-, interior, quasi and prime] ideals in left almost semigroups.

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1. Introduction

The notion of a rough set was originally proposed by Pawlak [14], which is an excellent mathematic tool to deal with granularity of information. The indiscernibility relation is the mathematical basis for the rough set theory. The indiscernible objects form an elementary set and all elementary sets form a partition of the universe. The theory of rough set is an extension of set theory. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set, and the upper approximation is the union of all equivalence classes which have a nonempty intersection with the set. Some authors have studied the algebraic properties of rough sets. Biswas and Nanda [3], introduced the notion of rough subgroups. Kuroki, in [8], introduced the notion of a rough ideal in a semigroup. Also, Kuroki and Mordeson in [7] studied the structure of rough sets and rough groups. Yaqoob et al. [2] introduced the concept of rough (m,n)-bi-ideals and generalized rough (m,n)-bi-ideals in semigroups and in this paper the concept of generalized rough sets is applied to the theory of semigroups, also see [1, 20]. Davvaz applied the rough set theory to rings [4]. Yao [15, 16, 17, 18, 19] introduced the concept of generalized rough sets. Further, Kondo [6], studied the structure of generalized rough sets. Kondo considered some fundamental properties of generalized rough sets induced by binary relations on algebras and do not restrict the universe to be finite and consider fundamental properties of generalized rough sets induced by binary relations.

This paper concerns the relationship between generalized rough sets and left almost semigroups. The left almost semigroups abbreviated as an LA-semigroup was first introduced by Kazim and Naseerudin [5]. Mushtaq [9, 10, 11, 12, 13] investigated the structure further and added many useful results to the theory of LA-semigroups. In this paper, we introduced the notion of a generalized rough LA-subsemigroup (resp. ideal, bi-ideal, interior ideal, quasi-ideal and prime ideal) of an LA-semigroup which is an extended notion of an LA-subsemigroup (resp. ideal, bi-ideal, interior ideal, quasi-ideal and prime ideal).

2. Preliminaries and Basic Definitions

Definition 2.1 ([5])**.** A groupoid $(S, \cdot)$ is called an LA-semigroup if it satisfies left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \quad \text{for all } a, b, c \in S.$$

Example 2.2 ([9])**.** Let $(\mathbb{Z}, +)$ denote the commutative group of integers under addition. Define a binary operation "$*$" in $\mathbb{Z}$ as follows:

$$a * b = b - a, \quad \text{for all } a, b \in \mathbb{Z},$$

where "$-$" denotes the ordinary subtraction of integers. Then $(\mathbb{Z}, *)$ is an LA-semigroup.

In an LA-semigroup the medial law holds: $(ab)(cd) = (ac)(bd)$, for all $a, b, c, d \in S$. In an LA-semigroup S with left identity, the paramedial law holds: $(ab)(cd) = (dc)(ba)$, for all $a, b, c, d \in S$. If an LA-semigroup contain a left identity then $a(bc) = b(ac)$ holds for all $a, b, c \in S$. ($cf.$ [12])

Definition 2.3 ([11])**.** A binary relation θ on an LA-semigroup S is called compatible if $a\theta b \Rightarrow as\theta bs$ and $sa\theta sb$ for all $s \in S$.

3. Generalized Rough Subsets in LA-semigroups

Let X be a nonempty set and θ be a binary relation on X. By $\wp(X)$ we mean the power set of X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \wp(X) \longrightarrow \wp(X)$ by

$$\theta_-(A) = \{x \in X : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\},$$

and

$$\theta_+(A) = \{x \in X : \exists\ y \in A,\ \text{such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \phi\},$$

where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the lower approximation and the upper approximation operations, respectively ([2, 6]).

Example 3.1 ([2])**.** Let $X = \{a, b, c\}$ and $\theta = \{(a, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$. Then $\theta N(a) = \{a\}$; $\theta N(b) = \{b, c\}$; $\theta N(c) = \{a, b, c\}$; $\theta_-(\{a\}) = \{a\}$; $\theta_-(\{b\}) = \phi$; $\theta_-(\{c\}) = \phi$; $\theta_-(\{a, b\}) = \{a\}$; $\theta_-(\{a, c\}) = \{a\}$; $\theta_-(\{b, c\}) = \{b\}$; $\theta_-(\{a, b, c\}) = \{a, b, c\}$; $\theta_+(\{a\}) = \{a, c\}$; $\theta_+(\{b\}) = \{b, c\}$; $\theta_+(\{c\}) = \{b, c\}$; $\theta_+(\{a, b\}) = \{a, b, c\}$; $\theta_+(\{a, c\}) = \{a, b, c\}$; $\theta_+(\{b, c\}) = \{b, c\}$; $\theta_+(\{a, b, c\}) = \{a, b, c\}$.

Theorem 3.2. *Let θ and λ be reflexive, transitive and compatible relations on an LA-semigroup S. If A and B are nonempty subsets of S, then the following hold:*

(1) $\theta_-(A) \subseteq A \subseteq \theta_+(A)$;
(2) $\theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B)$;
(3) $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$;
(4) $A \subseteq B$ *implies* $\theta_-(A) \subseteq \theta_-(B)$;
(5) $A \subseteq B$ *implies* $\theta_+(A) \subseteq \theta_+(B)$;
(6) $\theta_-(A \cup B) \supseteq \theta_-(A) \cup \theta_-(B)$;
(7) $\theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B)$;
(8) $\theta \subseteq \lambda$ *implies* $\theta_-(A) \supseteq \lambda_-(A)$;
(9) $\theta \subseteq \lambda$ *implies* $\theta_+(A) \subseteq \lambda_+(A)$.

Proof. (1) For all $a \in \theta_-(A)$, since θ is reflexive so $a\theta a$ implies $a \in A$. Thus $\theta_-(A) \subseteq A$. Now let $a \in A$ then $a\theta a$. Then by definition of θ_+, $a \in \theta_+(A)$. Hence $\theta_-(A) \subseteq A \subseteq \theta_+(A)$.

(2) Let $a \in \theta_+(A \cup B)$. Then

$$\begin{aligned} \theta N(a) \cap (A \cup B) \neq \varnothing &\iff (\theta N(a) \cap A) \cup (\theta N(a) \cap B) \neq \varnothing \\ &\iff \theta N(a) \cap A \neq \varnothing \text{ or } \theta N(a) \cap B \neq \varnothing \\ &\iff a \in \theta_+(A) \text{ or } a \in \theta_+(B) \\ &\iff a \in \theta_+(A) \cup \theta_+(B). \end{aligned}$$

Thus $\theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B)$.

(3) Let $a \in \theta_-(A \cap B)$. Then

$$\begin{aligned} \theta N(a) \subseteq A \cap B &\iff \theta N(a) \subseteq A \text{ and } \theta N(a) \subseteq B \\ &\iff a \in \theta_-(A) \text{ and } a \in \theta_-(B) \\ &\iff a \in \theta_-(A) \cap \theta_-(B). \end{aligned}$$

Thus $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$.

(4) Since $A \subseteq B$, so $A \cap B = A$. Thus by (3) we have

$$\theta_-(A) = \theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B).$$

This implies that $\theta_-(A) \subseteq \theta_-(B)$.

(5) Since $A \subseteq B$, so $A \cup B = B$. Thus by (2) we have

$$\theta_+(B) = \theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B).$$

This implies that $\theta_+(A) \subseteq \theta_+(B)$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so by (4) we have

$$\theta_-(A) \subseteq \theta_-(A \cup B) \text{ and } \theta_-(B) \subseteq \theta_-(A \cup B),$$

which yields $\theta_-(A) \cup \theta_-(B) \subseteq \theta_-(A \cup B)$.

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (5) we have

$$\theta_+(A \cap B) \subseteq \theta_+(A) \text{ and } \theta_+(A \cap B) \subseteq \theta_+(B),$$

which yields $\theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B)$.

(8) Since $\theta \subseteq \lambda$. Then for each $a \in \lambda_-(A)$, we have

$$\theta N(a) \subseteq \lambda N(a) \subseteq A$$

$$\Longrightarrow \theta N(a) \subseteq A$$
$$\Longrightarrow a \in \theta_-(A).$$

Thus $\lambda_-(A) \subseteq \theta_-(A)$.

(9) Let $a \in \theta_+(A)$, then $\theta N(a) \cap A \neq \varnothing$. Thus there exists $x \in \theta N(a) \cap A$. Since $\theta \subseteq \lambda$, we have

$$x \in \theta N(a) \subseteq \lambda N(a) \text{ and } x \in A.$$

Thus $x \in \lambda N(a) \cap A$ and so $a \in \lambda_+(A)$. Hence $\theta_+(A) \subseteq \lambda_+(A)$. □

Theorem 3.3. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. If A and B are nonempty subsets of S, then*

$$\theta_+(A)\theta_+(B) \subseteq \theta_+(AB).$$

Proof. Let c be any element of $\theta_+(A)\theta_+(B)$. Then $c = ab$ where $a \in \theta_+(A)$ and $b \in \theta_+(B)$. Thus there exist elements $x, y \in S$ such that

$$x \in A \text{ and } a\theta x \text{ ; } y \in B \text{ and } b\theta y.$$

Since θ is compatible relation on S, so $ab\theta xy$. As $xy \in AB$, so we have

$$c = ab \in \theta_+(AB).$$

Thus $\theta_+(A)\theta_+(B) \subseteq \theta_+(AB)$. □

Definition 3.4. Let θ be a compatible relation on an LA-semigroup S then for each $a, b \in S$, $\theta N(a)\theta N(b) \subseteq \theta N(ab)$. If

$$\theta N(a)\theta N(b) = \theta N(ab),$$

then θ is called complete compatible relation.

Theorem 3.5. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S and A, B are nonempty subsets of S. Then*

$$\theta_-(A)\theta_-(B) \subseteq \theta_-(AB).$$

Proof. Let c be any element of $\theta_-(A)\theta_-(B)$. Then $c = ab$ where $a \in \theta_-(A)$ and $b \in \theta_-(B)$. Thus we have $\theta N(a) \subseteq A$ and $\theta N(b) \subseteq B$. Since θ is complete compatible relation on LA-semigroup S, so we have

$$\theta N(ab) = \theta N(a)\theta N(b) \subseteq AB,$$

which implies that $ab \in \theta_-(AB)$. Thus $\theta_-(A)\theta_-(B) \subseteq \theta_-(AB)$. □

Theorem 3.6. *Let θ and λ be reflexive, transitive and compatible relations on an LA-semigroup S. If A is a nonempty subset of S, then*

$$(\theta \cap \lambda)_+(A) \subseteq \theta_+(A) \cap \lambda_+(A).$$

Proof. Note that $\theta \cap \lambda$ is also a reflexive, transitive and compatible relation on an LA-semigroup S. Let $c \in (\theta \cap \lambda)_+(A)$. Then $(\theta \cap \lambda)N(c) \cap A \neq \varnothing$. Let $a \in (\theta \cap \lambda)N(c) \cap A$. Then $a \in (\theta \cap \lambda)N(c)$ and $a \in A$. Now

$$(c, a) \in (\theta \cap \lambda) \Longrightarrow (c, a) \in \theta \text{ and } (c, a) \in \lambda.$$

Thus we have $a \in \theta N(c)$ and $a \in \lambda N(c)$. Since $a \in A$, so

$$a \in \theta N(c), a \in A \text{ and } a \in \lambda N(c), a \in A,$$

which implies that $c \in \theta_+(A)$ and $c \in \lambda_+(A)$, and so

$$c \in \theta_+(A) \cap \lambda_+(A).$$

Thus $(\theta \cap \lambda)_+(A) \subseteq \theta_+(A) \cap \lambda_+(A)$. □

Theorem 3.7. *Let θ and λ be reflexive, transitive and compatible relations on an LA-semigroup S. If A is a nonempty subset of S, then*

$$(\theta \cap \lambda)_-(A) = \theta_-(A) \cap \lambda_-(A).$$

Proof. Let c be any element of $(\theta \cap \lambda)_-(A)$. Then

$$\begin{aligned} (\theta \cap \lambda)N(c) &\subseteq A \\ &\Longleftrightarrow \theta N(c) \subseteq A \text{ and } \lambda N(c) \subseteq A \\ &\Longleftrightarrow c \in \theta_-(A) \text{ and } c \in \lambda_-(A) \\ &\Longleftrightarrow c \in \theta_-(A) \cap \lambda_-(A). \end{aligned}$$

Thus $(\theta \cap \lambda)_-(A) = \theta_-(A) \cap \lambda_-(A)$. □

4. Generalized rough ideals in LA-semigroups

A subset A of an LA-semigroup S is called an LA-subsemigroup of S if $AA \subseteq A$ and A is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$) and is called two sided ideal of S if it is both a left ideal and a right ideal. Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. Then a nonempty subset A of S is called a generalized upper rough LA-subsemigroup of S if $\theta_+(A)$ is an LA-subsemigroup of S and A is called a generalized upper rough left (right, two-sided) ideal of S if $\theta_+(A)$ is a left (right, two-sided) ideal of S.

Theorem 4.1. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. Then*

(1) *If A is an LA-subsemigroup of S, then A is generalized upper rough LA-subsemigroup of S.*
(2) *If A is a left (right, two sided) ideal of S, then A is generalized upper rough left (right, two-sided) ideal of S.*

Proof. (1) Let A be an LA-subsemigroup of S. Then by Theorem 3.2(1),

$$\varnothing \neq A \subseteq \theta_+(A).$$

By Theorem 3.3 and Theorem 3.2(5), we have

$$\theta_+(A)\theta_+(A) \subseteq \theta_+(AA) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is an LA-subsemigroup of S, that is A is a generalized upper rough LA-subsemigroup of S.

(2) Let A be a left ideal of S. Note that $\theta_+(S) = S$. Now by Theorem 3.3

$$S\theta_+(A) = \theta_+(S)\theta_+(A) \subseteq \theta_+(SA) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is a left ideal of S, that is A is a generalized upper rough left ideal of S. The other cases can be proved in a similar way. □

Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. Then a nonempty subset A of S is called a generalized lower rough LA-subsemigroup of S if $\theta_{-}(A)$ is an LA-subsemigroup of S and A is called a generalized lower rough left (right, two-sided) ideal of S if $\theta_{-}(A)$ is a left (right, two sided) ideal of S.

Theorem 4.2. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. Then*

(1) *If A is an LA-subsemigroup of S, then $\theta_{-}(A)$ is, if it is nonempty, an LA-subsemigroup of S.*
(2) *If A is a left (right, two-sided) ideal of S, then $\theta_{-}(A)$ is, if it is nonempty, a left (right, two-sided) ideal of S.*

Proof. (1) Let A be an LA-subsemigroup of S. Then by Theorem 3.5 and Theorem 3.2(4),

$$\theta_{-}(A)\theta_{-}(A) \subseteq \theta_{-}(AA) \subseteq \theta_{-}(A).$$

Thus $\theta_{-}(A)$ is, if it is nonempty, an LA-subsemigroup of S.

(2) Let A be a left ideal of S. Then by Theorem 3.5,

$$S\theta_{-}(A) = \theta_{-}(S)\theta_{-}(A) \subseteq \theta_{-}(SA) \subseteq \theta_{-}(A).$$

Thus $\theta_{-}(A)$ is, if it is nonempty, a left ideal of S. The other cases can be proved in a similar way. □

Theorem 4.3. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S, let A be a left ideal of S. Then*

(1) *$\theta_{+}(A^2)$ is a right ideal of S.*
(2) *If θ is complete, then $\theta_{-}(A^2)$ is a right ideal of S.*

Proof. (1) Let A be a left ideal of an LA-semigroup S. Now

$$\begin{aligned}
\theta_{+}(A^2)S &= \theta_{+}(A^2)\theta_{+}(S) && (\ \theta_{+}(S) = S\ ) \\
&\subseteq \theta_{+}(A^2S) && (\text{by Theorem } 3.3) \\
&= \theta_{+}[(AA)S] && \\
&= \theta_{+}[(SA)A] && (\text{left invertive law}) \\
&\subseteq \theta_{+}(AA) && (\text{because } SA \subseteq A) \\
&= \theta_{+}(A^2).
\end{aligned}$$

Hence we get $\theta_{+}(A^2)S \subseteq \theta_{+}(A^2)$. This shows that A^2 is an upper rough right ideal of S. By using Theorem 3.5, the proof of (2) can be seen in a similar way. □

Theorem 4.4. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S, let A be a left ideal of S. Then*

(1) *$[\theta_{+}(A)]^2$ is a right ideal of S.*
(2) *If θ is complete, then $[\theta_{-}(A)]^2$ is a right ideal of S.*

Proof. (1) Let A be a left ideal of an LA-semigroup S. Now

$$\begin{aligned}
[\theta_+(A)]^2\ S &= [\theta_+(A)]^2\ S \\
&= [\theta_+(A)]^2\ \theta_+(S) && (\ \theta_+(S) = S\ ) \\
&= [\theta_+(A)\ \theta_+(A)]\ \theta_+(S) \\
&= [\theta_+(S)\ \theta_+(A)]\ \theta_+(A) && \text{(left invertive law)} \\
&\subseteq \theta_+(SA)\ \theta_+(A) && \text{(by Theorem 3.3)} \\
&= \theta_+(A)\ \theta_+(A) && \text{(because } SA \subseteq A) \\
&= [\theta_+(A)]^2.
\end{aligned}$$

Hence we get $[\theta_+(A)]^2\ S \subseteq [\theta_+(A)]^2$. This shows that $[\theta_+(A)]^2$ is a right ideal of S. By using Theorem 3.5, the proof of (2) can be seen in a similar way. □

5. Generalized rough bi-ideals in LA-semigroups

An LA-subsemigroup A of an LA-semigroup S is called a bi-ideal of S if $(AS)A \subseteq A$. A subset A of an LA-semigroup S is called a generalized upper (lower) rough bi-ideal of S if $\theta_+(A)$ $(\theta_-(A))$ is a bi-ideal of S.

Theorem 5.1. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. If A is a bi-ideal of S, then it is a generalized upper rough bi-ideal of S.*

Proof. Let A be a bi-ideal of S. Then by Theorem 3.3 and Theorem 4.1(1), we have

$$(\theta_+(A)S)\theta_+(A) = (\theta_+(A)\theta_+(S))\theta_+(A) \subseteq \theta_+((AS)A) \subseteq \theta_+(A).$$

Thus we have $\theta_+(A)$ is a bi-ideal of S, that is A is a generalized upper rough bi-ideal of S. □

The following example shows that the converse of above theorem does not hold.

Example 5.2. Let $S = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" on S be defined as follows:

$\cdot$	1	2	3	4
1	4	2	3	4
2	2	2	2	2
3	3	2	3	3
4	3	2	3	3

Clearly, S is an LA-semigroup. But S is not a semigroup because $4 = 1 \cdot (1 \cdot 4) \neq (1 \cdot 1) \cdot 4 = 3$. Now let

$$\theta = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$$

be a reflexive, transitive and compatible relation on S such that

$$\theta N(1) = \{1\},\ \theta N(2) = \{2, 3\},\ \theta N(3) = \{3\} \text{ and } \theta N(4) = \{4\}.$$

Now for $\{3\} \subseteq S$, $\theta_+(\{3\}) = \{2, 3\}$. Here

$$(\{2, 3\}S)\{2, 3\} \subseteq \{2, 3\}$$

$$\text{but}\ \ (\{3\}S)\{3\} = \{2, 3\} \nsubseteq \{3\}.$$

It is clear that $\theta_+(\{3\})$ is a bi-ideal of S but the LA-subsemigroup $\{3\}$ of S is not a bi-ideal of S.

Theorem 5.3. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. If A is a bi-ideal of S, then $\theta_-(A)$ is, if it is nonempty, a bi-ideal of S.*

Proof. Let A be a bi-ideal of S. Then by Theorem 3.5 and Theorem 4.2(1), we have

$$(\theta_-(A)S)\theta_-(A) = (\theta_-(A)\theta_-(S))\theta_-(A) \subseteq \theta_-((AS)A) \subseteq \theta_-(A).$$

Thus we obtain that $\theta_-(A)$ is, if it is nonempty, a bi-ideal of S. □

The following example shows that the converse of above theorem does not hold.

Example 5.4. Consider the Example 5.2 and let

$$\theta = \{(1,1),(2,2),(3,3),(4,3),(4,4)\}$$

be a reflexive, transitive and complete compatible relation on S such that

$$\theta N(1) = \{1\},\ \theta N(2) = \{2\},\ \theta N(3) = \{3\} \text{ and } \theta N(4) = \{3,4\}.$$

Now for $\{2,4\} \subseteq S$, $\theta_-(\{2,4\}) = \{2\}$. Here $(\{2\}S)\{2\} \subseteq \{2\}$ but

$$(\{2,4\}S)\{2,4\} = \{2,3\} \nsubseteq \{2,4\}.$$

It is clear that $\theta_-(\{2,4\})$ is a bi-ideal of S but the subset $\{2,4\}$ of S is not a bi-ideal of S.

Theorem 5.5. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. If A and B are a right and a left ideal of S respectively, then*

$$\theta_+(AB) \subseteq \theta_+(A) \cap \theta_+(B).$$

Proof. Since A is a right ideal of S, $AB \subseteq AS \subseteq A$, and since B is a left ideal of S, $AB \subseteq SB \subseteq B$. Thus $AB \subseteq A \cap B$. Then by Theorem 3.2(7), we have

$$\theta_+(AB) \subseteq \theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B).$$

This completes the proof. □

Theorem 5.6. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. If A is a right and B is a left ideal of S, then*

$$\theta_-(AB) \subseteq \theta_-(A) \cap \theta_-(B).$$

Proof. Since A is a right ideal of S, $AB \subseteq AS \subseteq A$, and since B is a left ideal of S, $AB \subseteq SB \subseteq B$. Thus $AB \subseteq A \cap B$. Then by Theorem 3.2(3), we have

$$\theta_-(AB) \subseteq \theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B).$$

This completes the proof. □

6. Generalized rough interior ideals in LA-semigroups

A subset A of an LA-semigroup S is called an interior ideal of S if $(SA)S \subseteq A$. Let A be a nonempty subset of S. θ be a reflexive, transitive and compatible relation on an LA-semigroup S. Then A is called a generalized lower (upper) rough interior ideal of S, if $\theta_-(A)$ $(\theta_+(A))$ is an interior ideal of S.

Theorem 6.1. *Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. If A is an interior ideal of S, then A is a generalized upper rough interior ideal of S.*

Proof. Since A is an interior ideal of an LA-semigroup S, we have $(SA)S \subseteq A$. Then by Theorem 3.3, we have

$$(S\theta_+(A))S = (\theta_+(S)\theta_+(A))\theta_+(S) \subseteq \theta_+((SA)S) \subseteq \theta_+(A),$$

which yields that $\theta_+(A)$ is an interior ideal of S. □

Theorem 6.2. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. If A is an interior ideal of S, then $\theta_-(A)$ is, if it is nonempty, an interior ideal of S.*

Proof. Let A be an interior ideal of S. Then it follows from Theorem 3.5 that

$$(S\theta_-(A))S = (\theta_-(S)\theta_-(A))\theta_-(S) \subseteq \theta_-((SA)S) \subseteq \theta_-(A),$$

which implies that $\theta_-(A)$ is an interior ideal of S. □

We call A a generalized rough interior ideal of S if it is both a lower and upper generalized rough interior ideal.

7. Generalized rough quasi-ideals in LA-semigroups

A nonempty subset Q of an LA-semigroup S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$. Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. A subset Q of an LA-semigroup S is called a generalized lower rough quasi-ideal of S if $\theta_-(Q)$ is a quasi-ideal of S.

Theorem 7.1. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. If Q is a quasi-ideal of S, then Q is a generalized lower rough quasi-ideal of S.*

Proof. Let Q be a quasi-ideal of S. Now by Theorem 3.2(3) and Theorem 3.5, we get

$$\begin{aligned} \theta_-(Q)S \cap S\theta_-(Q) &= \theta_-(Q)\theta_-(S) \cap \theta_-(S)\theta_-(Q) \\ &\subseteq \theta_-(QS) \cap \theta_-(SQ) \\ &= \theta_-(QS \cap SQ) \\ &\subseteq \theta_-(Q). \end{aligned}$$

Thus we obtain that $\theta_-(Q)$ is a quasi-ideal of S, that is, Q is a generalized lower rough quasi-ideal of S. □

Theorem 7.2. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. Let L and R be a generalized lower rough left ideal and a generalized lower rough right ideal of S, respectively. Then $L \cap R$ is a generalized lower rough quasi-ideal of S.*

Proof. Let L and R be a generalized lower rough left ideal and a generalized lower rough right ideal of S, respectively. Then $S\theta_-(L) \subseteq \theta_-(L)$ and $\theta_-(R)S \subseteq \theta_-(R)$. We have

$$\theta_-(R)\theta_-(L) \subseteq S\theta_-(L) \cap \theta_-(R)S \subseteq \theta_-(L) \cap \theta_-(R) = \theta_-(L \cap R).$$

Then $\theta_-(L \cap R)$ is nonempty. We have

$$\begin{aligned} S\theta_-(L \cap R) \cap \theta_-(L \cap R)S &\subseteq S\theta_-(L) \cap \theta_-(R)S \\ &\subseteq \theta_-(L) \cap \theta_-(R) \\ &= \theta_-(L \cap R). \end{aligned}$$

Then $\theta_-(L \cap R)$ is a quasi-ideal of S. Hence $L \cap R$ is a generalized lower rough quasi-ideal of S. □

8. Generalized rough prime ideals in LA-semigroups

An ideal A of an LA-semigroup S is called a prime ideal if $xy \in A$ implies $x \in A$ or $y \in A$, for all $x, y \in S$. Let θ be a reflexive, transitive and compatible relation on an LA-semigroup S. Then a subset A of S is called a generalized lower (upper) rough prime ideal of S if $\theta_-(A)$ $(\theta_+(A))$ is a prime ideal of S.

Theorem 8.1. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S. If A is a prime ideal of S, then A is generalized upper rough prime ideal of S.*

Proof. Since A is a prime ideal of S, it follows from Theorem 4.1(2) that $\theta_+(A)$ is an ideal of S. Let $xy \in \theta_+(A)$ for some $x, y \in S$. Then

$$\theta N(xy) \cap A = \theta N(x)\theta N(y) \cap A \neq \phi,$$

so there exist elements

$$x' \in \theta N(x) \text{ and } y' \in \theta N(y) \text{ such that } x'y' \in A.$$

Since A is a prime ideal of S, so we have $x' \in A$ or $y' \in A$. Thus $\theta N(x) \cap A \neq \phi$ or $\theta N(y) \cap A \neq \phi$, and so $x \in \theta_+(A)$ or $y \in \theta_+(A)$. Therefore $\theta_+(A)$ is a prime ideal of S. □

Theorem 8.2. *Let θ be a reflexive, transitive and complete compatible relation on an LA-semigroup S and A is a prime ideal of S. Then $\theta_-(A)$ is, if it is nonempty, a prime ideal of S.*

Proof. Since A is an ideal of S, by Theorem 4.2(2), we have $\theta_-(A)$ is an ideal of S. Let $xy \in \theta_-(A)$ for some $x, y \in S$. Then

$$\theta N(xy) \subseteq A, \text{ which implies that } \theta N(x)\theta N(y) \subseteq \theta N(xy) \subseteq A.$$

We suppose that $\theta_-(A)$ is not a prime ideal of S. Then there exists $x, y \in S$ such that $xy \in \theta_-(A)$ but $x \notin \theta_-(A)$ and $y \notin \theta_-(A)$. Thus $\theta N(x) \nsubseteq A$ and $\theta N(y) \nsubseteq A$. Then there exists $x' \in \theta N(x)$, $x' \notin A$ and $y' \in \theta N(y)$, $y' \notin A$. Thus

$$x'y' \in \theta N(x)\theta N(y) \subseteq A.$$

Since A is a prime ideal of S, we have $x' \in A$ or $y' \in A$. It contradicts our supposition. This means that $\theta_-(A)$ is, if it is nonempty, a prime ideal of S. □

The following example shows that the converse of Theorem 8.1 and Theorem 8.2 does not hold.

Example 8.3. Let $S = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" on S be defined as follows:

$\cdot$	1	2	3	4
1	3	3	3	4
2	1	3	3	4
3	3	3	3	4
4	4	4	4	4

Clearly, S is an LA-semigroup. But S is not a semigroup because $1 = 2 \cdot (2 \cdot 1) \neq (2 \cdot 2) \cdot 1 = 3$. Now let

$$\theta = \{(1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

be a reflexive, transitive and complete compatible relation on S such that

$$\theta N(1) = \{1, 3, 4\},\ \theta N(2) = \{2, 3, 4\},\ \theta N(3) = \{3, 4\} \text{ and } \theta N(4) = \{4\}.$$

Now for $\{2, 4\} \subseteq S$, $\theta_-(\{2, 4\}) = \{4\}$ and $\theta_+(\{2, 4\}) = \{1, 2, 3, 4\}$. It is clear that $\theta_-(\{2, 4\})$ and $\theta_+(\{2, 4\})$ are prime ideals of S. The subset $\{2, 4\}$ is not an ideal and hence not a prime ideal.

We call A a generalized rough prime ideal of S, if it is both a lower and an upper generalized rough prime ideal of S.

Let θ and ϕ be binary relations on an LA-semigroup S. Then the product $\theta \circ \phi$ of θ and ϕ defined as follows:

$$\theta \circ \phi = \{(a, b) \in S \times S : (a, c) \in \theta \text{ and } (c, b) \in \phi \text{ for some } c \in S\}.$$

Lemma 8.4. *Let θ and ϕ are compatible relations on an LA-semigroup S. Then $\theta \circ \phi$ is also a compatible relation on S.*

Proof. Let $(a, b) \in \theta \circ \phi$ and $x \in S$. Then $(a, c) \in \theta$ and $(c, b) \in \phi$ for some $c \in S$. Since θ and ϕ are compatible relations on S, so $(xa, xc) \in \theta$ and $(xc, xb) \in \phi \Longrightarrow (xa, xb) \in \theta \circ \phi$. Similarly $(ax, bx) \in \theta \circ \phi$. Thus $\theta \circ \phi$ is a compatible relation on S. □

Theorem 8.5. *Let θ and ϕ be reflexive, transitive and compatible relations on an LA-semigroup S. If A is an LA-subsemigroup of S, then*

$$\theta_+(A)\phi_+(A) \subseteq (\theta \circ \phi)_+(A).$$

Proof. Let c be any element of $\theta_+(A)\phi_+(A)$. Then $c = ab$ with $a \in \theta_+(A)$ and $b \in \phi_+(A)$. Then there exists elements $x, y \in S$, such that $x \in \theta N(a) \cap A$ and $y \in \phi N(b) \cap A$. Thus $x \in \theta N(a)$, $y \in \phi N(b)$ and x, $y \in A$. Since A is an LA-subsemigroup of S, we have $xy \in A$. Then $(x, a) \in \theta$ and $(y, b) \in \phi$ and since θ and ϕ are compatible relations, we have

$$(xy, ay) \in \theta \text{ and } (ay, ab) \in \phi.$$

Thus we have $(xy, ab) \in \theta \circ \phi$, and so $xy \in (\theta \circ \phi)N(ab)$. Therefore we have

$$xy \in (\theta \circ \phi)N(ab) \cap A,$$

which yields $c = ab \in (\theta \circ \phi)_+(A)$. Thus we obtain that $\theta_+(A)\phi_+(A) \subseteq (\theta \circ \phi)_+(A)$. □

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